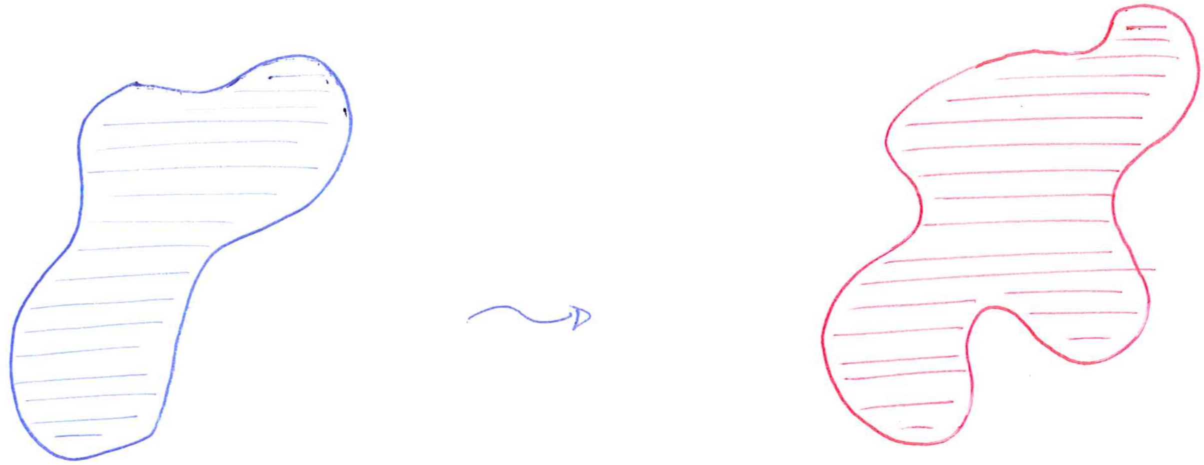
Volumetric deformable bodies



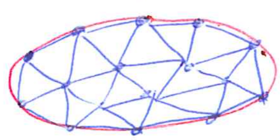
"Natural Configuration" "Deformed configuration"

Questions to be answered:

→ How do we represent a volumetric deformable body (in order to enable simulation of governing physics)

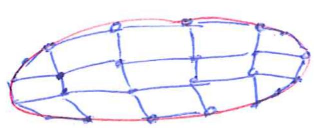
Options

→ Using triangle meshes in 2D, tetrahedral meshes in 3D



→ This is what we will focus on

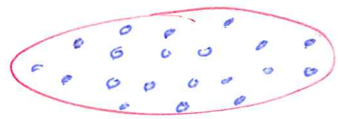
→ Using quadrilateral (hexahedral) meshes in 2D (3D)



⟹ Popular in engineering fields (quads/hexes can be "better" than triangles/tetrahedra in certain aspects)

Treatment is similar to triangle-based discretizations (but we will not explicitly discuss those).

→ Using "unorganized" point clouds

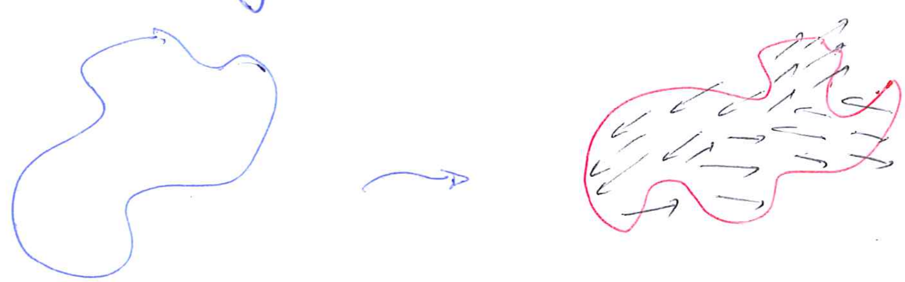


⇒ Benefits : Easy to split and merge (no mesh to reconnect!).

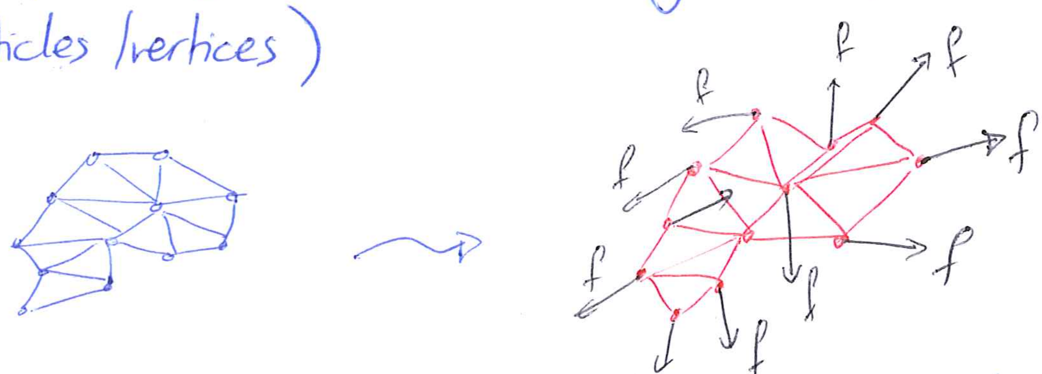
⇒ Drawbacks : May be slower than using meshes, theory is somewhat less developed.

→ How do forces arise? How do we evaluate them? Where are these forces applied?

⇝ In a "continuous" material perspective, forces are generated throughout the material



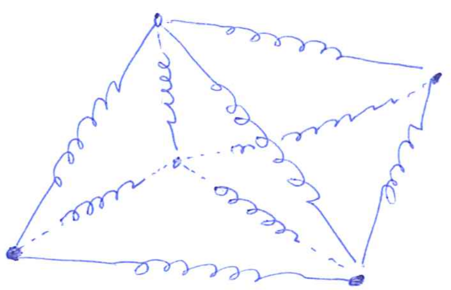
⇝ In a discrete representation of a body, we aggregate such forces on the discrete degrees of freedom only (particles / vertices)



This allows us to use all previously discussed time integration schemes, simply adjusting the definition for each force f_i (on particle p_i)

Possible approach:

Use mass-spring model to approximate volumetric material behavior:

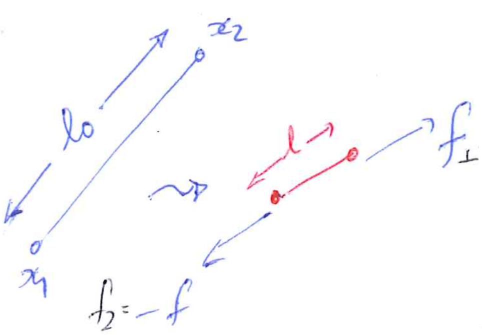


Pros: → Simple
→ Uniform treatment of either surfaces or volumes

Cons: → It is difficult to adjust spring stiffnesses to obtain a desired behavior
→ Overal behavior depends on spring connectivity patterns
→ Poor approximation of real materials
→ Altitude springs difficult to make work in 3D (many special cases).

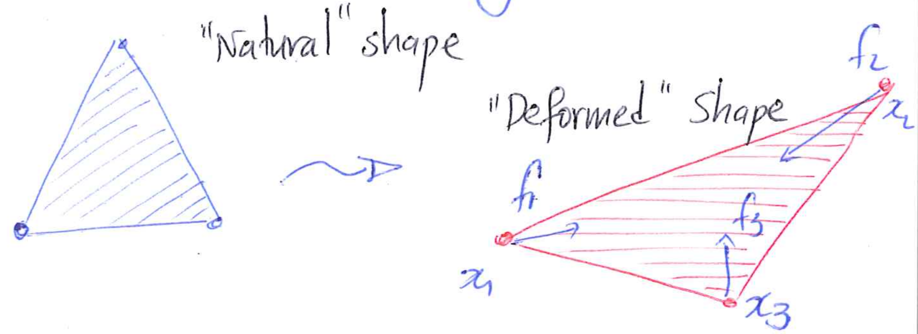
Thus we seek a material "building block" which is more natively volumetric. We start in 2D:

Spring:



$$f_1 = -k\left(\frac{l}{l_0} - 1\right)\frac{x_1 - x_2}{\|x_1 - x_2\|}$$

Elastic triangle

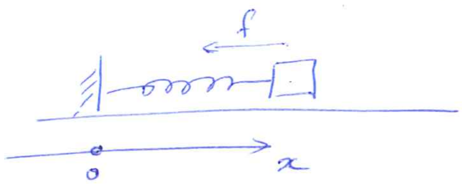


$$f_i \overset{?}{=} f_i(x_1, x_2, x_3) \; ?$$

Some observations:

→ Relation between spring (or "element") force & energy.

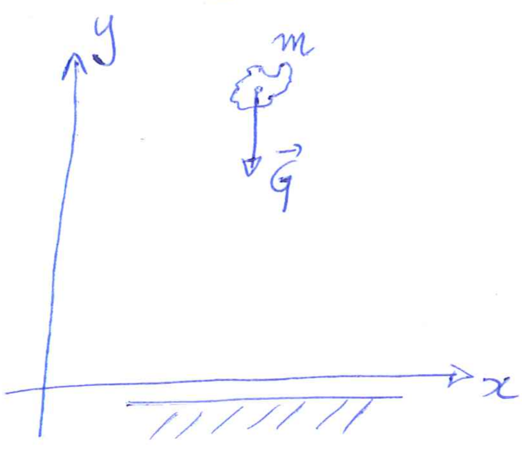
In 1D we know that the force (f) and potential energy of a zero length spring are given as:



$$f = -kx = f(x)$$
$$E = \frac{1}{2}kx^2 = E(x)$$

We observe that $f(x) = -E'(x)$!

This is not accidental... in cases where a force can be associated with a potential energy, the force is always given as the negative derivative (or gradient, in higher dimensions) of the potential energy. This is the same for the case of gravity, too:



Potential energy: $E = mgy$

Gravity force:

$$\vec{G} = -\nabla E = \left(-\frac{\partial E}{\partial x}, -\frac{\partial E}{\partial y}\right)$$
$$= (0, -mg) \quad !$$

We can certainly associate a potential energy with a spring in the 2D or 3D space. The expression for the energy is:

$$E = \frac{k\, l_0}{2}\left(\frac{l}{l_0}-1\right)^2 = E(\vec{x}_1,\vec{x}_2) \quad \left(\text{Compare with } E = \frac{k}{2}(l-l_0)^2 \text{ !}\right)$$

k in the first formula: Young's modulus; k in the second: Spring constant

Once again the force on each particle can be obtained by taking the negative partial derivative of E, w.r.t. the coordinates of that particle, i.e.

$$\vec{f}_i = -\frac{\partial E}{\partial \vec{x}_i} = \begin{pmatrix} -\frac{\partial E}{\partial x_i} \\ -\frac{\partial E}{\partial y_i} \\ -\frac{\partial E}{\partial z_i} \end{pmatrix}$$

Thus:

$$\vec{f}_1 = -\frac{\partial E}{\partial \vec{x}_1} = -\frac{\partial}{\partial \vec{x}_1}\left[\frac{k\, l_0}{2}\left(\frac{l(\vec{x}_1,\vec{x}_2)}{l_0}-1\right)^2\right]$$

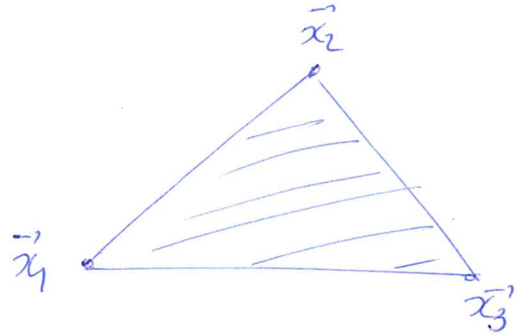
$$= -\frac{k\, \cancel{l_0}}{\cancel{2}}\cdot \cancel{2}\cdot\left(\frac{l}{l_0}-1\right)\cdot\frac{1}{\cancel{l_0}}\cdot\frac{\partial l}{\partial \vec{x}_1}$$

$$\left[\text{Lemma (w/o proof)}: \frac{\partial l}{\partial \vec{x}_1} = \frac{\partial \|\vec{x}_1-\vec{x}_2\|_2}{\partial \vec{x}_1} = \frac{\vec{x}_1-\vec{x}_2}{\|\vec{x}_1-\vec{x}_2\|}\right]$$

$$\Rightarrow \vec{f}_1 = -k\left(\frac{l}{l_0}-1\right)\frac{\vec{x}_1-\vec{x}_2}{\|\vec{x}_1-\vec{x}_2\|}$$ → Agrees with prior definition!

General idea

If we are able to describe a plausible "membrane energy" for a triangle $(\overset{\triangle}{x_1 x_2 x_3})$, we would derive the corresponding forces by taking the negative derivative $-\frac{\partial E}{\partial \vec{x}_i} = \vec{f}_i$



$\rightsquigarrow$ Energy $E(\vec{x}_1, \vec{x}_2, \vec{x}_3) = ?$

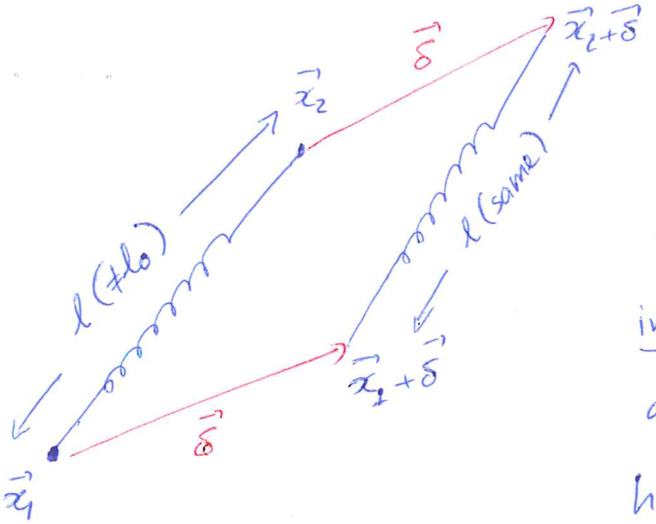
In designing such an energy, it would be useful to study (and possibly replicate) some of the properties/features of the "spring energy".

$$E = \frac{k l_0}{2}\left(\frac{l}{l_0} - 1\right)^2 = E(\vec{x}_1, \vec{x}_2) \quad \text{where } l(\vec{x}_1, \vec{x}_2) = \|\vec{x}_1 - \vec{x}_2\|_2$$

$\rightsquigarrow$ The dimensionality of the arguments $\vec{x}_1$ & $\vec{x}_2$ is 6 (in 3D). However, the energy depends on less information than these variables encode (just the length, i.e a single number).

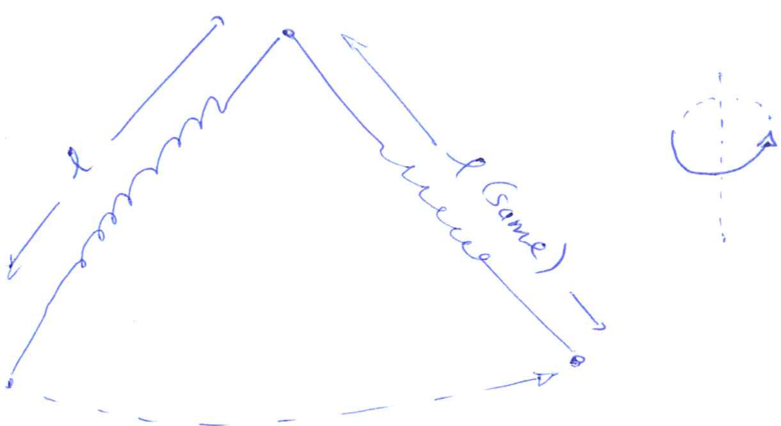
This is due to the fact that certain motions do not change (and should not change) the stored energy. Those are:

* Translations. If both endpoints are displaced by the same distance, and along the same direction, the length of the spring (and the energy) will not/should not change



This property is called translational invariance, and would be required of any reasonable discrete model that hopes to approximate a real material.

* Rotations: Rotating the entire spring about a given axis also leaves the length/energy unaffected



Rotational invariance is also a desired property (although it may require a non-trivial computational overhead to enforce).

⇝ The energy of the spring is a product of 2 factors

$$E = l_0 \cdot \underbrace{\frac{k}{2}\left(\frac{l}{l_0}-1\right)^2}$$

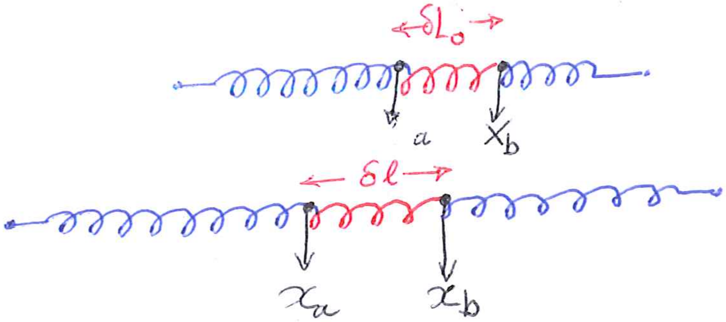
Total energy → E

length of spring → l_0

↳ "Energy density" (energy per unit length)

⇒ Only depends on the compression ratio $\frac{l}{l_0}$!

This compression ratio is a number that we could also compute <u>locally</u>, i.e. on a small section of the spring.

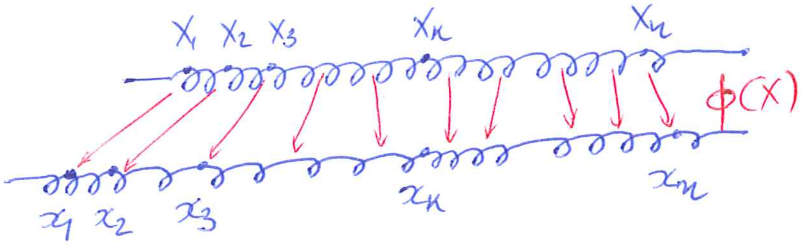


"Natural state"

"Expanded/deformed shape"

Let X_a, X_b (capital letters) be the endpoints of this short spring section, in its "natural" configuration, and let x_a, x_b be the endpoint locations after a certain deformation has occured

We can define the deformation function $\phi(X)$ as the function that maps the "natural" location of every point X^* to the "deformed" location of the same point x^* i.e. $x^* = \phi(X^*)$



(Here, we even allowed a spring to deform non-uniformly! $\phi(\cdot)$ can capture that)

Obviously, the function $\phi(X)$ offers a <u>very rich</u> representation of the motion; we can generate the deformed location of <u>any</u> point in the body (even if that was not a particle location). It also allows us to define the <u>local</u> compression ratio $\frac{\delta l}{\delta l_0}$ very compactly:

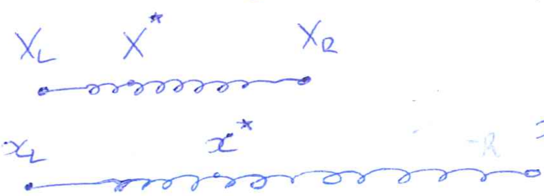
$$\frac{\delta l}{\delta l_0} = \frac{x_b - x_a}{X_b - X_a} = \frac{\phi(X_b) - \phi(X_a)}{X_b - X_a} \xrightarrow[X_b \to X_a]{} \phi'(X_a)$$

This allows us to write the energy of a spring as:

$$E = l_0 \cdot \underbrace{\frac{k}{2}\left(\phi'(x) - 1\right)^2}_{:= \psi(\phi') \;\to\; \text{The energy density function.}}$$

Note: This expression suggests that we would need to evaluate ϕ' at some specific location to determine the density ψ.

However, when modeling a single spring we typically make the simplifying assumption that any compression/expansion happens uniformly



By uniformity: $\dfrac{x^* - x_L}{\underbrace{x_R - x_L}_{l}} = \dfrac{X^* - X_L}{\underbrace{X_R - X_L}_{l_0}} \Rightarrow$

$$\Rightarrow x^* = x_L + \frac{l}{l_0}(X^* - X_L) = \phi(X^*) \Rightarrow \phi'(X) = \frac{l}{l_0} \text{ everywhere!}$$

If the deformation function ϕ was <u>really</u> such that ϕ' is not constant along the spring (i.e. the contraction is non-uniform) then we compute the energy by integrating (i.e. summing up "infinitesimal" sections)

$$E = \int_{X_L}^{X_R} \Psi(\phi')\,dX$$

(Again, if $\phi' = \phi'_{const}$, then $E = \int_{X_L}^{X_R} \Psi(\phi'_{const})\,dX = l_0 \cdot \Psi(\phi'_{const})$).