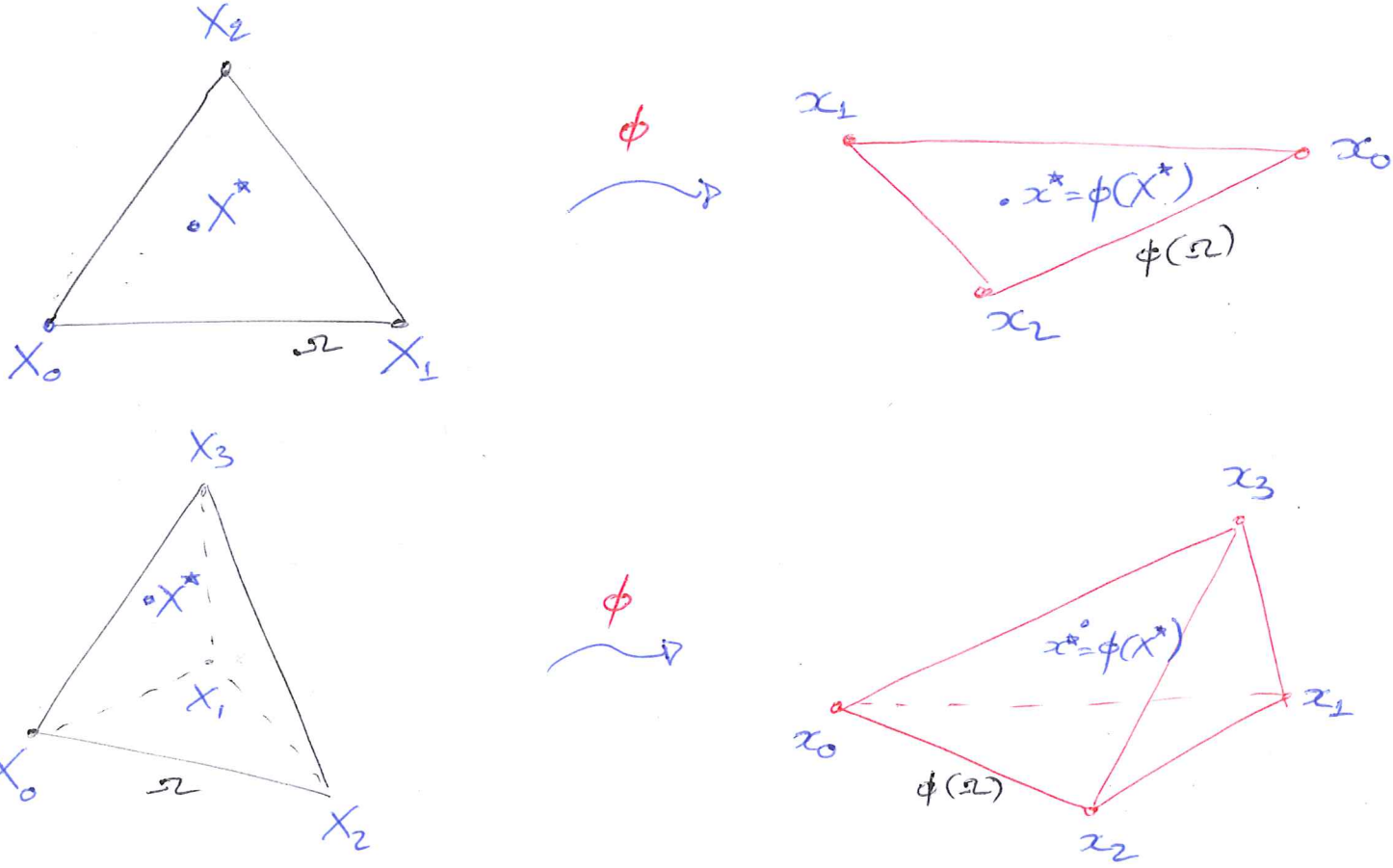
Review

As a building block of volumetric objects, we consider a triangle (2D) or a tetrahedron (3D)



The energy stored (as deformation-induced potential energy) on each triangle/tetrahedron is expressed as a volume integral

$$E = \int_{\Omega} \psi(F(X))\, dX$$

where $\psi()$ is an <u>energy density</u> function and $F = \frac{\partial \vec{\phi}}{\partial \vec{X}}$ is the Jacobian matrix of $\phi()$, and called the deformation gradient

↝ This definition automatically makes the energy be translation-invariant.

↝ If, additionally, $\Psi(F) = \Psi(RF)$ for all F and any rotation matrix R, the material is rotation-invariant.

↝ Isotropic models satisfy $\Psi(F) = \Psi(FR)$ as well.

Practical computation

Since we don't know $\phi(X)$ analytically, we approximate it as follows:

→ Assume that $\phi(X) = FX + t$ within each triangle/tetrahedron (F is the deformation gradient)

→ By setting $x_0 = \phi(X_0), \ldots, x_3 = \phi(X_3)$ we arrive at the numerical expression

$$\boxed{F = D_s D_m^{-1}}$$

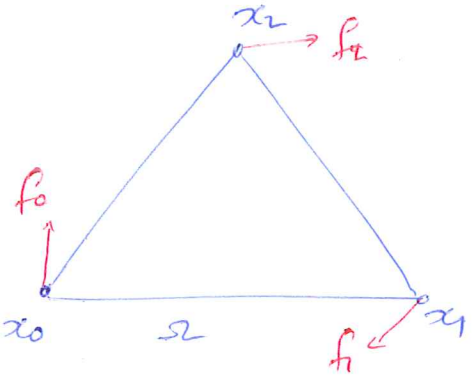
Where D_s : the deformed shape matrix (changes over time)

$$D_s = \left[\vec{x}_1 - \vec{x}_0 \;\middle|\; \vec{x}_2 - \vec{x}_0\right] \in \mathbb{R}^{2\times 2} \quad (2D) \qquad D_s = \left[\vec{x}_1 - \vec{x}_0 \;\middle|\; \vec{x}_2 - \vec{x}_0 \;\middle|\; \vec{x}_3 - \vec{x}_0\right] \in \mathbb{R}^{3\times 3} \quad (3D)$$

D_m : the reference shape matrix (never changes, can be precomputed)

$$D_m = \left[\vec{X}_1 - \vec{X}_0 \;\middle|\; \vec{X}_2 - \vec{X}_0\right] \in \mathbb{R}^{2\times 2} \quad (2D) \qquad D_s = \left[\vec{X}_1 - \vec{X}_0 \;\middle|\; \vec{X}_2 - \vec{X}_0 \;\middle|\; \vec{X}_3 - \vec{X}_0\right] \in \mathbb{R}^{3\times 3} \quad (3D)$$

Force computation



In theory

* Calculate element energy

$$E(\Omega) = \int_{\Omega} \Psi(F)\, dX = \quad \text{(} \Psi \text{ const on } \Omega \text{)}$$

$$= Vol(\Omega)\cdot \Psi(F^{\Omega})$$

$$= E^{\Omega}(x_0, x_1, x_2)$$

* Forces are the negative gradients of the energy

$$\vec{f}_i = -\partial E / \partial \vec{x}_i$$

In practice

The derivation above (after lots of calcul) reduces to the following compact form :

Define Force matrix G:

$$G = \left[\vec{f}_1 \,;\, \vec{f}_2\right] \text{ in 2D} \qquad G = \left[\vec{f}_1 \,;\, \vec{f}_2 \,;\, \vec{f}_3\right] \text{ in 3D}$$

(Note : $\vec{f}_0$ is excluded from G, in both cases)

Then, the following expression holds :

$$\boxed{G = -Vol(\Omega)\, P(F)\, D_m^{-T}}$$

Once $f_1, f_2 (f_3)$ (in 3D) have been computed, f_0 can be computed by noting that $\sum f_i = 0$, i.e:

- $\vec{f_0} = -\vec{f_1} - \vec{f_2}$ (2D)
- $\vec{f_0} = -\vec{f_1} - \vec{f_2} - \vec{f_3}$ (3D)

Notes:

- $P(F)$ is a matrix called the 1st Piola-Kirchhoff stress tensor. Given a formula for $\Psi(F)$, the expression for $P(F)$ is uniquely determined
- The volume $Vol(\Omega)$ of the undeformed triangle/tetrahedron is given by the formula

$$Vol(\Omega) = \begin{cases} \frac{1}{2}|\det D_m| & \text{in 2D} \\ \frac{1}{6}|\det D_m| & \text{in 3D} \end{cases}$$

- Many components of the formula $G = -Vol(\Omega)\, P(F)\, D_m^{-T}$ are constants and can be precomputed. In fact we can write $G = P \cdot B_m$, where

$$B_m = \begin{cases} -\frac{1}{2}|\det D_m|\, D_m^{-T} & \text{in 2D} \\ -\frac{1}{6}|\det D_m|\, D_m^{-T} & \text{in 3D} \end{cases}$$

Common material definitions

For all cases below:

μ & λ (Lamé parameters) are functions of the Young's modulus and Poisson's ratio

Linear Elasticity

$$\mathcal{E} = \frac{1}{2}(F + F^T) - I \qquad \text{"Small strain tensor"}$$

(Compare with $(\frac{l}{l_0} - 1)$ for springs)

$$\Psi = \mu \|\mathcal{E}\|_F^2 + \frac{\lambda}{2} tr^2(\mathcal{E})$$

(Note the definition: $\|M\|_F^2 := \sum_{ij} M_{ij}^2$ "Frobenius norm")

$$P(F) = 2\mu\mathcal{E} + \frac{\lambda}{2} tr(\mathcal{E}) \cdot I$$

Properties

* $F(x_0, x_1, x_2, x_3)$ is linear function of vertex positions $\underset{\sim}{x} = (\vec{x}_0, \ldots, \vec{x}_3)$

$\Rightarrow \mathcal{E}$ is affine on $\underset{\sim}{x}$ $\Rightarrow$ P is affine on $\underset{\sim}{x}$ $\Rightarrow$

$\Rightarrow G$ is affine on $\underset{\sim}{x}$ $\Rightarrow$ Forces $\underset{\sim}{f} := K\underset{\sim}{x} + b$

This makes computation cheap (K can be precomputed, is a constant matrix) and facilitates Backward Euler

* Linear elasticity is not rotationally invariant

$\Rightarrow$ Significant visible artifacts for large deformations.

St. Venant Kirchhoff material

→ Replaces small strain tensor with :

$$E = \frac{1}{2}(F^T F - I)$$ The "Green strain tensor"

$$\Psi(F) = \mu \|E\|_F^2 + \lambda \, tr^2(E)$$

$$P(F) = F\{2\mu E + \lambda \, tr(E) \cdot I\}.$$

Properties

⇒ Isotropic and rotation-invariant.

⇒ $f(x)$ is a nonlinear (in fact, a cubic polynomial) function (Backward Euler is tricky, computation is more expensive).

⇒ Issues : This material has several "nonphysical" rest states, e.g.

- When $F = -I$, or in general $F = R\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}$ (any "flipped" configuration), we have

$$E = 0 \rightsquigarrow P = 0 \rightarrow f = 0$$ (No forces!)

- When $F = 0$ (everything compressed to a point) we also have $P = 0$ ⇒ No force to trigger restoration.

⇒ Good for rotational invariance, but not for dramatic shape change!

Co-rotational Linear Elasticity

→ Tries to mimic linear elasticity as much as possible while "fixing" rotational invariance issue.

→ Based on "Polar decomposition":

Def Any matrix $A \in \mathbb{R}^{n \times n}$ can be decomposed as

$$A = R \cdot S \leftarrow \text{Symmetric.}$$

$\uparrow$ Rotation matrix, $R^T R = I$

Practical Implementation: If we know the SVD of A

$$A = U \Sigma V^T$$

$$= \underbrace{U V^T}_{:=R} \underbrace{V \Sigma V^T}_{:=S} = RS.$$

Corotated formulation:

→ Let $F = RS$ be the polar decomposition of F

→ $\varepsilon^R = S - I$ "Rotated strain tensor"

→ $\psi(F) = \mu \| \varepsilon^R \|_F^2 + \frac{\lambda}{2} tr^2(\varepsilon^R)$

→ $P(F) = R \left\{ 2\mu \varepsilon^R + \lambda tr(\varepsilon^R) I \right\}$

Properties

⇒ Isotropic and rotation invariant!

⇒ Nonlinear! (R & S are nonlinear functions of x!)

⇒ Approximations (linear) that facilitate Backward Euler do exist ⇒ "Warped stiffness" techniques.

⇒ Bulk of cost: Computing the SVD or polar decomposition of F

⇒ Does not resist compression too strongly: A finite force would suffice to collapse a body to a single point. (i.e $F=0 \Rightarrow f=$ finite)

In the real world as $F \to 0$, we would have $\Psi \to \infty$, $f \to \infty$.

Neo-hookean Material (or, Neohookean elasticity)

$$\Psi(F) = \frac{\mu}{2}\left[tr(F^TF) - tr(I)\right] - \mu \log(J) + \frac{\lambda}{2}\log^2(J)$$

$$P(F) = \mu(F - F^{-T}) + \lambda \log(J) F^{-T}$$

Here $J = \det(F)$ = volume change ratio = $\frac{\text{Volume of } \phi(\Omega)}{\text{Volume of } \Omega}$

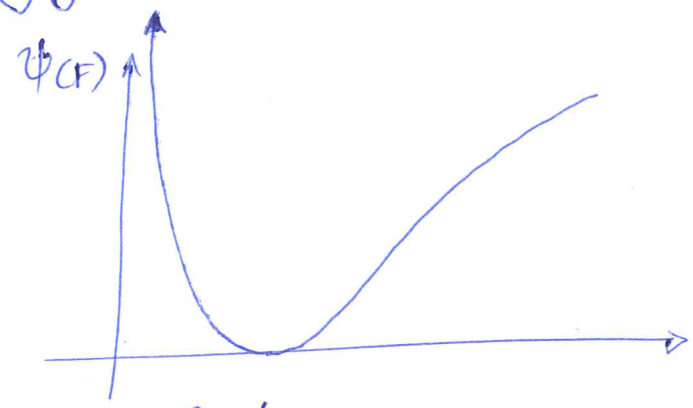
(Incompressibility implies $J = 1$)

Properties

⇒ Nonlinear, isotropic, rotation-invariant

⇒ Very good approximation of natural rubber-like materials (and reasonably good approximation of flesh).

⇒ Strongly resists compression to zero volume



(It requires infinite energy to collapse a material to a single point)

⇒ The last property, although faithful to reality, creates issues with robustness:

If $J \ll 1$ forces are <u>extremely strong</u>

(may need to use CFL number $\ll 1$ for stability)

If $J < 0$ forces are <u>undefined</u>! ⇒ Simulation Halts!

⇒ Could arise from very fast moving boundary trajectories

⇒ Could arise from nonphysical artist-imposed kinematic constraints.

Consequence:

- When using neohookean we may be forced to move kinematic constraints slowly or may be limited to simulations without nonphysical constraints
- Remedy: Invertible methods (see. ref.), or just use co-rotated elasticity.

Damping

Commonly used: Linear damping model

Define: $\dot{D}_s = \left[\vec{v}_1 - \vec{v}_0 \;\vdots\; \vec{v}_2 - \vec{v}_0 \;\vdots\; \vec{v}_3 - \vec{v}_0 \right]$

$$\dot{F} = \dot{D}_s \cdot D_m^{-1}$$

$$\dot{\varepsilon}^d = \frac{1}{2}\left(\dot{F} + \dot{F}^T\right)$$

$$P^d = 2\alpha \varepsilon^d + \beta\, tr(\varepsilon^d) \cdot I$$

$$G^d = -Vol(\Omega)\, P^d D_m^{-T} = \left[f_1^d \;\vdots\; f_2^d \;\vdots\; f_3^d \right]$$

As before: $f_0^d = -f_1^d - f_2^d - f_3^d$

Parameter Selection

We typically set:

$$\alpha = \gamma \cdot \mu \qquad \beta = \gamma \cdot \lambda$$

$\gamma :=$ Rayleigh coefficient.

Stability We treat γ as we would for the ratio b/k on a single spring.

i.e. if $\Delta t < 2k/b$ for a single spring

then $\Delta t < 2/\gamma$ for the volumetric forces