Simulation of fluids

→ Can be used to model

- ↳ Water
- ↳ Fire
- ↳ Smoke, etc.

In all of the above, there exists a fluid medium that governs the behavior of the phenomenon; for water simulation this is obvious, for smoke and fire the fluid medium is the air (or the mixture of air and combustible gases/combustion products). while, for example, the visible part of smoke is due to particulate matter that is dragged around by the moving air.

Additionally for the above phenomena there may be additional physical properties of relevance (other than the motion of the fluid). eg:

→ Pressure, e.g. $p = \rho g h$ in a calm pool of water
(ρ = density of water)

→ The shape of the interface between 2 fluids (e.g, air and water).

→ The temperature of the fluid (important in fire & smoke)
⇒ determines color of flame, for example
⇒ Affects upwards

→ The concentration of smoke particles

⇒ This affects how "thick" the smoke is rendered as.

The following properties have a significant effect in how the governing equations for the fluid are formulated:

→ Compressible vs. Incompressible

Water is almost always modeled as incompressible: The density remains <u>constant</u> throughout the fluid volume (Reasonable expectation for all but very limited scenarios, e.g. underwater explosions or deep oceans)

Air is certainly more compressible than water (about 10^4 times!) but for practical smoke/fire applications it is typically treated as incompressible, too. Air would have to be modeled as a compressible fluid for atmospheric or meteorological applications, or to model explosions.

→ Viscous vs. inviscid fluids

The motion of a fluid is governed by (a) inertial factors, such as the tendency of water particles to preserve momentum, including forces which promote incompressibility, and (b) viscous/friction forces which dampen and decelerate the flow

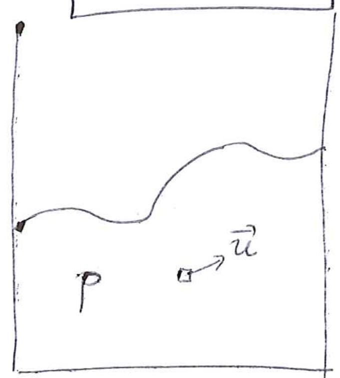
(other effects such as surface tension are omitted, here)

→ In viscous fluids, the friction forces dominate. Examples are: Honey, lava, motion of glaciers etc. The flavor of the governing equations are quite different in these extreme cases

→ In inviscid fluids, friction forces are negligible. For simplicity we will consider this case; it is a reasonable approximation for water, and other practical fluids (e.g, gasoline, ethanol). In fact, some fluids (e.g liquid helium) are so inviscid that other strange effects (e.g. surface tension dominates) become very pronounced

The inviscid Navier-Stokes equations (also called "Euler equations")

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = g} \quad \text{and} \quad \boxed{\nabla \cdot \vec{u} = 0}$$



$\vec{u}(\vec{x},t)$ = The velocity of the fluid at time t, in the location $\vec{x}$

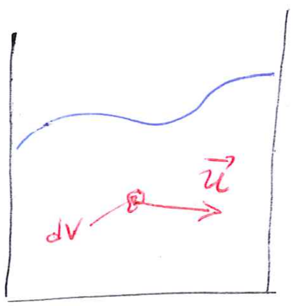
$p(\vec{x},t)$ = Fluid pressure at time t and location $\vec{x}$

ρ = Fluid density (assumed constant)

g = Acceleration of gravity ($\approx 9.81\ m/sec^2$).

Before discussing how we would solve this partial differential equation, we give some justification for it, as follows

First consider a small "blob"/"drop" of water, inside a larger volume. The motion of this small (infinitesimal) amount of water is essentially governed by $f = ma$. Let $\vec{u}$ denote the velocity of this tiny water volume and $m = \rho\, dV$ be the mass.



There are several forces we should consider:

→ Gravity: $\vec{G} = m\vec{g}$ $\quad$ ($\vec{G}$ = gravity, $\vec{g}$ = grav. acceleration)

$$= \rho\, dV \vec{g}$$

→ Pressure force

The fluid exhibits a distribution of pressures, based on factors such as:

→ Gravity (fluid feels the weight of the fluid above it)

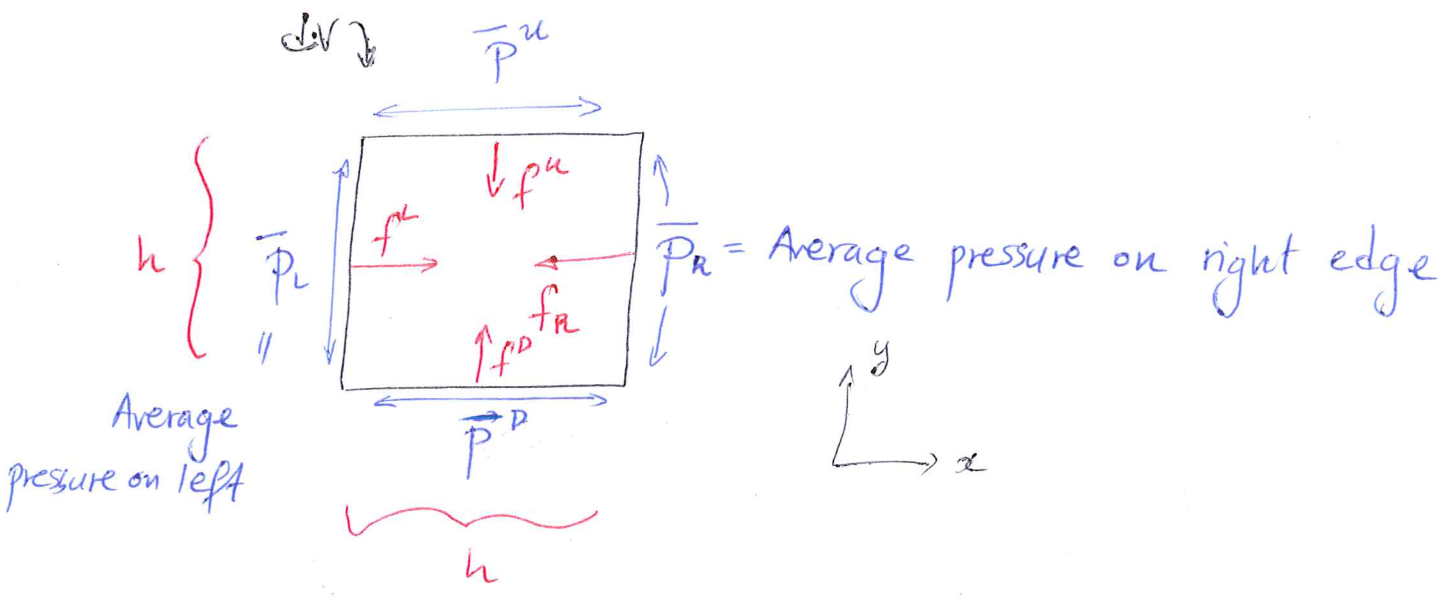
→ Inertia (Water rushing in to a location gives rise to pressure).

We will discuss how pressure (p) is computed, later, but for now, we can try to build some intuition about how, given a pressure field, we can compute the force excerted on this tiny water volume

Pressure is associated with the ratio

$$p = \frac{\text{force}}{\text{area}}$$

Let us assume that the tiny volume dV is shaped like a square (in 2D). Then



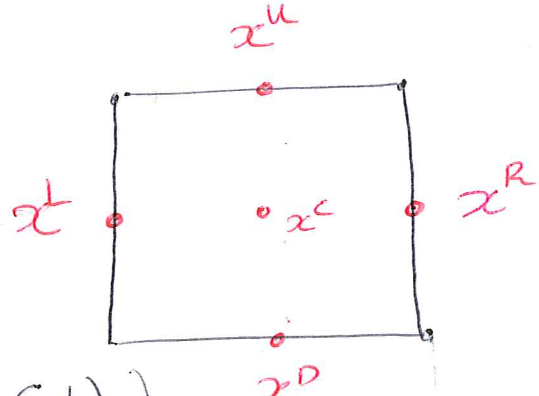
The pressure-induced force is always perpendicular to the fluid boundary. Thus, the pressure forces on the 4 square edges are

$$f^R = \begin{pmatrix} -p^R h \\ 0 \end{pmatrix}, \quad f^L = \begin{pmatrix} p^L h \\ 0 \end{pmatrix}, \quad f^u = \begin{pmatrix} 0 \\ -p^u h \end{pmatrix}, \quad f^D = \begin{pmatrix} 0 \\ p^D h \end{pmatrix}$$

Adding up these 4 contributions, we get the total pressure force as:

$$\vec{f} = -h \begin{pmatrix} \bar{p}^R - \bar{p}^L \\ \bar{p}^U - \bar{p}^D \end{pmatrix}$$

At this point, we approximate $\bar{p}^R \approx p(x^R)$, $\bar{p}^U \approx p(x^U)$, etc. Where the points x^L, x^R, x^U, x^D are the <u>centers</u> of the respective edges



Then $$\vec{f} \approx -h \begin{pmatrix} p(x^R) - p(x^L) \\ p(x^u) - p(x^D) \end{pmatrix}$$

$$= \underbrace{-h^2}_{= -dV} \begin{pmatrix} \frac{p(x^R) - p(x^L)}{h} \\ \frac{p(x^u) - p(x^D)}{h} \end{pmatrix}$$

Using the definition of the partial derivative

$$\frac{\partial p}{\partial x}(x^*, y^*) = \lim_{h \to 0} \frac{p(x^* + \frac{h}{2}, y^*) - p(x^* - \frac{h}{2}, y^*)}{h}$$

We get that $\dfrac{p(x^R)-p(x^L)}{h} \xrightarrow[h\to 0]{} \dfrac{\partial p}{\partial x}(x^c)$

Similarly, for small h we have $\dfrac{p(x^u)-p(x^D)}{h} \approx \dfrac{\partial p}{\partial y}(x^c)$

Thus: $\vec{f} = -dV \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix}$ or $\boxed{\vec{f} = -dV \cdot \nabla p}$

(Note: This derivation made several assumptions, for convenience of illustration. However the relation $\vec{f} = -dV \nabla p$ can be proven rigorously and generally).

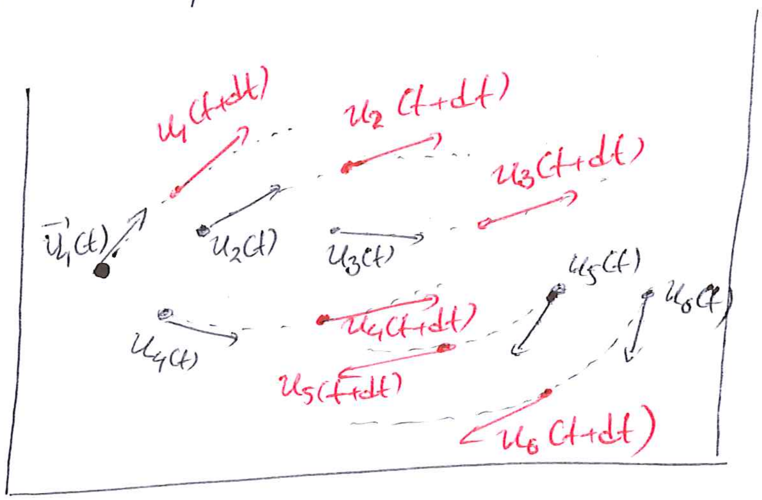
→ Lastly, apart from gravity & pressure forces, there are also <u>friction</u> forces, but we will not address them, as we focus on inviscid fluids

So far: $m\vec{a} = \vec{f}$ has become

$$\rho dV \vec{a} = \rho dV \vec{g} - dV \nabla p \Rightarrow$$

$$\vec{a} + \frac{1}{\rho}\nabla p = g \quad \text{or} \quad \boxed{\frac{D\vec{v}}{Dt} + \frac{1}{\rho}\nabla p = g} \qquad (*)$$

In this last equation, we used the special symbol $\frac{D\vec{v}}{Dt}$ to denote the acceleration of an infinitesimal water volume. If our representation of the fluid wa chosen to be a set of moving water volumes, this would be all that was ne ssary and equation (*) could be used directly



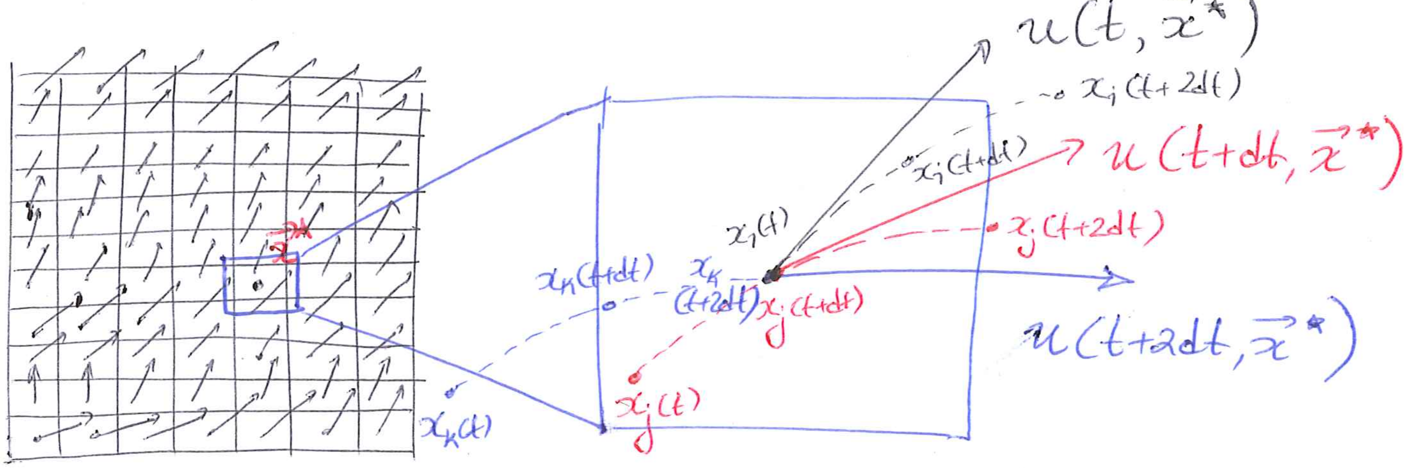
This is called a <u>Lagrangian</u> formulation of fluid dynamics; our state variables are the position ($\vec{x}_i(t)$) and velocity ($\vec{u}_i(t)$) of every infinitesimal water blob. This <u>is</u> a usable approach (see e.g. SPH methods), but:

→ Computing derivatives (e.g. ∇p, along with other derivatives of $\vec{u}$, which will be needed) is hard.

→ The water blobs are irregularly and non-uniformly positioned

→ Incompressibility is somewhat tricky to enforce.

An alternative approach is the <u>Eulerian</u> formulation: Instead of tracking every fluid particle, we record what happens at a <u>specific location in space</u>, regardless of what water particle happens to be at that location at a specific time:



I.e. We record what the velocity is at a <u>specific location</u> $\underline{\underline{x}}^*$ at various points in time (although it could be <u>different</u> water particles that pass through x^* at any given point in time).

We call this description of velocity

$$u(\vec{x}, t)!$$

Now, let's focus on a <u>specific</u> water particle (q_i) which moves along the trajectory $\vec{x}_i(t)$. Then, the velocity of this <u>specific</u> particle is

$$\vec{u}_i(t) = \vec{u}(\vec{x}_i(t), t)$$

Now, what we symbolized as $\frac{D\vec{u}}{Dt}$ before is really $\frac{d}{dt}\left[\vec{u}_i(t)\right]$. Using the chain rule, we have:

$$\frac{d}{dt}\left[\vec{u}_i(t)\right] = \frac{d}{dt}\left[\vec{u}(\vec{x}_i(t), t)\right]$$

At this point it is useful to write every component of the velocity separately:

$$\vec{u}(\vec{x}, t) = \begin{pmatrix} u(x,t) \\ v(x,t) \\ w(x,t) \end{pmatrix}$$

Thus
$$\frac{d}{dt}\left[\vec{u}(\vec{x}_i(t), t)\right] = \begin{pmatrix} \frac{d}{dt}\, u(\vec{x}(t), t) \\ \frac{d}{dt}\, v(\vec{x}(t), t) \\ \frac{d}{dt}\, w(\vec{x}(t), t) \end{pmatrix} =$$

$$= \begin{pmatrix} u_x \cdot x'(t) + u_y\, y'(t) + u_z\, z'(t) + u_t \\ v_x\, x'(t) + v_y\, y'(t) + v_z\, z'(t) + v_t \\ w_x\, x'(t) + w_y\, y'(t) + w_z\, z'(t) + w_t \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} x'(t) & y'(t) & z'(t) \end{pmatrix}}_{=\vec{u}^T} \overbrace{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}}^{:= \nabla \vec{u}} + \begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} := \frac{\partial u}{\partial t}$$

Thus $$\frac{Du}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}(\nabla \vec{u})$$

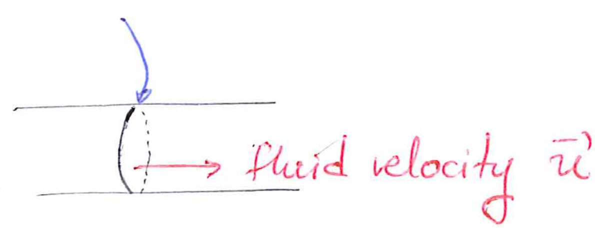
$$(\text{or}) = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}$$

Substituting into (*) we get the final form of (one of) the Euler equations:

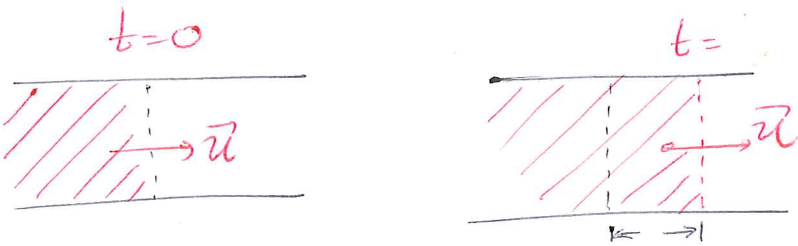
$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = g}$$

Incompressibility

Def Flow past a surface



$$\text{flow} = \frac{\text{Volume of fluid through A in time } t}{t}$$

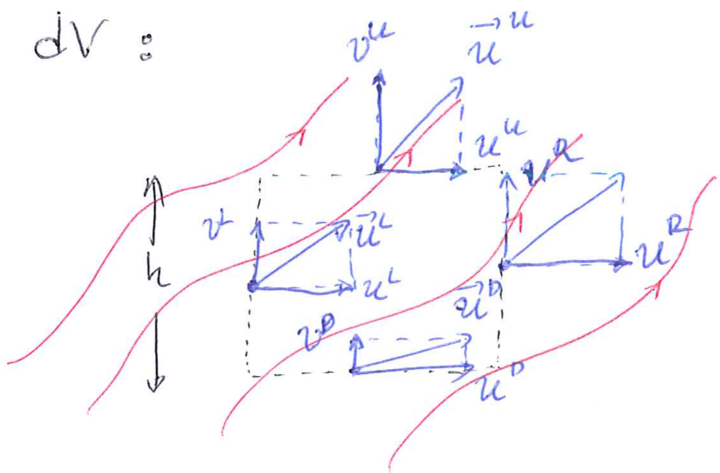


↳ length $= \vec{u} \cdot T$

Volume $= \vec{u} \cdot T \cdot A$

i.e. $\text{flow} = \vec{u} \cdot A$

Now, consider an infinitesimal square volume dV :



In order for the fluid to be incompressible, the total flow in/out of this square must be <u>zero</u>. i.e :

Volume Inflow From (Left) + Volume Inflow From (Right)
+ Volume Inflow From (Up) + Volume Inflow From (Down) = 0

$$\Rightarrow \quad u^L \cdot h - u^R h + v^D h - v^u h = 0$$

$$\xrightarrow{\times\left(-\frac{1}{h^2}\right)} \quad \frac{u^R - u^L}{h} + \frac{v^u - v^D}{h} = 0$$

$$\xrightarrow[h \to 0]{} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \longrightarrow \quad \boxed{\operatorname{div} \vec{u} = 0}$$

or $\boxed{\nabla \cdot \vec{u} = 0}$

The "divergence" of the vector valued function
$\vec{u}(x,y,z) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$ is defined as $\operatorname{div} \vec{u} \ (= \nabla \cdot \vec{u}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$