Review

The dynamics of inviscid, incompressible fluids are governed by the Euler equations:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = g}$$

and $\boxed{\nabla \cdot \vec{u} = 0}$

Where:

$\vec{u}(\vec{x}, t)$ = Velocity of the fluid at location $\vec{x}$, and at time t (Written, componentwise $\vec{u} = (u, v, w)$)

$p(\vec{x}, t)$ = Pressure at location $\vec{x}$, and time t

ρ = Fluid density ($\approx 10^3\ kg/m^3$ for water
$\approx 1.3\ kg/m^3$ for air)

g = Acceleration of gravity ($\approx 9.81\ m/sec^2$)

We saw some preliminary justification for the equations above, e.g

→ We saw that $\vec{f}_p = -dV \nabla p$ is the pressure-induced force that an infinitesimal volume receives, and

→ The condition $\nabla \cdot \vec{u} = 0 \Leftrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ enforces incompressibility.

Lastly, we saw that the terms:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \qquad \left(:= \frac{Du}{Dt} \text{ the "material derivative"} \right)$$

are directly related to the acceleration $\vec{a}_i(t) = \dfrac{d\vec{u}_i(t)}{dt}$ of a specific fluid particle P_i. The difference is that we employ a description of the velocity field $\vec{u}(\vec{x},t)$ which is <u>not</u> particle-centric (we record velocities at <u>fixed locations in space</u> rather than velocities of moving particles).

If $\vec{x}_i(t)$ is the trajectory of a particle over time (thus, $\vec{u}_i(t) = \vec{x}_i'(t)$) the relation between the 2 velocity descriptions is:

$$u_i(t) = u(x_i(t), t)$$

and, using the chain rule (in a careful application, we get)

$$\frac{d\vec{u}_i(t)}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\nabla\vec{u} = \begin{pmatrix} u_t + u u_x + v u_y + w u_z \\ v_t + u v_x + v v_y + w v_z \\ w_t + u w_x + v w_y + w w_z \end{pmatrix} \quad \left(:= \frac{D\vec{u}}{Dt} \right)$$

At this point we will examine a generalization of this duality between particle-centric and location-centric descriptions:

Imagine that a physical medium (e.g, a fluid) is composed of many, infinitesimally small, particles $\{P_i\}$, each moving along a trajectory $\vec{x}_i(t)$. Assume that each particle "carries" along with it some physical property q_i (we will be intentionally abstract: This property could be any scalar- or vector-valued entity, e.g. temperature, color, density, orientation, translucency, etc).

Let us assume that, on a particle-by-particle basis this property changes in accordance to a law such as:

$$\frac{d\, q_i(t)}{dt} = Q(\dots)$$

Now, if we define the <u>field</u> of time varying values of this property as $q(\vec{x},t)$ (:= the value of property q at location $\vec{x}$ & time $= t$), this law translates as

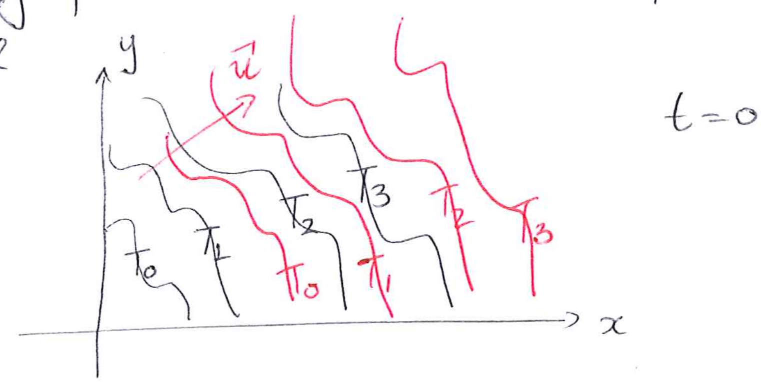
(we assume q is a scalar function, e.g. Temperature):

$$Q(\ldots) = \frac{d q_i(t)}{dt} = \frac{d}{dt}\left[q(\vec{x}_i(t), t) \right] =$$

$$= \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial q}{\partial z} \cdot \frac{dz}{dt}$$

$$= q_t + q_x \cdot u + q_y \cdot v + q_z \cdot w = q_t + \vec{u} \cdot \nabla q := \frac{Dq}{Dt}$$

Example: Assume that every particle has a fixed temperature associated with it, and all particles move with the same constant velocity $\vec{u}$. Also, assume that the temperature of a given particle never changes, i.e.



$$\frac{dq_i}{dt} = 0 \rightsquigarrow \frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = 0$$

The solution of this last PDE is given as:

$$q(\vec{x}, t) = q(\vec{x} - t\vec{u}, 0) = q_0(\vec{x} - t\vec{u})$$

↳ i.e. Temperature field at $t=0$.

Thus, the original temperature field at $t=0$ just "scrolls" along the direction of the velocity.

(of course if the temperature of each particle was variable over time, this solution would have been different).

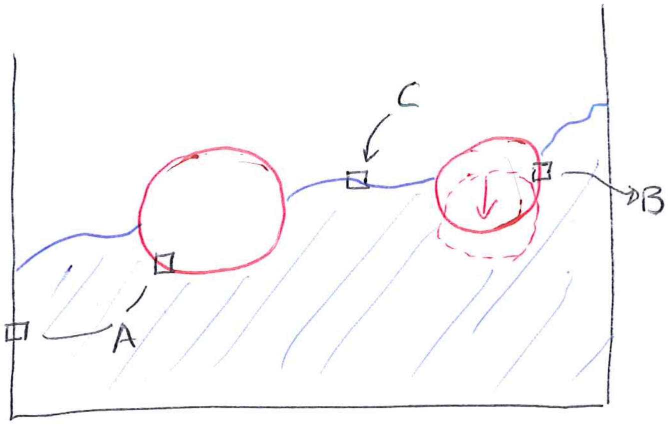
The process we just described is called "advection": A velocity field carries around values of a physical property, as time evolves. The material derivative of $\vec{u}$ (from the Euler equations) is also an instance of this phenomenon: In this case, what is "carried around" by the velocity field is velocity values themselves! In fact each component (u,v,w) is advected, as q was in the previous description:

$$\left.\begin{aligned} \frac{Du}{Dt} &= u_t + \vec{u}\cdot\nabla u \\ \frac{Dv}{Dt} &= v_t + \vec{u}\cdot\nabla v \\ \frac{Dw}{Dt} &= w_t + \vec{u}\cdot\nabla w \end{aligned}\right\} \quad \text{or} \quad \frac{D\vec{u}}{Dt} = \vec{u}_t + \vec{u}\cdot\nabla\vec{u}$$

We will return to the principle of advection and the duality between particle-centric and space-centric descriptions, later.

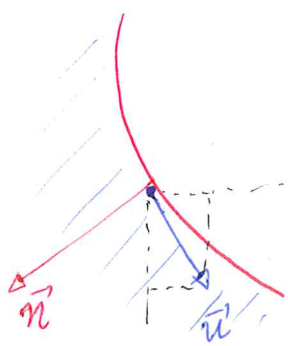
(other things that get advected: Concentration (smoke), Levelset Values (water))

Boundary conditions



Fluids exhibit interesting motion because the interaction with their surroundings imposes certain constraints, which trigger visually interesting shapes and behaviors. We summarize such constraints here:

A. Interface between fluid and container walls or stationary objects



The fluid must be disallowed to enter the object, thus the normal component of the velocity at the point of contact must be zero.

i.e. $\boxed{\vec{u} \cdot \vec{n} = 0}$ (the no-stick condition)

Note that this prevents both (a) penetration of fluid into the solid and (b) separation of fluid from the solid.

Note: This is a plausible model for inviscid solids! I.e. we do not try to prevent "slippage", or tangential motion of the fluid. For a viscous fluid, a different condition might be appropriate, e.g. the "no slip" condition $\vec{u} = 0$ (no motion in any direction).

B. Interface between fluid and moving objects.

The no-stick condition becomes

$$(\vec{u}-\vec{u}_{solid})\cdot\vec{\eta} = 0$$

$$\text{or} \quad \vec{u}\cdot\vec{\eta} = \vec{u}_{solid}\cdot\vec{\eta}$$

Again, this forces the fluid to stay in contact with the solid, without interpenetration or separation

C. Free surface (e.g. water).

This is, really, the interface between 2 fluids: air and water. Due to the massively smaller density of air w.r.t water, we can assume the air pressure is equal to the canonical atmospheric pressure p_{atm} everywhere in the extent that air occupies. Practically, we set that pressure to be zero: Since p shows up in Euler equations just via its derivatives, any constant offset in the pressure values does not influence the physics; thus it is ok to define (by convention)

$$\boxed{p=0}$$

anywhere along the air-water interface (and, throughout the air, too).

[Note: We need to be more careful with bubbles! The pressure of air would be important in this case].

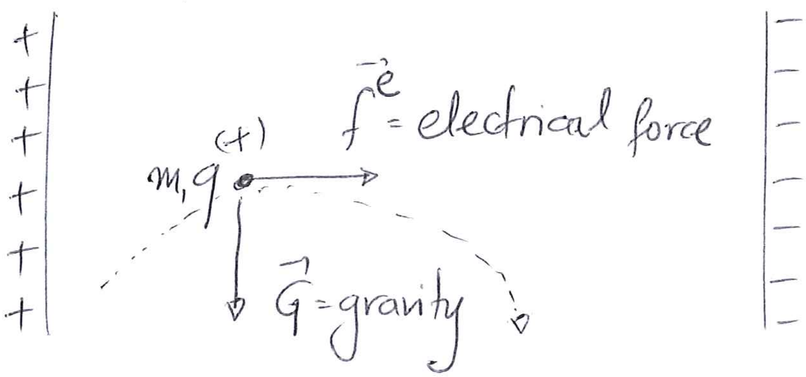
Operator splitting & Euler equations

Consider an ODE of the form

Unknown function: $y(t)$

$$\frac{dy}{dt} = f(y,t) + g(y,t)$$

example: Motion of a "heavy" charged particle in an electric field:



→ Subject to Gravity (G)

→ Subject to electrostatic force (f^e).

Newton's law:

$$m\vec{v}'(t) = \vec{G} + \vec{f}^e$$

$$\Rightarrow \vec{v}'(t) = \frac{\vec{G}}{m} + \frac{\vec{f}^e}{m}$$

e.g. Forward Euler:

$$\vec{v}^{(n+1)} = \vec{v}^{(n)} + \Delta t \left(\frac{G}{m} + \frac{f^e}{m} \right).$$

Idea: Let's do this in 2 steps:

Step 1: Use only gravity, compute "intermediate" velocity $\hat{v}^{(n+1)}$:

$$\hat{v}^{(n+1)} = v^{(n)} + \Delta t \left(\frac{G}{m} \right)$$

Step 2: Use this intermediate velocity as if it was the velocity at time n (!!) and take a forward Euler step only using the electrostatic force. Call your result the final time $= t^{n+1}$ velocity:

$$v^{(n+1)} = \hat{v}^{(n+1)} + \Delta t \left(\frac{f^e}{m} \right)$$

$$= v^{(n)} + \Delta t \left(\frac{G}{m} + \frac{f^e}{m} \right)$$

We note that the result is the same as if we had taken one Forward Euler step with the combined force $G + f^e$!

This was somewhat lucky though... generally we cannot expect the result to be exactly the same. E.g.:

$$y'(t) = f(y,t) + g(y,t) \quad \ldots$$

Using a single F.E. Step

$$y^{(n+1)} = y^{(n)} + \Delta t\left[f(y^n,t^n) + g(y^n,t^n)\right]$$

Using Splitting

$\left(\begin{array}{l} y'(t) = f(y,t) \text{ followed by} \\ y'(t) = g(y,t) \end{array}\right)$

$$\hat{y}^{(n+1)} = y^{(n)} + \Delta t\, f(y^n,t^n)$$

$$y^{(n+1)} = \hat{y}^{(n+1)} + \Delta t\, g(\hat{y}^{(n+1)}, t^n)$$

$$= y^n + \Delta t\left[f(y^n,t^n) + g(y^n + \Delta t f(y^n,t^n), t^n)\right]$$

$\neq$

<u>Observation</u>! (Refer to R.Bridson's notes for details)

Although the two formulas are <u>not</u> the same, we can show that the discrepancy between the 2 scales like $O(\Delta t^2)$! This is the same discrepancy that using F.E. incurs (compared to computing the analytic solution). Thus this discrepancy is tolerable, and well worth for the convenience it offers.

Note: After splitting the equation:

$$y'(t) = f(y) + g(y) \rightarrow \begin{cases} y'(t) = f(y) \\ \text{followed by} \\ y'(t) = g(y) \end{cases}$$

we <u>do not</u> need to use the same integration method for both split equations (ok to mix & match)

Specifically, for Euler equations:

To advance

$$\left.\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho}\nabla p = \vec{g} \\ &\nabla\cdot\vec{u} = 0 \end{aligned}\right\}$$ from t^n to t^{n+1}, do the following:

Step 1: (Advection step)

Integrate: $(t^n \to t^{n+1})$

$$\frac{d\vec{u}}{dt} + \vec{u}\cdot\nabla\vec{u} = 0$$

Input: $\vec{u}^{(n)}$

Output: $\tilde{u}$ (intermediate)

(Method: To be discussed).

Step 2: (Body forces)

Integrate $(t^n \to t^{n+1})$

$$\frac{\partial \vec{u}}{\partial t} = \vec{g}$$

Input: $\tilde{u}$

Output: $\hat{u}$

(Method: usually Forward Euler)

Step 3: Pressure & Incompressibility

$$\frac{d\vec{u}}{dt} + \frac{1}{\rho}\nabla p = 0$$

s.t. $\nabla\cdot\vec{u} = 0$

Input: $\hat{u}, p^{(n)}$

Output: $\vec{u}^{(n+1)}, p^{(n+1)}$

(Method: To be discussed).