CS412 Introduction to Numerical Methods – Spring 2013

Practice Midterm #2

1. MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) Which of the following piecewise polynomial interpolants have continuous second derivatives?
   (Circle or underline ALL correct answers)

   (a) Cubic splines, using the not-a-knot method.
   (b) Cubic splines, using the complete spline method
   (c) Hermite cubic splines.

   (2) Which of the following statements, regarding norms, are true?
   (Circle or underline ALL correct answers)

   (a) A valid matrix norm always has to be induced by a valid vector norm.
   (b) $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_\infty$ for any $\mathbf{x} \in \mathbf{R}^n$.
   (c) $\|\mathbf{A}^{-1}\| \geq \frac{1}{\|\mathbf{A}\|}$ for any matrix $\mathbf{A} \in \mathbf{R}^{n\times n}$.

   (3) Which of the following is NOT a valid reason for choosing piecewise cubic, instead of piecewise linear interpolation?
   (Circle or underline the ONE most correct answer)

   (a) Piecewise cubics can also achieve continuity of the derivatives of the interpolant.
   (b) Piecewise cubics generally converge faster than linear interpolants when increase the number of data points.
   (c) Piecewise cubics are cheaper to compute.
   (d) Piecewise cubics can predict derivative values better than linear interpolants.

   (4) When should we use Chebyshev points for polynomial interpolation?
   (Circle or underline the ONE most correct answer)

   (a) We should use them if we intend to use Lagrange interpolation.
   (b) We should use them if we have the flexibility to pick a specific set of x-values, and we know that both f(x) and f'(x) are bounded.
   (c) We should always use Chebyshev points, this is the best method

(5) Which of the following are valid reasons for using piecewise polynomial interpolation, as opposed to using a single polynomial?
(Circle or underline ALL correct answers)

(a) Spurious oscillations associated with high-degree polynomials can be avoided by using lower-degree piecewise polynomials.

(b) Polynomial splines have well defined derivatives of any order.

(c) Piecewise polynomials can be extended to include more data points, while it is impossible to update a single polynomial interpolant incrementally to include additional points.

(6) Which of the following statements, regarding cubic splines, are true?
(Circle or underline ALL correct answers)

(a) The cubic spline method requires only function values, not derivatives, and produces a result with continuous second derivatives.

(b) Hermite splines require both function values and derivative values, but the computation is very easy and can be performed independently on each interval.

(c) Although Hermite splines are easier to compute, the interpolation error only scales proportionately to h^3, while cubic splines have the error decreasing proportionately to h^4

(7) Which of the following statements, regarding norms, are true?
(Circle or underline ALL correct answers)

(a) For any matrix norm $\|\cdot\|$ induced by a vector norm, we must have $\|\mathbf{I}\| = 1$, where $\mathbf{I}$ is the identity matrix.

(b) $\|\mathbf{A}^2\| \leq \|\mathbf{A}\|^2$ for any matrix $\mathbf{A}$.

(c) If $\kappa(\cdot)$ denotes the condition number, then $\kappa(\alpha\mathbf{A}) = |\alpha|\kappa(\mathbf{A})$, where $\alpha \in \mathbf{R}$ and $\mathbf{A}$ is any matrix.

(d) $\|\mathbf{Ax}\| = \|\mathbf{A}\|\|\mathbf{x}\|$ for any matrix $\mathbf{A}$ and vector $\mathbf{x}$.

2. CRITICAL THINKING PROBLEMS. Answer each of the following questions in **approximately 1 short paragraph**.

   (a) Explain the difference between the "not-a-knot" method and the "complete spline" approach for generating cubic splines. Describe how the two methods differ in the input they require, and in the equations used to define the spline interpolant.

   (b) Assume we use a Hermite cubic spline $s(x)$ to approximate a function $f(x)$, sampled at intervals of equal length $= h$. The interpolation error is:

   $$|s(x) - f(x)| \leq \frac{1}{384} \|f''''(x)\|_{\infty} h^4$$

   If $f(x) = 3\sin(2x)$, find the interval size h that guarantees the interpolation error is always less than 10^{-4}.

   (c) Name two reasons why we may prefer to use piecewise polynomial interpolation, rather than using a single polynomial interpolant with Chebyshev points.

   (d) Show that the Frobenius norm is *not* induced by any vector norm. **Hint:** What is the norm of the identity matrix?

3. Show the following properties:

   (a) $\|\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{x}$ for any vector $\mathbf{x} \in \mathbf{R}^n$.

   (b) $\|\mathbf{Q}\|_2^2 = 1$, for any orthogonal matrix $\mathbf{Q}$. Remember that an orthogonal matrix satisfies $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$.

4. Using any of the methods we discussed in class, find a cubic polynomial s(x), defined over [0;1] that satisfies:

$$\begin{aligned} s(0) &= 2 \\ s'(0) &= -1 \\ s(1) &= 1 \\ s'(1) &= -3 \end{aligned}$$

Note: In case you decide to use the Hermite basis polynomials, those are given below:

$$\begin{aligned} h_{00}(x) &= 2x^3 - 3x^2 + 1 \\ h_{01}(x) &= -2x^3 + 3x^2 \\ h_{10}(x) &= x^3 - 2x^2 + x \\ h_{11}(x) &= x^3 - x^2 \end{aligned}$$

5. Two vector norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be *equivalent*, if constants $c_1, c_2 > 0$ exist, such that

$$\|\mathbf{x}\|_a \leq c_1 \|\mathbf{x}\|_b \quad \text{and} \quad \|\mathbf{x}\|_b \leq c_2 \|\mathbf{x}\|_a$$

for any vector $\mathbf{x} \in \mathbf{R}^n$.

Show that the matrix norms induced by these vector norms are also *equivalent*, i.e. constants $K_1, K_2 > 0$ exist such that

$$\|\mathbf{M}\|_a \leq K_1 \|\mathbf{M}\|_b \quad \text{and} \quad \|\mathbf{M}\|_b \leq K_2 \|\mathbf{M}\|_a$$

for any matrix $\mathbf{M} \in \mathbf{R}^{n \times n}$.

(Note that the constants K_1, K_2 are generally different than c_1, c_2).