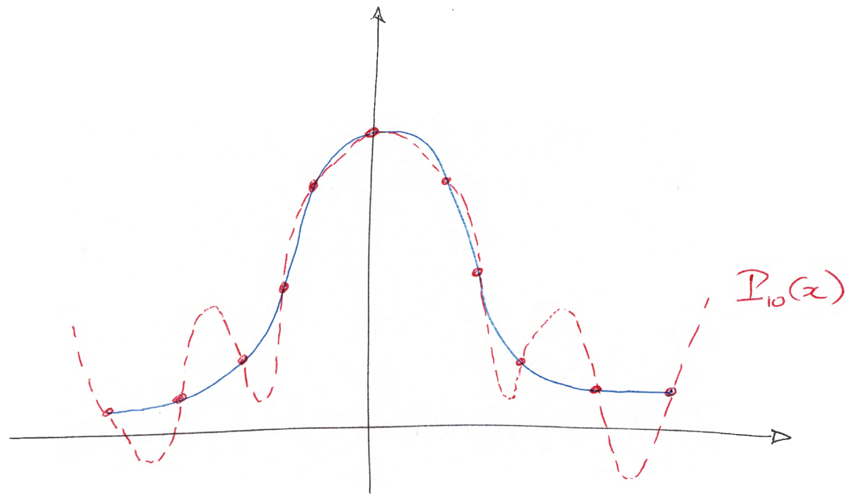
Surprisingly, increasing the number of sample points to 11, and passing a 10-degree polynomial through them does not help:



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Thus, in this case, the polynomials $P_k(x)$ do *not* uniformly converge to $f(x)$ as we add more points. A possible improvement stems from the expression for the error:

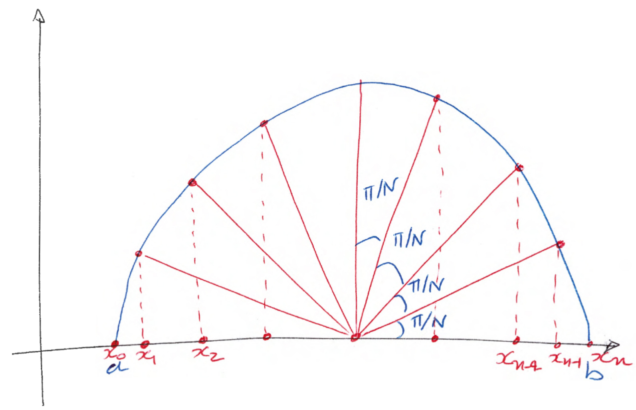
$$f(x) - P(x) = \underbrace{\frac{f^{(n)}(\theta)}{n!}}_{\mathbf{A}} \underbrace{(x - x_1)\dots(x - x_n)}_{\mathbf{B}}$$

The quantity indicated by **A** is beyond our control; this is determined by the function that we seek to approximate. However, if we have the flexibility to choose the x-locations of the data points we collect prior to polynomial interpolation, there is a possibility of minimizing the quantity **B**.

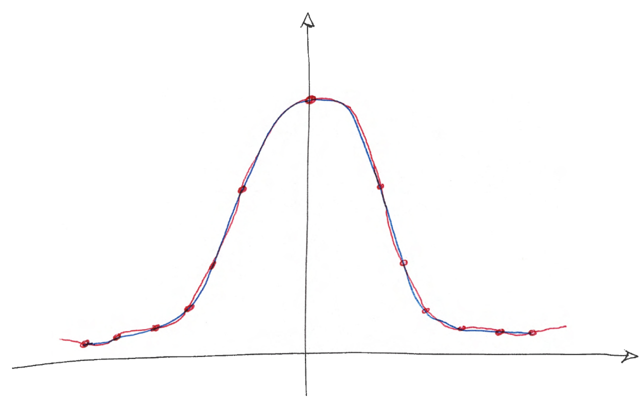
The value of the product $(x - x_1)\dots(x - x_n)$ is minimized by selecting the x_i's as the *Chebyshev points*. If the interpolation interval is $[a, b]$, the Chebyshev points are given by:

$$x_i = a + (b - a)\cos^2(\frac{i\pi}{2N}),\ i = 0, 1, 2, \dots, N$$

Graphically, these point are the projections on the x-axis of the $(N + 1)$ data points located along the half circle with diameter the interval $[a, b]$ at equal arc-lengths:



Now, we can re-try Runge's function using Chebyshev points:



In fact, it is possible to show that, using Chebyshev points, we can guarantee that:

$$|f(x) - P(x)| \xrightarrow[n\to\infty]{} 0$$

provided that over $[a, b]$ both $f(x)$ and its derivative $f'(x)$ remains bounded. (The benefit is that this condition does not place restrictions on higher-order derivatives of $f(x)$).

Although using Chebyshev points mitigates some of the drawbacks of high-order polynomial interpolants, this is still a non-ideal solution, since:

- We do not always have the flexibility to pick the x_i's.
- Polynomial interpolants of high degree typically require more than $O(n)$ computational cost to construct.
- Local changes in the data points affect the entire extent of the interpolant.

Piecewise polynomial inteprolants

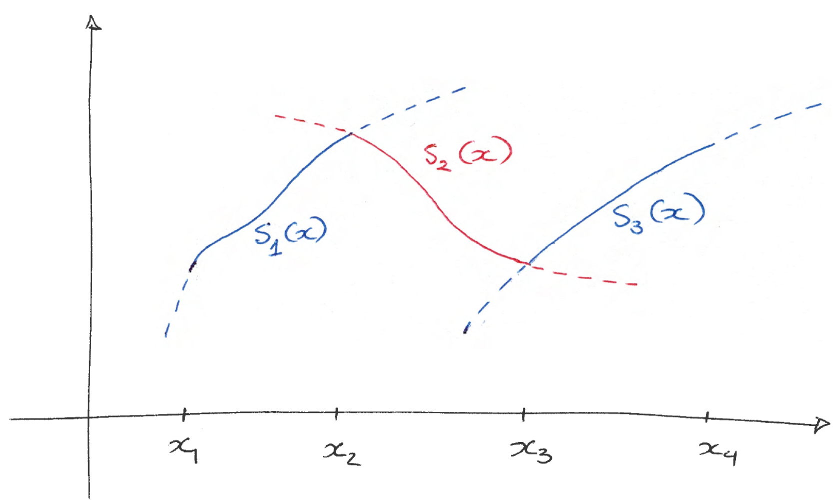
Although the use of Chebyshev points improves the applicability of polynomial interpolation, this method has significant practical limitations. A better, more flexible remedy is to use *piecewise polynomials*. Assume that the x-values $\{x_i\}_{i=1}^n$ are sorted in ascending order:

$$a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

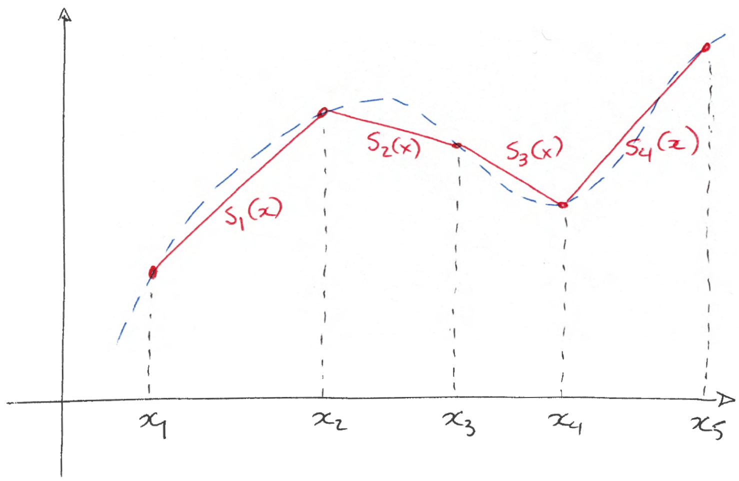
Let us write $I_k = [x_k, x_{k+1}]$ for the k-th interval of interpolation, and let $h_k = |x_{k+1} - x_k|$ denote the length of that interval.

We now define a finite sequence of polynomials $S_1(x), S_2(x), \ldots, S_{n-1}(x)$ and use each of them to define the interpolant $S(x)$ at the respective interval I_k:

$$S(x) = \begin{cases} S_1(x), & x \in I_1 \\ S_2(x), & x \in I_2 \\ \quad\vdots & \\ S_{n-1}(x), & x \in I_{n-1} \end{cases}$$



The benefit of using piecewise-polynomial interpolants is that each $S_k(x)$ can be relatively low-order and thus non-oscillatory and easier to compute. The simplest piecewise polynomial interpolant is a piecewise linear curve:



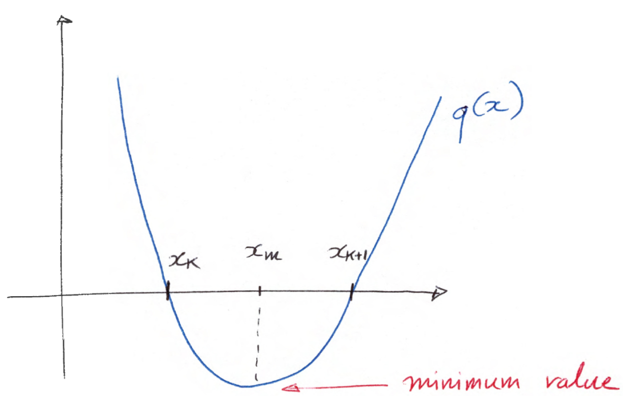
In this case, every S_k can be written out explicitly as:

$$S_k(x) = y_k + \frac{y_{k+1} - y_k}{x_{k+1} - x_k}(x - x_k)$$

The next step is to examine the error $e(x) = f(x) - S_k(x)$ in the interval I_k. From the theorem we presented in the last lecture, we have that, for any $x \in I_k$, there is a $\theta_k = \theta(x_k)$ in I_k, such that:

$$e(x) = f(x) - S_k(x) = \frac{f''(\theta_k)}{2} \underbrace{(x - x_k)(x - x_{k+1})}_{q(x)} \tag{9}$$

We are interested in the *maximum* value of $|q(x)|$ in order to determine a bound for the error. $q(x)$ is a quadratic function which crosses zero at x_k and x_{k+1}, thus the extreme value is obtained at the midpoint $x_m = \frac{x_{k+1}+x_k}{2}$.



Thus $|q(x)| \leq |q(x_m)| = \frac{x_{k+1}-x_k}{2} = \frac{h_k}{4}$ for all $x \in I_k$.

Then, using equation (9) we obtain:

$$\begin{aligned}
|f(x) - S_k(x)| &\leq \max_{x\in I_k} |\frac{f''(x)}{2}| \cdot \max_{x\in I_k} |(x - x_k)(x - x_{k+1}| \\
&= \max_{x\in I_k} |\frac{f''(x)}{2}| \cdot \frac{h_k^2}{4} \\
\Rightarrow |f(x) - S_k(x)| &\leq \frac{1}{8} \max_{x\in I_k} |f''(x)| \cdot h_k^2 \text{ for all } x \in I_k
\end{aligned}$$

Additionally, if we assume all data points are equally spaced, i.e.

$$h_1 = h_2 = \cdots = h_{n-1} = h(= \frac{b-a}{n-1})$$

we can additionally write:

$$|f(x) - S(x)| \leq \frac{1}{8} \max_{x\in[a,b]} |f''(x)| \cdot h^2$$

We often express the quantity on the right-hand side using the *infinity norm* of a given function, defined as:

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

Thus, using this notation:

$$|f(x) - S(x)| \leq \frac{1}{8} \|f''\|_\infty \cdot h^2$$

Note that:

- As $h \to 0$, the maximum discrepancy between f and S vanishes (proportionally to h^2).
- As we introduce more points, the quality of the approximation increases consistently since the criterion above only considers the second derivative $f''(x)$ and not any higher order.

Lecture of: 7 Mar 2013

A possible improvement from piecewise linear polynomial approximations is given by *Piecewise cubic* interpolation: