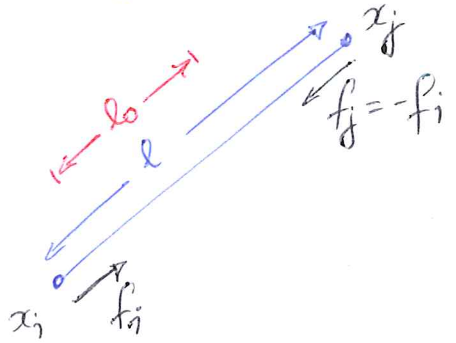
Review

* Spring in n-dimensional space



Force : $f_i = -k\left(\frac{l}{l_0} - 1\right)\frac{x_i - x_j}{\|x_i - x_j\|}$

Energy : $E = \underbrace{l_0}_{\text{Length}} \cdot \underbrace{\frac{k}{2}\left(\frac{l}{l_0} - 1\right)^2}_{\psi\left(\frac{l}{l_0}\right)}$

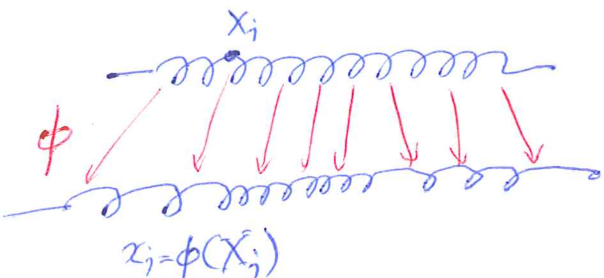
$\psi\left(\frac{l}{l_0}\right) \Rightarrow$ energy density

$\frac{l}{l_0}$ $\downarrow$ compression ratio

Relation of force-energy :

$$\vec{f_i} = -\frac{\partial E}{\partial \vec{x_i}} = \begin{pmatrix} -\partial E/\partial x_i \\ -\partial E/\partial y_i \\ -\partial E/\partial z_i \end{pmatrix}$$

The above works for a spring that is uniformly extended. If the extension is non-uniform, we consider the deformation function $\phi(X)$, which maps the "natural" (or "reference") position of each point, to its deformed location $x = \phi(X)$



Example :

$\Rightarrow$ Spring moved along x-axis :

$x = \phi(X) = X + c$

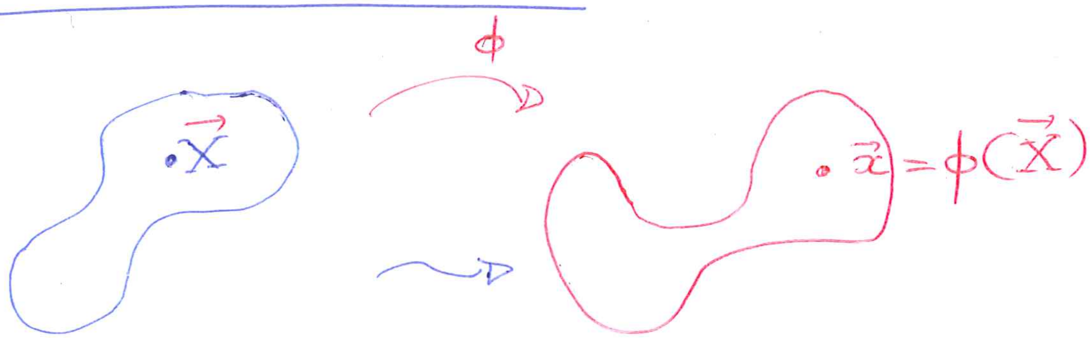
$\Rightarrow$ Spring moved and twice extended

$x = \phi(X) = 2X + c.$

We associate the compression ratio $\left(\frac{\ell}{\ell_0}\right)$ with the derivative $\phi'(X)$: If the spring is uniformly extended, then $\phi'(X) = \text{const} = \ell/\ell_0$

Finally, energy $E = \int_{X_L}^{X_R} \Psi(\phi'(X))\, dX$ (X_L, X_R are the spring endpoints' reference locations).

Extension to volumetric solids



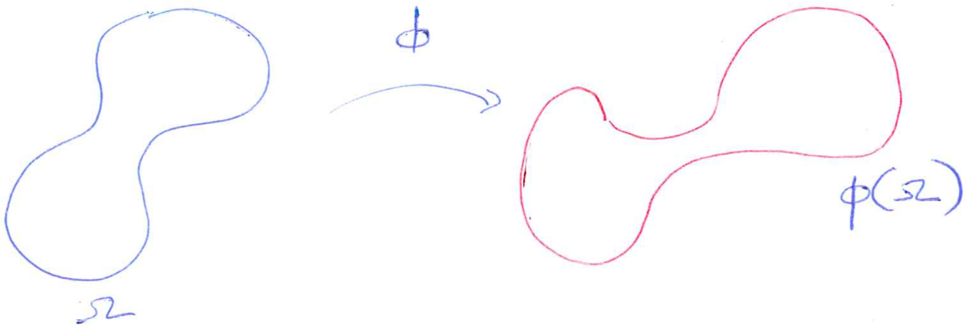
Reference shape   Deformed shape

In this case $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$\vec{X} \longmapsto \vec{x} = \phi(\vec{X})$
$\phi\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \phi_1(X,Y,Z) \\ \phi_2(X,Y,Z) \\ \phi_3(X,Y,Z) \end{pmatrix}$

The analogue of ϕ' (as used in the spring energy) is the Jacobian matrix (called the "deformation gradient" F).

$$F = \frac{\partial \vec{\phi}}{\partial \vec{X}} = \begin{pmatrix} \frac{\partial \phi_1}{\partial X} & \frac{\partial \phi_1}{\partial Y} & \frac{\partial \phi_1}{\partial Z} \\ \frac{\partial \phi_2}{\partial X} & \frac{\partial \phi_2}{\partial Y} & \frac{\partial \phi_2}{\partial Z} \\ \frac{\partial \phi_3}{\partial X} & \frac{\partial \phi_3}{\partial Y} & \frac{\partial \phi_3}{\partial Z} \end{pmatrix}$$

Similarly the energy stored in a deformed body is calculated as:



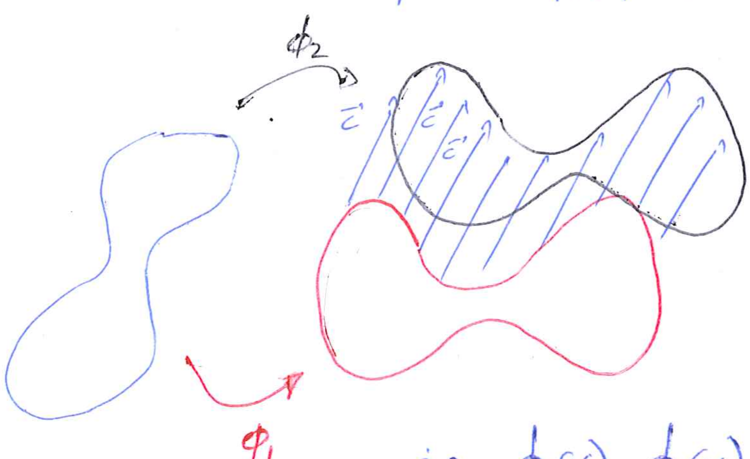
$$E = \int_{\Omega} \Psi(F)\, dX$$

The exact definition of $\Psi(F)$ is a defining property of the material being modeled (e.g. steel, concrete, flesh, etc).

<u>Properties</u>:

→ Translational invariance

⇒ Consider 2 different deformations, which differ by a translational shift (ϕ_1, ϕ_2)



Then:

$$F_1 = \frac{\partial \phi_1}{\partial X} = \frac{\partial(\phi_2 + c)}{\partial X}$$

$$= \frac{\partial \phi_2}{\partial X} = F_2$$

i.e. $\phi_2(X) = \phi_1(X) + c$

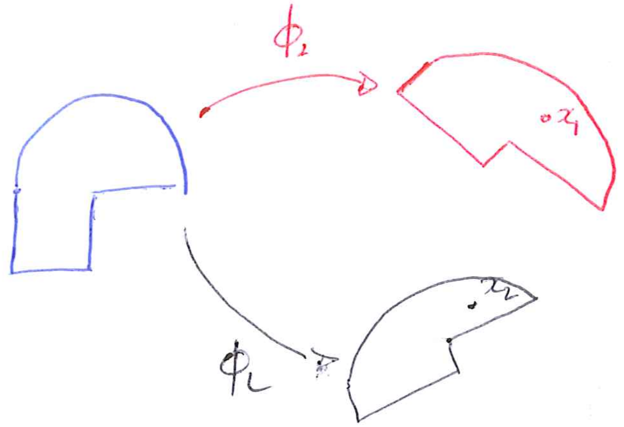
Thus, defining the energy density as a function of F automatically guarantees translational invariance, as:

$$F_1 = F_2 \rightarrow \Psi(F_1) = \Psi(F_2)$$

$$\rightarrow E[\phi_1] = \int \Psi(F_1) = \int \Psi(F_2) = E[\phi_2]$$

→ <u>Rotational invariance</u>

Consider 2 deformations that differ by a rotation, e.g.



$$\vec{x}_2 = R\vec{x}_1 + \vec{t}$$

$$\Downarrow \partial/\partial\vec{X}$$

$$\boxed{F_2 = R F_1}$$

If the energy density satisfies $\Psi(F) = \Psi(RF)$ for <u>any</u> rotation matrix R, then rotational invariance is automatically guaranteed

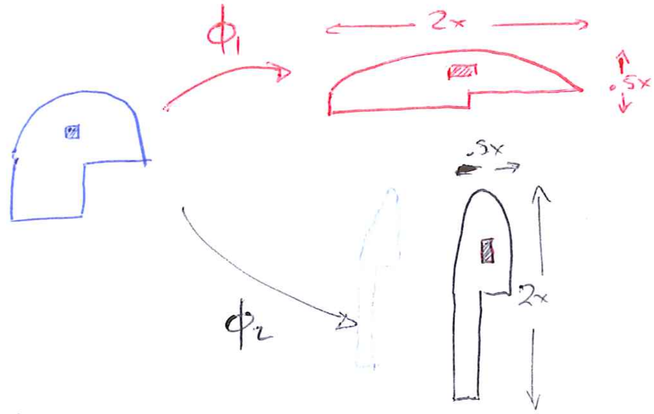
e.g. $\Psi(F) = tr(F^T F)$

$$\Psi(RF) = tr(F^T \underbrace{R^T R}_{=I} F) = tr(F^T F) = \Psi(F)$$

Rotational invariance is a desired property of Ψ (although the cost of enforcing it may be non-trivial).

→ Isotropy

Consider 2 deformations where the same ratios of expansion/contraction are applied, but along different directions



Here we can prove that

$$\boxed{F_1 = F_2 \cdot R}$$

post-multiply w/ rotation matrix

For certain materials we would expect the energy of these 2 deformations to be the <u>same</u>. Consider the deformation of an infinitesimal square section (mark by the shaded square) to see how this would be justified. This behavior is called <u>isotropy</u> and is characteristic of substances like rubber, jello, etc. Other materials such as woven fabrics, steel-reinforced concrete & muscles are <u>not</u> isotropic (there are directions along the material that resist deformation more than others)

If we want a material to be isotropic, we need to ensure that $\Psi(FR) = \Psi(F)$ for all rotations R.

If we want to enforce <u>both</u> rotation invariance and isotropy, we need $\Psi(R_1 F R_2) = \Psi(F)$ for any rotations R_1, R_2.

e.g. $\Psi = \det(F)$

$$\Psi(R_1 F R_2) = \det(R_1 F R_2) = \underbrace{\det(R_1)}_{=1} \det(F) \underbrace{\det(R_2)}_{=1} = \Psi(F)$$

We will see more examples of energy densities later, but here are two very simple academic examples:

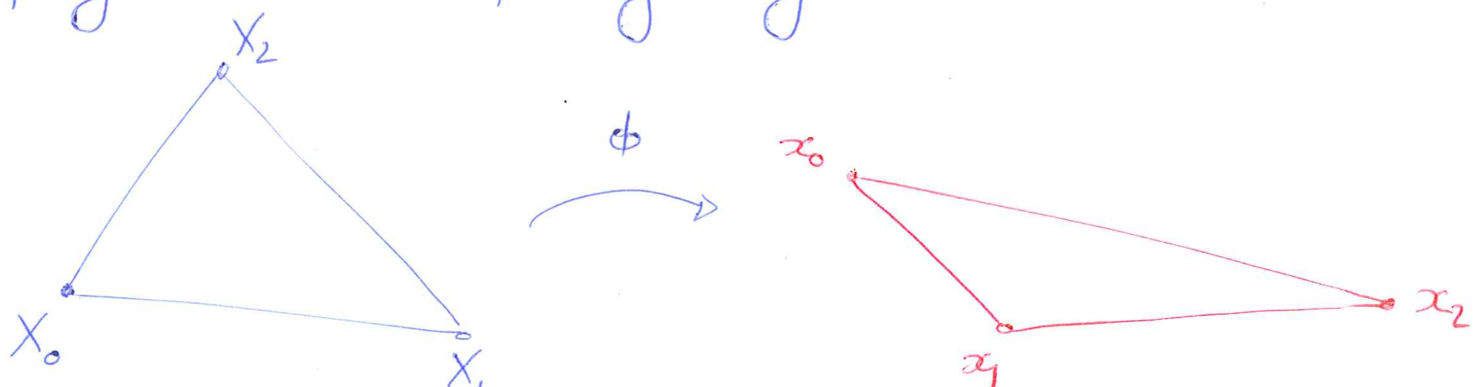
$$\Psi = \mathrm{tr}(F^T F) = \|F\|_F^2 \rightsquigarrow \text{"Zero rest-area membrane"}$$
(Remember, $\|M\|_F^2 = \sum_{i,j=1}^{n} M_{ij}^2$)

We can also show that

$J := \det(F)$ is the % change in the volume of an infinitesimal region. An incompressible material satisfies $J=1$

$$\Psi = c \cdot (J-1)^2$$ $\rightsquigarrow$ A material that resists volume change (but nothing beyond that).

Going back to a deforming triangle:



Here, we do not have access to the deformation function $\phi(X)$; we only know how it transforms the vertices $x_i = \phi(X_i)$. In order to formulate an energy, we fit a model for ϕ to these values, which is given by a linear relation

$$\vec{x} = \phi(\vec{X}) = A\vec{X} + \vec{t}$$

(A: 2x2, $\vec{t}$: 2-vector.)

Observations :

→ Such a model fit will be <u>specific</u> to the triangle $\underline{t}$ in question, i.e, we can write $\phi^{(t)}(X) = A^{(t)}X + t^{(t)}$ <u>per</u> triangle

→ Since $F = \frac{\partial \phi}{\partial X} = \frac{\partial}{\partial X}(AX+t) = A^{=const}$, the model really takes the form $\vec{x} = \phi(\vec{X}) = F \cdot \vec{X} + t$

(Once again F is <u>constant</u> per triangle, but different from triangle to triangle).

→ We can <u>find</u> F, as follows

$$\left.\begin{aligned} x_0 &= FX_0 + t \\ x_1 &= FX_1 + t \\ x_2 &= FX_2 + t \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 - x_0 &= F(X_1 - X_0) \\ x_2 - x_0 &= F(X_2 - X_0) \end{aligned}$$

Finally we can write the last 2 equations as a <u>single</u> matrix equation:

$$\left[\vec{x}_1 - \vec{x}_0 \;\vdots\; \vec{x}_2 - \vec{x}_0\right] = F\left[\vec{X}_1 - \vec{X}_0 \;\vdots\; \vec{X}_2 - \vec{X}_0\right]$$

2 columns

or $$\underbrace{\begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{bmatrix}}_{:= D_s} = F \underbrace{\begin{bmatrix} X_1 - X_0 & X_2 - X_0 \\ Y_1 - Y_0 & Y_2 - Y_0 \end{bmatrix}}_{:= D_m}$$

D_S = deformed shape matrix (variable over time)

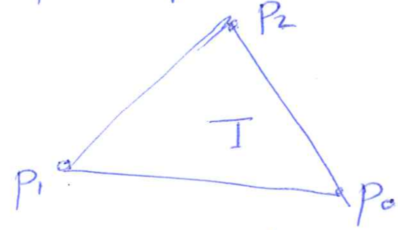
D_m = reference shape matrix (constant)

since $D_S = F D_m \Rightarrow \boxed{F = D_S D_m^{-1}}$

Implementation detail

Since D_m is constant, we precompute & store D_m^{-1} ahead of time.

Force definition process



Stage 1 : Determine $F = D_S D_m^{-1}$ from current positions

$$\left(F = F(\vec{x}_0, \vec{x}_1, \vec{x}_2) = D_S(\vec{x}_0, \vec{x}_1, \vec{x}_2) \cdot D_m^{-1} \right)$$

Stage 2 : Plug into $\Psi(F)$

$$\psi(\vec{x}_0, \vec{x}_1, \vec{x}_2) = \Psi(F(\vec{x}_0, \vec{x}_1, \vec{x}_2))$$

Stage 3 : Energy of entire triangle T

$$E^T = \int_T \underbrace{\Psi(F)}_{\text{constant on } T!} dX = Vol(T) \cdot \Psi(F^{(T)})$$

$$= Vol(T) \cdot \psi(\vec{x}_0, \vec{x}_1, \vec{x}_2)$$

Stage 4 : Compute forces on nodes as

$$\vec{f}_i = -\frac{\partial}{\partial \vec{x}_i} E(\vec{x}_0, \vec{x}_1, \vec{x}_2)$$

Doing all these steps explicitly would be tricky, since the final expression for $E(x_0, x_1, \ldots)$ can be elaborate and tricky to differentiate.

Fortunately, we can show that there is a <u>compact</u> expression for the forces, that results from this derivation. If we define the "force matrix" G, as follows:

$$G = \left[\vec{f}_1 \,\vdots\, \vec{f}_2\right]$$ we can show that

$$\boxed{G = -\mathrm{Vol}(T) \cdot P(F) \cdot D_m^{-T}}$$

and the force on the last node is given by $\vec{f}_0 = -\vec{f}_1 - \vec{f}_2$ (as forces need to balance out).

P is called the <u>1st Piola-Kirchhoff stress</u> matrix, and for every definition of $\Psi(F)$ there is a corresponding definition of $P(F)$.

<u>Note</u>: It can be shown that $\mathrm{Vol}(T) = \begin{cases} \frac{1}{2}\,|\det D_m| & \text{in 2D} \\ \frac{1}{6}\,|\det D_m| & \text{in 3D} \end{cases}$

We typically precompute $B_m = -\mathrm{Vol}(T) D_m^{-T}$ ahead of time, and then calculate $G = P \cdot B_m$.

Examples of material models:

Note: For all of the following material models, we reference the 2 constants μ & λ, called the Lamé parameters of the material. Their values are given as:

$$\mu = \frac{k}{2(1+\nu)} \qquad \lambda = \frac{k\nu}{(1+\nu)(1-2\nu)}$$

where k = Young's modulus (a measure of stiffness)

ν = Poisson's ratio (ranging from $0 \leq \nu < 0.5$ for most materials)

$\nu = 0$ Fully compressible (e.g. cork)

$\nu = 0.5$ Fully incompressible (jello, rubber)

$\nu < 0$ exotic materials ...

Remember: Frobenius norm of a matrix:

$$\|M\|_F^2 = \sum_{i,j} M_{ij}^2$$

Linear elasticity

$$\Psi(F) = \mu \|\varepsilon\|_F^2 + \frac{\lambda}{2} tr^2(\varepsilon)$$

$$P(F) = 2\mu\varepsilon + \lambda tr(\varepsilon) \cdot I$$

where $\varepsilon = \frac{1}{2}(F+F^T) - I$ "small strain tensor"

$\Rightarrow$ Not rotationally invariant

(strictly speaking, not isotropic either.)

Benefit: Forces are a linear function of x_i's!

$$D_s = [\vec{x}_1 - \vec{x}_0 \mid \vec{x}_2 - \vec{x}_0] \Rightarrow \text{linear on } x$$

$$F = D_s \underbrace{D_m^{-1}}_{\text{const}} \Rightarrow \text{linear on } x$$

$$\varepsilon = \frac{1}{2}(F+F^T) - I \Rightarrow \text{linear on } x$$

$$P = 2\mu\varepsilon + \lambda tr(\varepsilon) \cdot I \Rightarrow \text{linear on } x.$$

$$[f_1 \vdots f_2] = G = -Vol(T) P D_m^{-T} \Rightarrow \text{linear on } x$$

Ultimately, we have

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \underbrace{K}_{\text{const.}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + b$$

which makes things easy with Backward Euler.

St-Venant Kirchhoff material

$$\Psi(F) = \mu \|E\|_F^2 + \frac{\lambda}{2} \mathrm{tr}^2(E)$$

$$P(F) = F\left\{2\mu E + \lambda \mathrm{tr}(E)\cdot I\right\}$$

where $E = \frac{1}{2}(F^TF - I)$ "Green strain tensor"

⇒ Isotropic, rotationally invariant

⇒ $f(x)$ is a nonlinear function

Co-rotational (linear) elasticity

Let $F = U\cdot\Sigma\cdot V^T$ be the singular value decomposition of. Then

$$F = U(V^TV)\Sigma V^T = \underbrace{(UV^T)}_{R}\underbrace{(V\Sigma V^T)}_{S} = RS$$

R ↑ rotation, S ↑ symmetric

is called the polar decomposition of F

$$\Psi(F) = \mu \|\varepsilon^R\|_F^2 + \frac{\lambda}{2} \mathrm{tr}^2(\varepsilon^R)$$

$$P(F) = R\left\{2\mu\varepsilon^R + \lambda \mathrm{tr}(\varepsilon^R) I\right\}$$

where $\varepsilon^R = S - I$ (S: from polar decomposition $F=RS$).

⇒ Isotropic, rotationally invariant

⇒ Easier to make work w/ Backward Euler than St-Venant-Kirchhoff.