Angle	sin(θ)	sin(θ)
...	...	...
44^0	0.695	0.719
45^0	0.707	0.707
46^0	0.719	0.695
47^0	0.731	0.682
...	...	...

If we were asked to approximate the value of $\sin(44.6^0)$ it would be natural to consider deriving an estimate from these tabulated values, rather than attempting to write an analytic expression for this quantity.

Lecture of: 7 Feb 2013

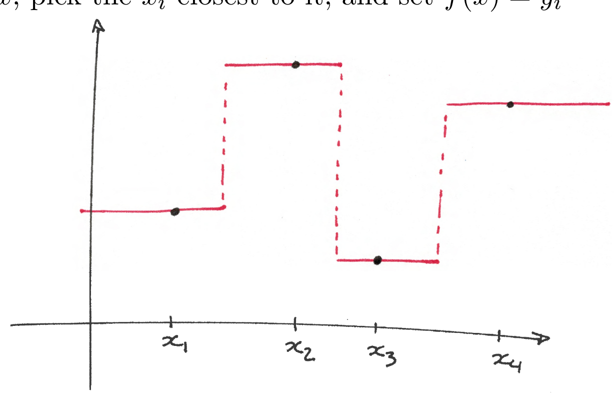
Interpolation methods attempt to answer questions about the value of $f(x)$ at points *other* than the ones it was sampled at. An obvious question would be to ask what is an estimate for $f(x^*)$ for a value x^* different than any sample we have collected; similar questions can be asked about the derivatives $f'(x^*), f''(x^*), \ldots$ at such locations.

The question of how to reconstruct a smooth function $f(x)$ that agrees with a number of collected sample values is not a straightforward one, especially since there is more than one way to accomplish this task. First, let us introduce some notation: Let us write $x_1, x_2, \ldots, x_N$ for the x-locations where f is being sampled and denote the known value of $f(x)$ at $x = x_1$ as $y_1 = f(x_1)$, at $x = x_2$ as $y_2 = f(x_2)$, etc. Graphically, we seek to reconstruct a function $f(x)$, $x \in [a, b]$ such that the plot of f passes through then following points:

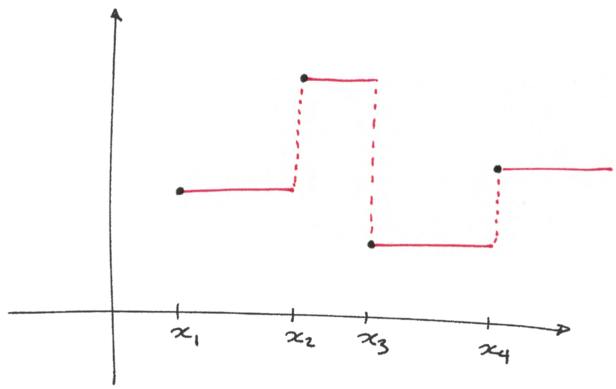
$$(x_1, y_1) \ , \ (x_2, y_2) \ , \ \ldots \ , \ (x_N, y_N)$$

Here are some possible ways to do that:

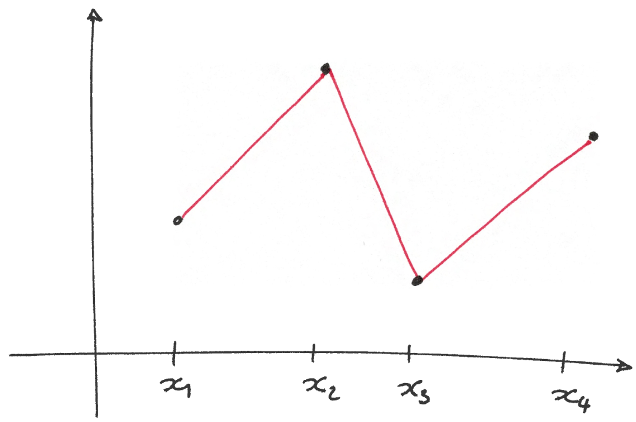
- For every x, pick the x_i closest to it, and set $f(x) = y_i$



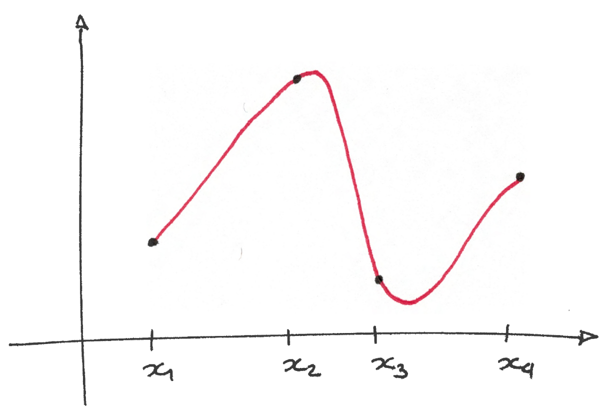
Or, simply pick the value to the "left":



- Try connecting every two horizontally neighboring points with a straight line



- Or, try to find a smoother curve that connects them all ...



It is not trivial to argue that any particular one of these alternatives is "better", without having some knowledge of the nature of $f(x)$, or the purpose this reconstructed function will be used for. For example:

- It may appear that the discontinuous approximation generated by the "pick the closest sample" method is awkward and not as well behaved. However, the real function $f(x)$ being sampled could have been just as discontinuous to begin with, for example if $f(t)$ denoted the transaction amount for the customer of a bank being served at time $= t$.
- Sometimes, we may know that the real $f(x)$ is supposed to have some degree of smoothness. For example, if $f(t)$ is the position of a moving vehicle in a highway, we would expect both $f(t)$ and $f'(t)$ (velocity), possibly even $f''(t)$ (acceleration) to be continuous functions of time. In this case, if we seek to estimate $f'(t)$ at a given time, we may prefer the piecewise-linear reconstruction. If $f''(t)$ is needed, the even smoother method might be preferable.

Polynomial Interpolation

A commonly used approach is to use a properly crafted polynomial function:

$$f(x) = P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

to interpolate the points $(x_0, y_0), \ldots, (x_k, y_k)$.

Benefits of using polynomials for interpolation include:

- Polynomials are relatively simple to evaluate. In fact, this can be done with just n multiplications and n additions by writing:

$$P_n(x) = a_0 + x(a_1 + x(a_2 + x(a_3 \cdots + a_{n-1} + x a_n)))$$

- We can easily compute derivatives P_n', P_n'' if desired.
- Reasonably established procedure to determine the a_i's.
- Polynomial approximations are *familiar* from, e.g. Taylor series.

And some disadvantages:

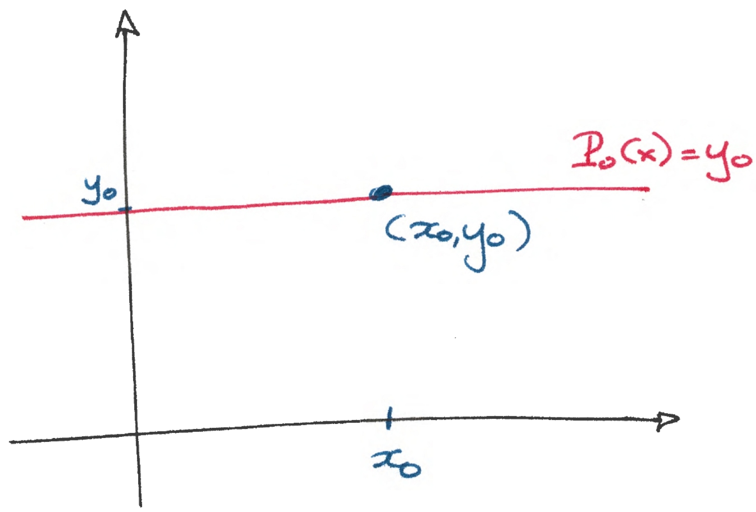
- Fitting polynomials can be problematic when:

(i) We have *many* data points (k is large), or

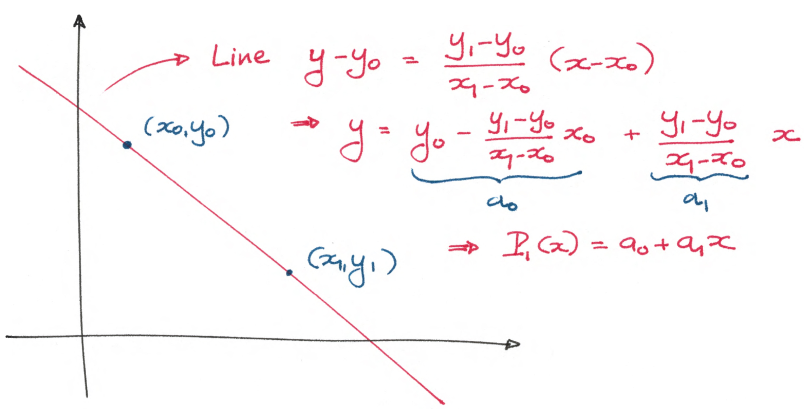
(ii) Some of the samples are too close together ($\|x_i - x_j\| = \text{small}$).

In the interest of simplicity (and not only), we try to find the most basic, yet adequare, $P_n(x)$ that interpolates $(x_0, y_0), \ldots, (x_k, y_k)$. For example:

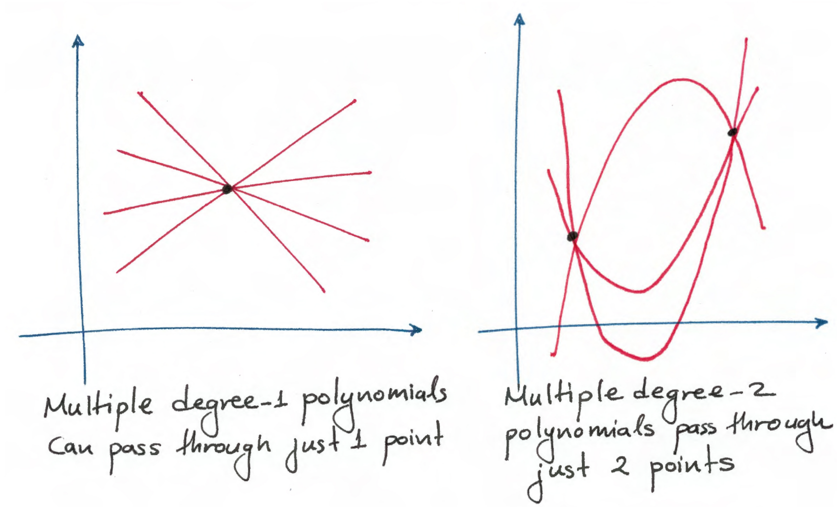
- If $k = 0$ (only one data sample), we have to interpolate through (x_0, y_0) only. A 0-degree polynomial (constant) will achieve that if $P_0(x) = y_0$.



- If $k = 1$, we have 2 points (x_1, y_1) and (x_2, y_2). A 0-degree polynomial $P_0(x) = a_0$ will not always be able to pass through both points (unless $y_1 = y_2$), but a 1-degree polynomial $P_1(x) = a_0 + a_1 x$ always can.



These are not the only polynomials that accomplish the task, e.g.



The problem with using a degree higher than the minimum necessary is that:

- More than one solution becomes available, with the "right" one being unclear
- Wildly varying curves become permissible, producing questionable approximations.

Lecture of: 12 Feb 2013

In fact, we can show that using a polynomial $P_n(x)$ of degree n is the best choice when interpolating $n+1$ points. In this case, the following properties are assured:

- Existence: Such a polynomial always exists. (Assuming that all the x_i's are different! It would be impossible for a function to pass through 2 points on the same vertical line.) We will show this later by constructing such a function.
- Uniqueness: We can sketch a proof:

Assume that

$$\begin{aligned} P_n(x) &= p_0 + p_1 x + \cdots + p_n x^n \\ Q_n(x) &= q_0 + q_1 x + \cdots + q_n x^n \end{aligned}$$

both interpolate every (x_i, y_i), i.e. $P_n(x_i) = Q_i(x_i) = y_i \ \forall i$. Define another n-degree polynomial

$$r_0 + r_1 x + \cdots + r_n x^n = R_n(x) = P_n(x) - Q_n(x).$$

Apparently, $R_n(x_i) = 0 \ \forall i = 1, 2, \ldots, n+1$. From algebra, we know that every polynomial of degree n has at most n real roots *unless* it is the zero polynomial, i.e. $r_0 = r_1 = \cdots = r_n = 0$. Since we have $R_n(x) = 0$ for n_1 distinct values, we must have $R_n(x) \equiv 0$ or $P_n(x) \equiv Q_n(x)$.