Volumetric deformable bodies

"Natural Configuration"

"Deformed configuration"

Questions to be answered:

→ How do we represent a volumetric deformable body (in order to enable simulation of governing physics)

Options
→ Using triangle meshes in 2D, tetrahedral meshes in 3D

→ This is what we will focus on
→ Using quadrilateral (hexahedral) meshes in 2D (3D)

→ Popular in engineering fields (quads/hexes can be "better" than triangles/tetrahedra in certain aspects)
Treatment is similar to triangle-based discretizations (but we will not explicitly discuss those).
Using "unorganized" point clouds

- Benefits: Easy to split and merge (no mesh to reconnect!).
- Drawbacks: May be slower than using meshes, theory is somewhat less developed.

- How do forces arise? How do we evaluate them? Where are these forces applied?

- In a "continuous" material perspective, forces are generated throughout the material.

- In a discrete representation of a body, we aggregate such forces on the discrete degrees of freedom only (particles/vertices).

This allows us to use all previously discussed time integration schemes, simply adjusting the definition for each force $f_i$ (on particle $p_i$).
Possible approach:

Use mass-spring model to approximate volumetric material behavior.

\[ f_i = -k \left( \frac{p_i}{p_0} - 1 \right) \frac{x_i - x_j}{||x_i - x_j||} \]

Pros: → Simple
→ Uniform treatment of either surfaces or volumes

Cons: → It is difficult to adjust spring stiffnesses to obtain a desired behavior
→ Overall behavior depends on spring connectivity patterns
→ Poor approximation of real materials
→ Attitude springs difficult to make work in 3D (many special cases).

Thus we seek a material "building block" which is more natively volumetric. We start in 2D:

Spring:

Elastic triangle

"Natural" shape

"Deformed" Shape

\[ f_i(x_1, x_2, x_3) \]
Some observations:

Relation between spring (or "element") force & energy.

In 1D we know that the force \( f \) and potential energy of a zero length spring are given as:

\[
f = -kx = f(x) \\
E = \frac{1}{2} kx^2 = E(x)
\]

We observe that \( f(x) = -E'(x) \)!

This is not accidental... in cases where a force can be associated with a potential energy, the force is always given as the negative derivative (or gradient, in higher dimensions) of the potential energy. This is the same for the case of gravity, too:

Potential energy: \( E = mg \cdot y \)

Gravity force:

\[
G = -\nabla E = \left( -\frac{\partial E}{\partial x}, -\frac{\partial E}{\partial y} \right) = (0, -mg)
\]
We can certainly associate a potential energy with a spring in the 2D or 3D space. The expression for the energy is:

$$E = \frac{k l_0}{2} \left( \frac{l}{l_0} - 1 \right)^2 = E(x_1, x_2) \quad \text{(Compare with } E = \frac{k}{2}(l-l_0)^2 !)$$

Young's modulus

Once again, the force on each particle can be obtained by taking the negative partial derivative of $E$, wrt. the coordinates of that particle, i.e.

$$\vec{f}_i = -\frac{\partial E}{\partial \vec{x}_i} = \begin{pmatrix} -\frac{\partial E}{\partial x_1} \\ -\frac{\partial E}{\partial y_1} \\ -\frac{\partial E}{\partial z_1} \end{pmatrix}$$

Thus:

$$\vec{f}_i = -\frac{\partial E}{\partial \vec{x}_i} = -\frac{\partial}{\partial \vec{x}_1} \left[ \frac{k l_0}{2} \left( \frac{l(x_1, x_2)}{l_0} - 1 \right)^2 \right]$$

$$= -\frac{k l_0}{2} \left( \frac{l}{l_0} - 1 \right) \cdot \frac{1}{l_0} \cdot \frac{\partial l}{\partial \vec{x}_1}$$

Lemma (w/o proof): $\frac{\partial l}{\partial \vec{x}_1} = \frac{\partial \| \vec{x}_1 - \vec{x}_2 \|_2}{\partial \vec{x}_1} = \frac{\vec{x}_1 - \vec{x}_2}{\| \vec{x}_1 - \vec{x}_2 \|}$

$$\Rightarrow \vec{f}_i = -k \left( \frac{l}{l_0} - 1 \right) \cdot \frac{\vec{x}_1 - \vec{x}_2}{\| \vec{x}_1 - \vec{x}_2 \|} \quad \text{Agrees with prior definition!}$$
General idea

If we are able to describe a plausible "membrane energy" for a triangle \((x_1, x_2, x_3)\), we would derive the corresponding forces by taking the negative derivative \(-\frac{\partial E}{\partial x_i} = f_i\).

\[ \Delta \text{ Energy } E(x_1, x_2, x_3) = ? \]

In designing such an energy, it would be useful to study (and possibly replicate) some of the properties/features of the "spring energy".

\[ E = \frac{k l_0}{2} \left( \frac{l}{l_0} - 1 \right)^2 = E(x_i, x_j) \quad \text{where } l(x_i, x_j) = ||x_i - x_j||_2 \]

The dimensionality of the arguments \(x_1, x_2\) is \( \leq 5 \) (in 3D).

However, the energy depends on less information than these variables encode (just the length, i.e. a single number). This is due to the fact that certain motions do not change (and should not change) the stored energy. Those are:

* Translations. If both endpoints are displaced by the same distance and along the same direction, the length of the spring (and the energy) will not/should not change.
This property is called translational invariance, and would be required of any reasonable discrete model that hopes to approximate a real material.

*Rotations*: Rotating the entire spring about a given axis also leaves the length/energy unaffected.

Rotational invariance is also a desired property (although it may require a non-trivial computational overhead to enforce).

The energy of the spring is a product of 2 factors

\[ E = l_0 \cdot \frac{k}{2} \left( \frac{l}{l_0} - 1 \right)^2 \]

- Total energy
- Length of spring

\[ \text{"Energy density" (energy per unit length)} \]

\[ \frac{l}{l_0} \]

\[ \text{"Only depends on the compression ratio"} \]
This compression ratio is a number that we could also compute locally, i.e. on a small section of the spring.

"Natural state"

"Expanded/deformed shape"

Let \( X_a, X_b \) (capital letters) be the endpoints of this short spring section, in its "natural" configuration, and let \( x_a, x_b \) be the endpoint locations after a certain deformation has occurred. We can define the deformation function \( \phi(x) \) as the function that maps the "natural" location of every point \( X^* \) to the "deformed" location of the same point \( x^* \), i.e. \( x^* = \phi(X^*) \).

(Here, we even allowed a spring to deform non-uniformly! \( \phi() \) can capture that.)

Obviously, the function \( \phi(X) \) offers a very rich representation of the motion; we can generate the deformed location of any point in the body (even if that was not a particle location). It also allows us to define the local compression ratio \( \frac{\ell}{\ell_0} \) very compactly.
\[ \frac{\delta l}{\delta l_0} = \frac{x_b - x_a}{x_b - x_a} = \frac{\phi(x_b) - \phi(x_a)}{x_b - x_a} \quad x_b \to x_a \rightarrow \phi'(x_a) \]

This allows us to write the energy of a spring as:

\[ E = l_0 \cdot \frac{k}{2} \left( \phi'(x) - 1 \right) \]

\[ := \phi(\phi') \rightarrow \text{The energy density function.} \]

Note: This expression suggests that we would need to evaluate \( \phi' \) at some specific location to determine the density \( \Phi \).

However, when modeling a single spring we typically make the simplifying assumption that any compression / expansion happens uniformly:

```
X_L \quad X^* \quad X_R
```

By uniformity:

\[ \frac{x^* - x_L}{x_R - x_L} = \frac{x^* - X_L}{X_R - x_L} \rightarrow \]

\[ \Rightarrow x^* = x_L + \frac{l}{l_0} (X^* - X_L) = \phi(X^*) \Rightarrow \phi'(X) = \frac{l}{l_0} \text{ everywhere!} \]
If the deformation function $\phi$ was really such that $\phi'$ is not constant along the spring (i.e., the contraction is non-uniform), then we compute the energy by integrating (i.e., summing up "infinitesimal" sections)

$$E = \int_{x_L}^{x_R} \Psi(\phi') \, dx$$

(Again, if $\phi' = \phi_{\text{const}}$, then $E = \int_{x_L}^{x_R} \Psi(\phi_{\text{const}}) \, dx = l_0 \cdot \Psi(\phi_{\text{const}})$.)