CS412 Spring Semester 2013

Homework Assignment #2

Due Tuesday February 26th 2013, in class

Sum of all problems : 120%, Maximum possible score : 100%.

For the following problems, even if you do not answer a given question, you are still allowed to use its result in order to answer a subsequent question.

1. [40%] Let $c_0, c_1, \ldots, c_n$ be the coefficients of the Newton interpolation method. That is, the interpolating polynomial is given as

$$P_n(x) = \sum_{i=0}^{n} c_i n_i(x), \quad \text{where} \quad n_i(x) = \prod_{j=0}^{i-1}(x - x_j)$$

   (a) [10%] Explain why the polynomial $P_n(x)$ can be recursively evaluated as follows (you can look at your 2/14 lecture notes for guidance):

   - Define $Q_n(x) = c_n$
   - Recursively define $Q_{k-1}(x) = c_{k-1} + (x - x_{k-1})Q_k(x)$
   - After k steps, the polynomial interpolant is given as $P_n(x) = Q_0(x)$.

   (b) [10%] Using this process, how many operations do we need to evaluate $P_n(x)$ for an arbitrary value of x?

   (c) [20%] Show that

$$Q'_{k-1}(x) = Q_k(x) + (x - x_{k-1})Q'_k(x)$$

   and explain how these results can be used to compute values of $P'_n(x)$ with a number of arithmetic operations proportional to n.

2. [60%] Use **all three methods** we discussed in class (the Vandermonde matrix method, Lagrange interpolation and Newton interpolation) to find a cubic polynomial that interpolates the following data points:

$$(-2, -3)$$
$$(-1, 1)$$
$$(0, -1)$$
$$(1, -3)$$

   After applying each of these methods, convert the polynomial to the standard form $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ (if it is not already in this form), and verify that all 3 methods produced the same result.

   Note: When using the Vandermonde matrix method, it is acceptable to write the linear system of equations whose solution are the coefficients $a_0, \ldots, a_3$, and simply verify that the coefficients computed by the other methods satisfy this system.

3. [20%] [Bonus question] Consider the $n+1$ data points $(x_0, y_0), \ldots, (x_n, y_n)$, and the Lagrange polynomials $\{l_i(x)\}_{i=0}^{n}$ associated with these points. Show that, if we were to write the i-th Lagrange polynomial in the canonical form

$$l_i(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

then, the coefficients $c_0, c_1, \ldots, c_n$ are the elements of the i-th column of $\mathbf{V}^{-1}$ ($\mathbf{V}$ is the Vandermode matrix associated with these points).

Hint: Use the Vandermonde method to construct a polynomial with the properties that define $l_i(x)$, namely that $l_i(x_j) = 0$ when $i \neq j$ and $l_i(x_i) = 1$.