# CS412 Introduction to Numerical Methods – Spring 2013

## Practice Midterm #1

1. MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) What would happen if we try to use an $N$-degree polynomial to interpolate $N$ data points?
   **(Circle or underline ALL correct answers)**

   (a) This is the normal case. A unique polynomial can be determined.
   (b) Such an interpolant generally exists, but it is not unique.
   (c) Such an interpolant will generally not exist.

   (2) Which of the following are benefits of the secant method for solving $f(x) = 0$?
   **(Circle or underline ALL correct answers)**

   (a) It works without problems even near a solution $x = a$ that satisfies $f'(a) = 0$.
   (b) It does not require knowledge of the first derivative $f'(x)$.
   (c) It may be less computationally expensive than Newton's method, in cases where $f(x)$ is cheap to compute, but $f'(x)$ is much more expensive.

   (3) If the error $e_k$ in a method for solving nonlinear equations satisfies the inequality $|e_{k+1}| \leq C|e_k|^d$, we say that the order of convergence is equal to $d$. Which of the following statements are true?
   **(Circle or underline ALL correct answers)**

   (a) When $d = 1$, the condition $C < 1$ is also necessary for convergence.
   (b) When $d = 2$, the condition $C < 1$ is also necessary for convergence.
   (c) If $d = 2$ and $C = 0.9$ the number of correct significant digits in our approximation will roughly double after each iteration.

   (4) Which of the following are true when comparing Newton's method to the Secant method for solving the equation $f(x) = 0$?
   **(Circle or underline ALL correct answers)**

   (a) The Secant method requires knowledge of the derivative $f'(x)$ while Newton's method does not require it.
   (b) An iteration of Newton's method is always computationally cheaper than an iteration of the Secant method.
   (c) The fact that the order of convergence is $d \approx 1.6$ for the Secant method, and $d = 2$ for Newton's method is *not* something we would consider a critical disadvantage for the Secant method.

(5) Which of the following can be claimed as advantages of the Vandermonde method for polynomial interpolation?
**(Circle or underline ALL correct answers)**

(a) Once the coefficients have been computed, evaluating either the polynomial or its derivative can be done very efficiently.

(b) It is easy to incrementally update the interpolant if we need to add one extra data point.

(c) Computing the coefficients with the Vandermonde method is more efficient than using divided differences.

(6) What would happen if we tried to use Newton's method to solve an equation $f(x) = 0$ that does *not* have any solutions (e.g., $x^2 + 1 = 0$)?
**(Circle or underline the ONE most correct answer)**

(a) Newton would converge to the value $x$ where $f(x)$ is closest to zero, i.e. the minimum if $f(x) > 0$ or the maximum if $f(x) < 0$.

(b) Newton would iterate forever without converging anywhere.

(c) It would only converge if started from an approriate initial guess.

(7) Assume we want to construct a polynomial function that interpolates the $N + 1$ data points:

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N).$$

In addition, assume that the $x$-values $x_0, x_1, \ldots, x_N$ are constant, but the $y$-values $y_0, y_1, \ldots, y_N$ may change frequently. Which one of the following is true?
**(Circle or underline the ONE most correct answer)**

(a) The Newton method would be ideal, since only a small number of entries from the divided difference table would need to be updated when the $y$-values change.

(b) The Lagrange method would be a good candidate, especially if we do not need to evaluate the polynomial too many times, since the interpolating polynomial could be updated very easily.

(c) With the Vandermonde matrix approach we would be able to update the interpolating polynomial with only a computational cost proportional to the number of data points.

2. SHORT ANSWER SECTION. Answer each of the following questions in no more than 1-2 sentences.

   (a) Describe one scenario where you would prefer using the Bisection method instead of Newton's method to solve an equation $f(x) = 0$.

   (b) Give an example of a function $g(x)$ which is *not* a contraction.

   (c) Are there some circumstances that may justify using the Vandermonde system approach instead of Lagrange interpolation to compute a polynomial interpolant for $N$ given data points?

   (d) Name two iterative methods for solving nonlinear equations that are *not* fixed point iterations.

   (e) If the *relative error* of a computation is 0.001, and the computed value is $x = 3.5$, write the interval where the "exact" value lies.

3. Consider the fixed point iteration method $x_{k+1} = g(x_k)$ which we apply to find the solution to the nonlinear equation $f(x) = 0$. Now, consider the different fixed point iteration method, described by the recurrence:

$$x_{k+1} = h(x_k) = \frac{1}{2}\left[x_k + g(x_k)\right] \tag{1}$$

   (a) Show that if $g(x)$ is a contraction, then $h(x)$ is a contraction, too.

   (b) Show that if the iteration $x_{k+1} = g(x_k)$ converges linearly, then the iteration $x_{k+1} = h(x_k)$ will converge linearly as well.

   (c) Explain why even if the iteration $x_{k+1} = g(x_k)$ converges *quadratically*, the iteration $x_{k+1} = h(x_k)$ will generally have only linear convergence.

4. Consider the nonlinear equation $f(x) = (x-1)^3 = 0$. The only solution to this equation is obviously $x = 1$. Show that when we apply Newton's method to this equation:

   (a) The iteration always converges, and

   (b) The order of convergence is linear.

5. Use Newton interpolation (and divided differences) to determine the cubic polynomial that interpolates the following data points:

$$(-1,-7)$$
$$(0,-3)$$
$$(1,-1)$$
$$(2,5)$$