The Bisection method

Lecture of:
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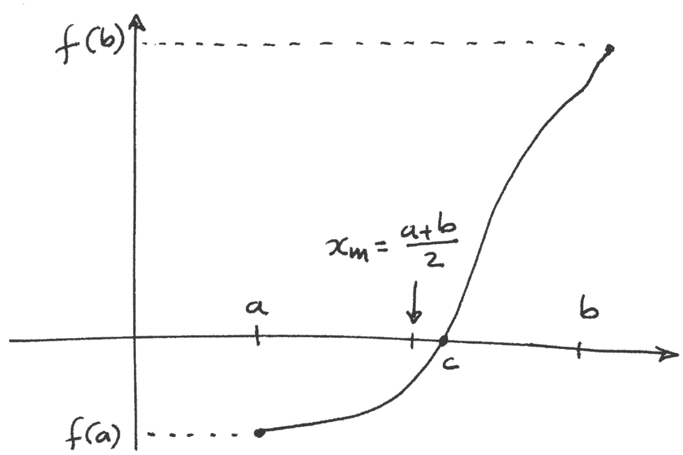
Newton's method is a popular technique for the solution of nonlinear equations, but alternative methods exist which may be preferable in certain situations. The *Bisection method* is yet another technique for finding a solution to the nonlinear equation $f(x) = 0$, which can be used provided that the function f is continuous. The motivation for this technique is drawn from Bolzano's theorem for continuous functions:

Theorem (Bolzano) : If the function $f(x)$ is continuous in $[a, b]$ and $f(a)f(b) < 0$ (i.e. the function f has values with different signs at a and b), then a value $c \in (a, b)$ exists such that $f(c) = 0$.



The bisection algorithm attempts to locate the value c where the plot of f crosses over zero, by checking whether it belongs to either of the two sub-intervals $[a, x_m]$, $[x_m, b]$, where x_m is the midpoint

$$x_m = \frac{a+b}{2}$$

The algorithm proceeds as follows:

- If $f(x_m) = 0$, we have our solution (x_m) and the algorithm terminates.
- In the much more likely case that $f(x_m) \neq 0$ we observe that $f(x_m)$ *must* have the opposite sign than one of $f(a)$ or $f(b)$ (since they have opposite signs themselves). Thus
  - Either $f(a)f(x_m) < 0$, or
  - $f(x_m)f(b) < 0$.

We pick whichever of these 2 intervals satisfies this condition, and continue the bisection process with it.

The bisection algorithm is summarized (in pseudocode) as follows:

Algorithm 1 Bisection search on $[a, b]$

procedure BISECTIONSEARCH(f, a, b)
  $a_0 \leftarrow a,\ b_0 \leftarrow b,\ I_0 \leftarrow [a_0, b_0]$ ▷ I_k denote intervals
  for $k = 0, 1, 2, \dots, N$ **do**
    $x_m \leftarrow \frac{a_k + b_k}{2}$
    if $x_m = 0$ **then**
      x_m is the desired solution, **return**.
    else if $f(a_k)f(x_m) < 0$ **then**
      $I_{k+1} := [a_{k+1}, b_{k+1}] \leftarrow [a_k, x_m]$
    else if $f(x_m)f(b_k) < 0$ **then**
      $I_{k+1} := [a_{k+1}, b_{k+1}] \leftarrow [x_m, b_k]$
    end if
  end for
  return the approximate solution $x_{\text{approx}} = \frac{a_N + b_N}{2}$
end procedure

Convergence Let us conventionally define the "approximation" at x_k after the k-th iteration as the midpoint

$$x_k := \frac{a_k + b_k}{2}$$

of I_k. Since the actual solution $f(a) = 0$ satisfies $a \in I_k$, we have

$$|x_k - a| \leq \frac{1}{2}|I_k|$$

where $|I_k|$ symbolizes the length of the interval I_k. Since the length of the current search interval gets divided in half in each iteration, we have

$$|e_k| = |x_k - a| \leq \left(\frac{1}{2}\right)^k |I_0|$$

We interpret this behavior as *linear* convergence; although we cannot strictly guarantee that $|e_{k+1}| \leq L|e_k|$ $(L < 1)$ at each iteration, this definition can also be iterated to yield

$$|e_k| \leq L^k |e_0|$$

which is qualitatively equivalent to the expression for Bisection. Since the order of convergence is linear, we expect to gain a fixed number (or fixed fraction) of significant digits at each iteration; since $0.5^{10} \approx 0.001$ we can actually say that the Bisection method yields about 3 additional correct significant digits after every 10 iterations.

The Bisection procedure is very robust, by virtue of Bolzano's theorem, and despite having only linear convergence can be used to find an approximation within any desired error tolerance. It is often used to localize a good initial guess which can then be rapidly improved with a Fixed Point Iteration method such as Newton's. *Note that bisection search is not a fixed point iteration itself!*

The Secant method

The *secant method* is yet another iterative technique for solving nonlinear equations; it closely mimics Newton's method, but relaxes the requirement that an analytic expression for the derivative $f'(x)$ must be provided. It operates as follows:

- We bootstrap the iteration not only with one initial guess (x_0), but also with a second improved approximation x_1.

- At the k-th step of the iteration, we first approximate

  $$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

  Remember that since

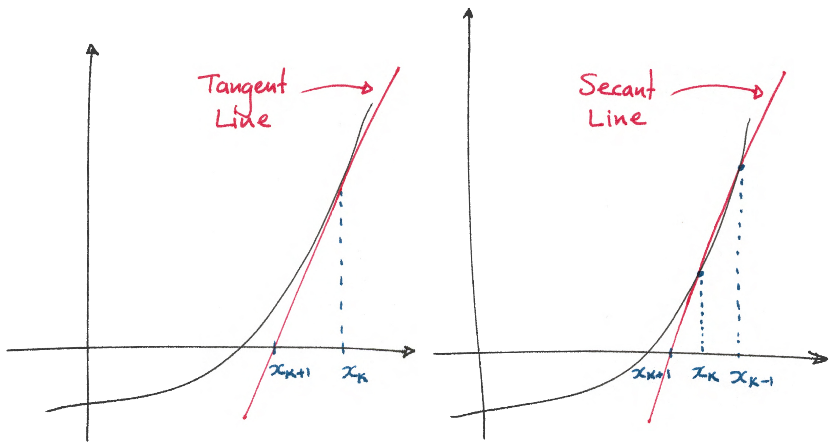
  $$f'(x_k) = \lim_{y \to x_k} \frac{f(x_k) - f(y)}{x_k - y}$$

  as the iterates x_{k-1}, x_k get closer to one another (while they both approach the solution) this approximation becomes more and more accurate.

  We then replace this particular approximation for $f'(x_k)$ in Newton's method $x_{k+1} = x_k - f(x_k)/f'(x_k)$ to obtain:

  $$x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}}$$

Geometrically, Newton's method approximates $f(x)$ at each step by the *tangent* line to the graph of $f(x)$, while the method we just described approximates f by the *secant* line as illustrated below:



We can show that, once we are "close enough" to the solution, the error e_k for the secant method satisfies

$$|e_{k+1}| \leq c|e_k|^d, \quad \text{where } d = \frac{1+\sqrt{5}}{2} \approx 1.6$$

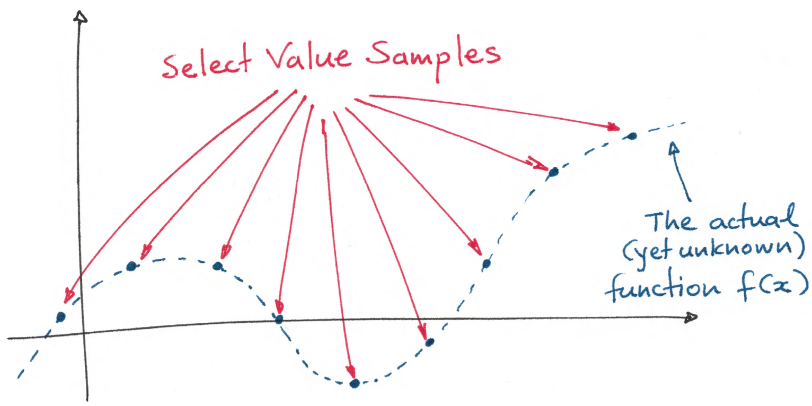
Thus, the secant method provides *superlinear* convergence. In practice, it may need a few more iterations (about 50% more?) than Newton, but we need to weigh in the fact that each iteration is likely cheaper, since no derivatives of f need to be evaluated.

We also note that, despite the fact that the Secant method does not include $f'(x)$ in its formula, it has exactly the same issues as Newton's method does when $f'(x) \approx 0$, especially close to (or at) a solution. Although the denominator is not exactly the derivative, it is (hopefully) a good approximation to the derivative, thus when $f'(x) \approx 0$, the Secant method will also be exposed to the danger of dividing by near-zero quantities.

Interpolation

Corresponding textbook chapter(s): §4.2,4.3

We are often interested in a certain function $f(x)$, but despite the fact that f may be defined over an entire interval of values $[a, b]$ (which may be the entire real line) we only know its precise value at select points $x_1, x_2, \ldots, x_N$:



There may be several good reasons why we could only have a limited number of values for $f(x)$, instead of its entire graph:

- Perhaps we do not have an analytic formula for $f(x)$ because it is the result of a complex process that is only observed experimentally. For example $f(x)$ could correspond to a physical quantity (temperature, density, concentration, velocity, etc) which varies over time in a laboratory experiment. Instead of an explicit formula, we use a measurement device to capture sample values of f at predetermined points in time.

- Or, perhaps we do have a formula for $f(x)$, but this formula is not trivially easy to evaluate. Consider for example:

$$f(x) = \sin(x) \quad \text{or} \quad f(x) = \ln(x) \quad \text{or} \quad f(x) = \int_0^x e^{-t^2} dt$$

  Perhaps evaluating $f(x)$ with such a formula is a very expensive operation and we want to consider a less expensive way to obtain a "crude approximation". In fact, in years when computers were not as ubiquitous as today, trigonometric tables were very popular. For example