CS412 Spring Semester 2013

Solutions to Midterm #1

Thursday 21 February 2013

1. [30% = 6 questions × 5% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) An iterative method for solving $f(x) = 0$ has an error that satisfies the inequality $|e_{k+1}| < C|e_k|^{1.6}$. Which of the following are true? **(Circle or underline ALL correct answers)**

   (a) Convergence is guaranteed if $C < 1$.

   (b) Convergence is guaranteed if we start our iteration close enough to the solution.

   (c) The correct significant digits are expected to double after each iteration.

   (2) Which of the following facts about polynomial interpolation are true? **(Circle or underline ALL correct answers)**

   (a) The Vandermonde matrix approach makes it difficult to evaluate derivatives of the constructed interpolant.

   (b) An interpolated polynomial $P_n(x)$ constructed via Newton's method can be evaluated at a given value x with cost proportional to the polynomial degree n.

   (c) If all we need to do after constructing the polynomial interpolant is to evaluate $P_n(x)$ at a *single* location x, the Lagrange interpolation method would be very appropriate for the task.

   (3) What is the best reason one might want to use the Secant method, instead of Newton's method (for solving $f(x) = 0$)? **(Circle or underline the ONE most correct answer)**

   (a) The Secant method avoids the risk of division by zero, when $f'(x) = 0$ near the solution.

   (b) The Secant method can be used without having a formula for $f'(x)$.

   (c) Newton's method exhibits slower convergence than the Secant method.

(4) Which of these statements about the Bisection method are true?
(Circle or underline ALL correct answers)

(a) In order to use the Bisection method, we need to have a continuous function $f(x)$, with continuous derivative $f'(x)$.

(b) The order of convergence for bisection is linear.

(c) We can use the Bisection method without any knowledge of the derivative $f'(x)$.

(5) What happens if we use Newton's method to solve $f(x) = 0$, and the derivative $f'(x)$ happens to be zero at the exact location of the solution?
(Circle or underline the ONE most correct answer)

(a) Newton should be avoided; the Secant method would be more robust in this case.

(b) We may still be able to use Newton's method, if we have the explicit formula for $f(x)$. The order of convergence would however degrade to just linear in this case.

(c) Newton would still have a quadratic order of convergence, but we would need to start with an initial guess that is very close to the solution.

(6) Given all the nice properties of Newton interpolation, why would we even care to consider the Vandermonde matrix as a good option for an interpolation problem?
(Circle or underline the ONE most correct answer)

(a) We would never use Vandermonde, since Newton is better in all cases.

(b) For small enough problems, all three methods are very inexpensive, and it may be simpler to just solve the Vandermonde matrix rather than dealing with the slightly more intricate Divided Differences table.

(c) Although the Vandermonde matrix approach requires more effort to compute the polynomial interpolant, evaluating this polynomial is dramatically cheaper if it is constructed via the Vandermonde matrix as opposed to Newton interpolation.

2. [25% = 5 questions × 5% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 1-2 sentences.

(a) Write the iterative formula of Newton's method for solving the nonlinear equation $3x = \sin(2x) + 1$.

We reformulate the equation as $f(x) = 3x - \sin(2x) - 1 = 0$. *The derivative is* $f'(x) = 3 - 2\cos(2x)$. *Newton's method then becomes:*

$$x_{k+1} = x_k + \frac{f(x_k)}{f'(x_k)} = x_k + \frac{3x_k - \sin(2x_k) - 1}{3 - 2\cos(2x_k)}$$

(b) If the *relative error* of a computation is no more than 0.002, and the *exact* value is $x = 50$, write the interval where our computed approximation must lie.

Let $x = 50$ *denote the exact result of the computation and* $\hat{x}$ *be the computed approximation. The relative error is* $e_{rel} = |\hat{x} - x|/|x|$, *and:*

$$\frac{|\hat{x} - x|}{|x|} \leq 0.002 \Rightarrow \frac{|\hat{x} - 50|}{50} \leq 0.002 \Rightarrow |\hat{x} - 50| \leq 0.1 \Rightarrow \hat{x} \in [49.9, 50.1]$$

(c) State (a) one advantage of the Vandermonde matrix method over Lagrange interpolation and (b) one advantage of Newton interpolation vs. Lagrange interpolation.

A polynomial constructed with the Vandermonde matrix method can be subsequently evaluated in $O(n)$ *time (vs.* $O(n^2)$ *for Lagrange), and provides easy access to derivatives.*
Newton iterpolation also allows for $O(n)$ *evaluation cost, and can provide access to derivatives with relatively small cost.*

(d) Assume that a nonlinear equation $f(x) = 0$ is such that Newton's method *always* converges. Can you think of any scenario where using the Bisection method instead might be a better option?

If $f'(x) = 0$ *at the solution, at best Newton's method will degrade to linear convergence (same as bisection) or in the worst case even risk dividing by zero. Using Bisection might then be the safer approach.*
We may also choose bisection if we don't know an explicit formula for $f'(x)$, *and have reason to suspect that the Secant method does not*

have the same unconditional guarantees for convergence as Newton's method does.

(e) Assume that the fixed point iteration methods

$$x_{k+1} = g(x_k) \quad \text{and} \quad x_{k+1} = h(x_k)$$

are *both* guaranteed to converge, and their order of convergence is quadratic. Would the fixed point iteration

$$x_{k+1} = \frac{g(x_k) + h(x_k)}{2}$$

also exhibit quadratic convergence, or not? Explain.

This new fixed point iteration is written as $x_{k+1} = q(x_k)$*, where* $q(x) = \frac{g(x)+h(x)}{2}$*. If the first two fixed point iterations have quadratic convergence, we have that* $g'(a) = h'(a) = 0$ *at the solution* $x = a$*. Consequenly,* $q'(a) = 0$ *as well, and the newest fixed point iteration will convergen quadratically as well.*

3. [15%] Using Newton interpolation, find a cubic polynomial that interpolates the four points:

$$(-2, -5),\ (-1, -6),\ (1, -8),\ (2, 3)$$

We construct the table of divided differences

x_i	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
-2	-5			
-1	-6	-1		
1	-8	-1	0	
2	3	0	4	1

Reading the coefficients of the Newton polynomials from the diagonal of this table, the final interpolant becomes:

$$\begin{aligned} p(x) &= -5 \cdot 1 + (-1) \cdot (x+2) + 0 \cdot (x+2)(x+1) + 1 \cdot (x+2)(x+1)(x-1) \\ &= x^3 + 2x^2 - 2x - 9 \end{aligned}$$

4. [15%] Consider the nonlinear equation $f(x) = e^{3x} - 2 = 0$. Prove that Newton's method will always converge if started from a *positive* initial guess, and that convergence will be quadratic.

(**Hint** : If you write Newton's method in the form $x_{k+1} = g(x_k)$, you can prove the first part of the question by showing that $g(x)$ is a contraction for $x > 0$, and the second part by showing that $g'(a) = 0$, where a is the solution).

The iteration function $g(x)$ for Newton's method is:

$$g(x) = x + \frac{f(x)}{f'(x)} = x - \frac{e^{3x} - 2}{3e^{3x}} = x - \frac{1}{3} + \frac{2}{3}e^{-3x}$$

and its derivative is

$$g'(x) = 1 - 2e^{-3x}$$

The condition $|g'(x)| < 1$ is equivalently written as

$$\begin{aligned} |1 - 2e^{-3x}| < 1 \Leftrightarrow \\ \Leftrightarrow -1 < 1 - 2e^{-3x} < 1 \\ \Leftrightarrow -2 < -2e^{-3x} < 0 \\ \Leftrightarrow 1 > e^{-3x} > 0 \end{aligned}$$

which is true for all $x > 0$, indicating that g is a contraction in this case. The solution a satisfies $f(a) = 0 \Rightarrow e^{3a} - 2 = 0 \Rightarrow e^{3a} = 2$. Thus

$$g'(a) = 1 - 2e^{-3a} = 1 - \frac{2}{e^{3a}} = 1 - \frac{2}{2} = 0$$

Consequently, convergence has to be quadratic.

5. [15%] The Lagrange method interpolates through the $n+1$ data points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

by constructing the interpolating polynomial $P_n(x)$ as

$$P_n(x) = y_0 l_0(x) + y_1 l_1(x) + \cdots + y_n l_n(x)$$

where the *Lagrange polynomials* $l_i(x)$ are defined as:

$$l_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

(a) [8%] Show that, for any value of x we have

$$l_0(x) + l_1(x) + l_2(x) + \cdots + l_n(x) = 1$$

[**Hint:** What would be the polynomial that interpolates these $n+1$ data points, if *all* their y-values were equal to the same constant, i.e. $y_0 = y_1 = \cdots = y_n = c$?]

(b) [7%] Show that, for any value of x we have

$$x_0 l_0'(x) + x_1 l_1'(x) + x_2 l_2'(x) + \cdots + x_n l_n'(x) = 1$$

[**Hint:** How would you interpolate through the aforementioned $n+1$ data points, if all those locations were sampled on the straight line f(x)=x (i.e. $x_0 = y_0, x_1 = y_1, \ldots, x_n = y_n$)?]

(a) Let us use the Lagrange method to interpolate through the following points

$$(x_0, c), (x_1, c), (x_2, c), \ldots, (x_n, c)$$

Obviously, the unique n-degree interpolant is simply the constant polynomial $p(x) = c$. The Lagrange method equivalently gives

$$c \cdot l_0(x) + c \cdot l_1(x) + c \cdot l_2(x) + \cdots + c \cdot l_n(x) = c$$
$$\Rightarrow l_0(x) + l_1(x) + l_2(x) + \cdots + l_n(x) = 1$$

(a) Now, let us interpolate through the following points

$$(x_0, x_0), (x_1, x_1), (x_2, x_2), \ldots, (x_n, x_n)$$

We can see that in this case the unique n-degree interpolant is the linear polynomial $p(x) = x$. The Lagrange method equivalently gives

$$x_0 l_0(x) + x_1 l_1(x) + x_2 l_2(x) + \cdots + x_n l_n(x) = x$$

Differentiating this equation we obtain

$$x_0 l_0'(x) + x_1 l_1'(x) + x_2 l_2'(x) + \cdots + x_n l_n'(x) = 1$$