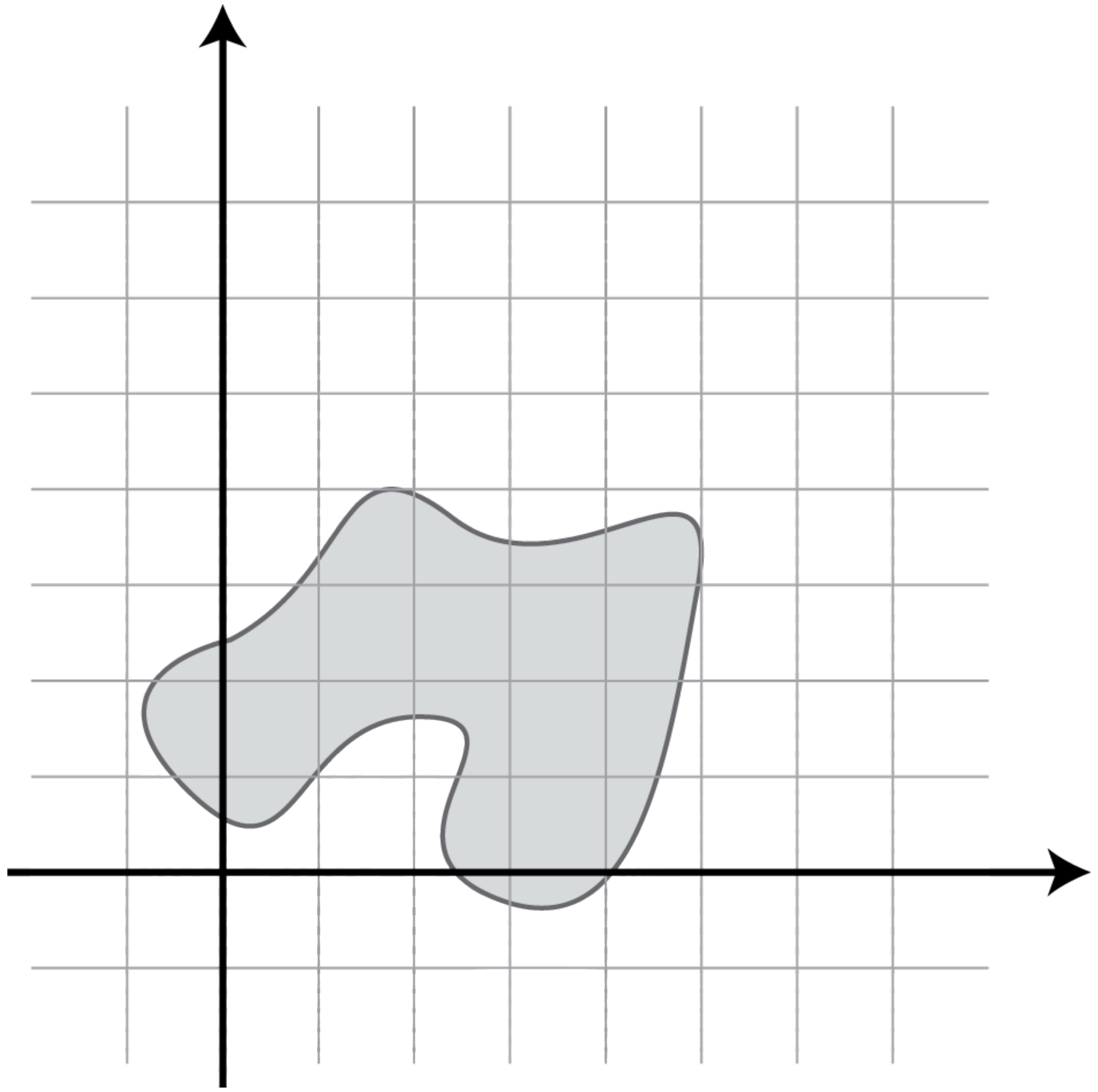
# **2D/3D Elasticity** - The deformation map



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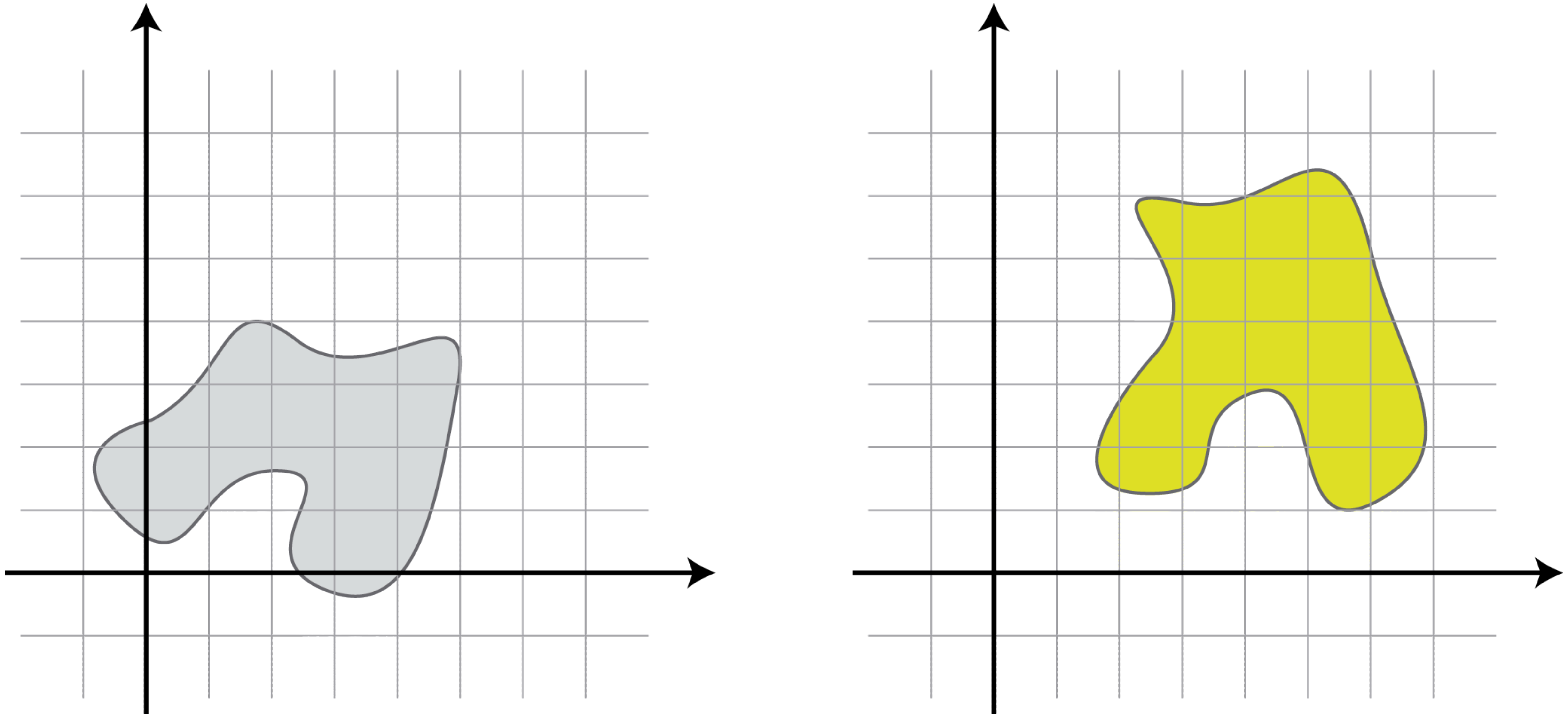

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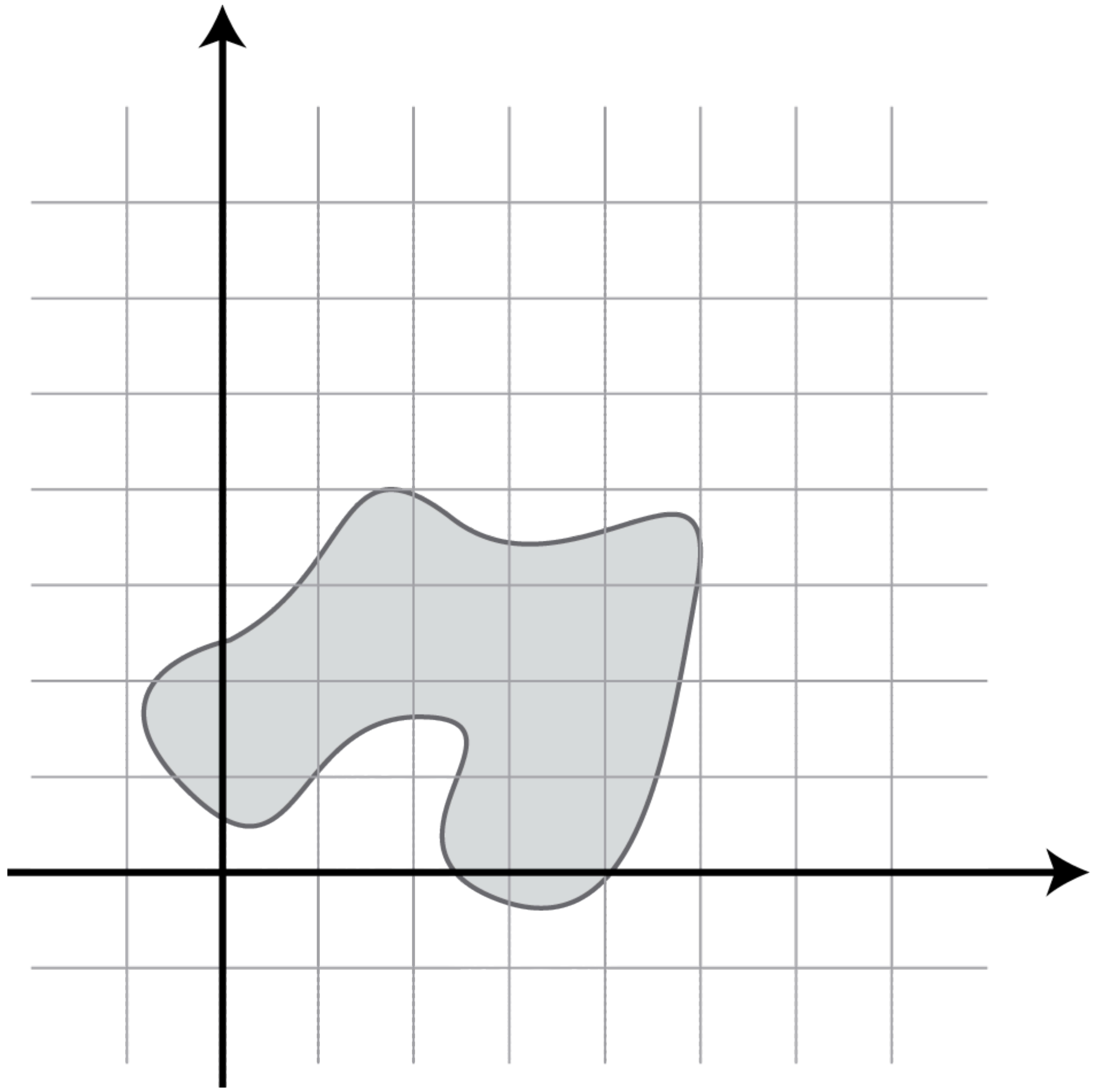
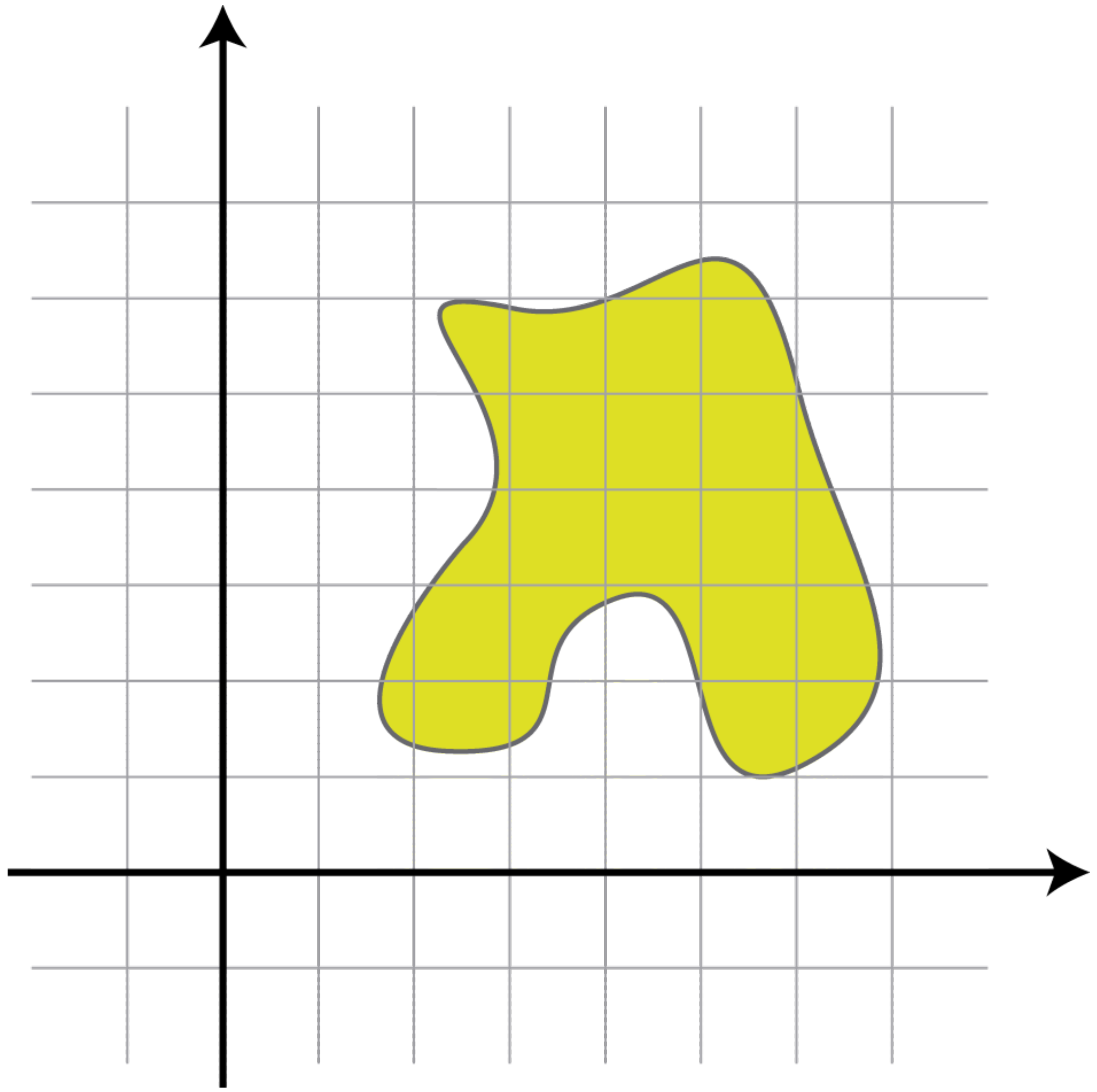


*Undeformed configuration*
*(**material** coordinates)*



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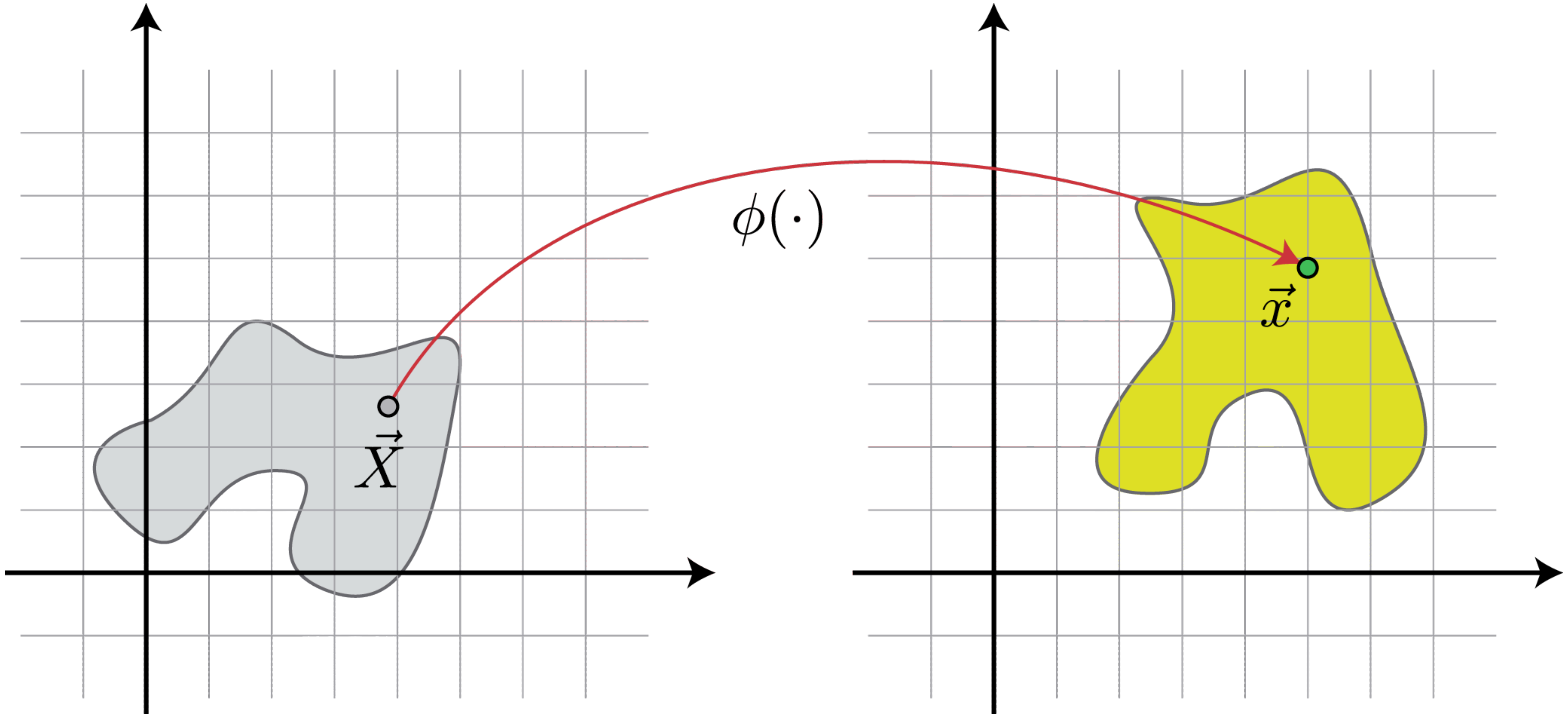

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$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \phi(\vec{X}) = \begin{pmatrix} x(X, Y, Z) \\ y(X, Y, Z) \\ z(X, Y, Z) \end{pmatrix}$$

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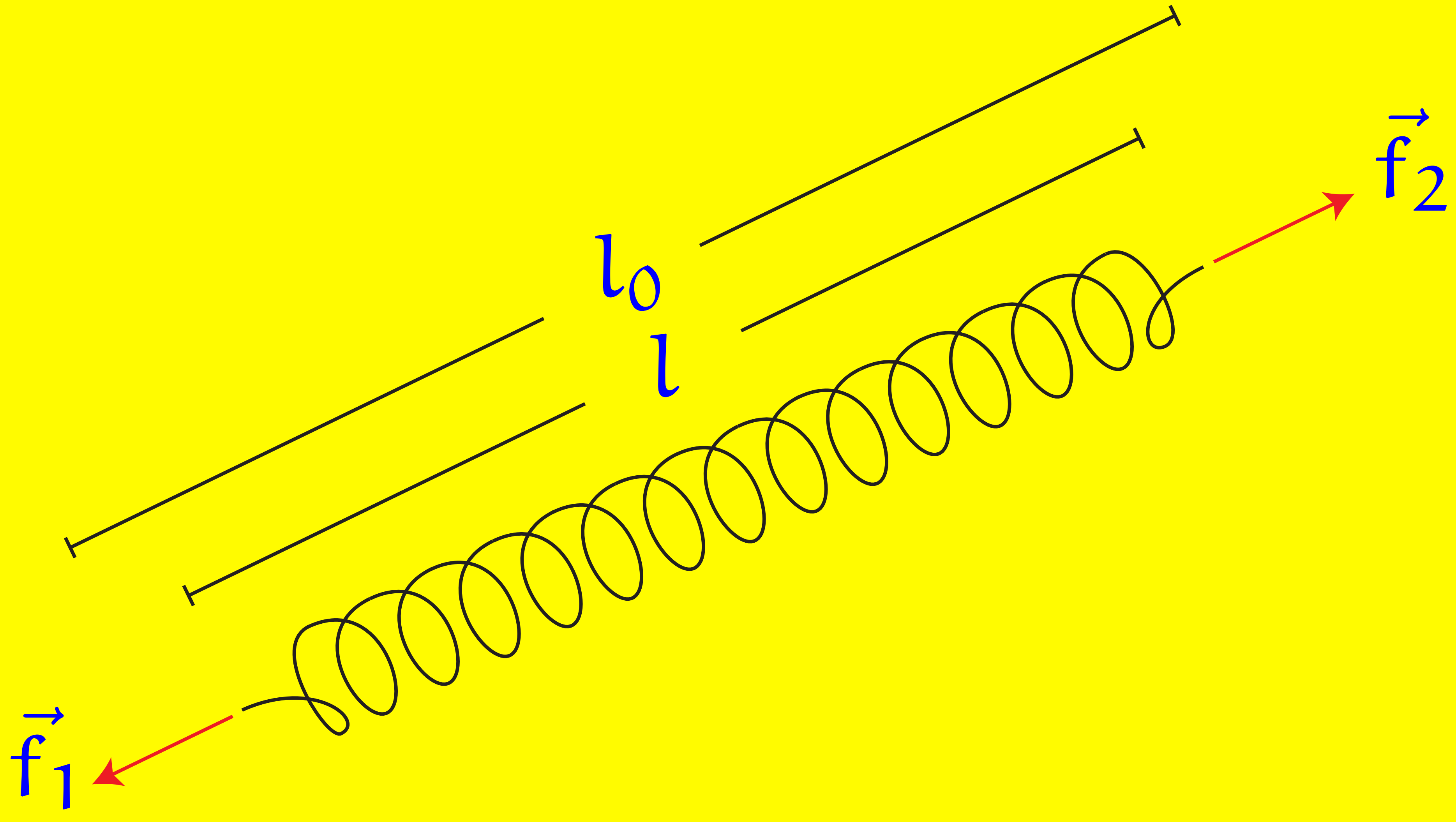

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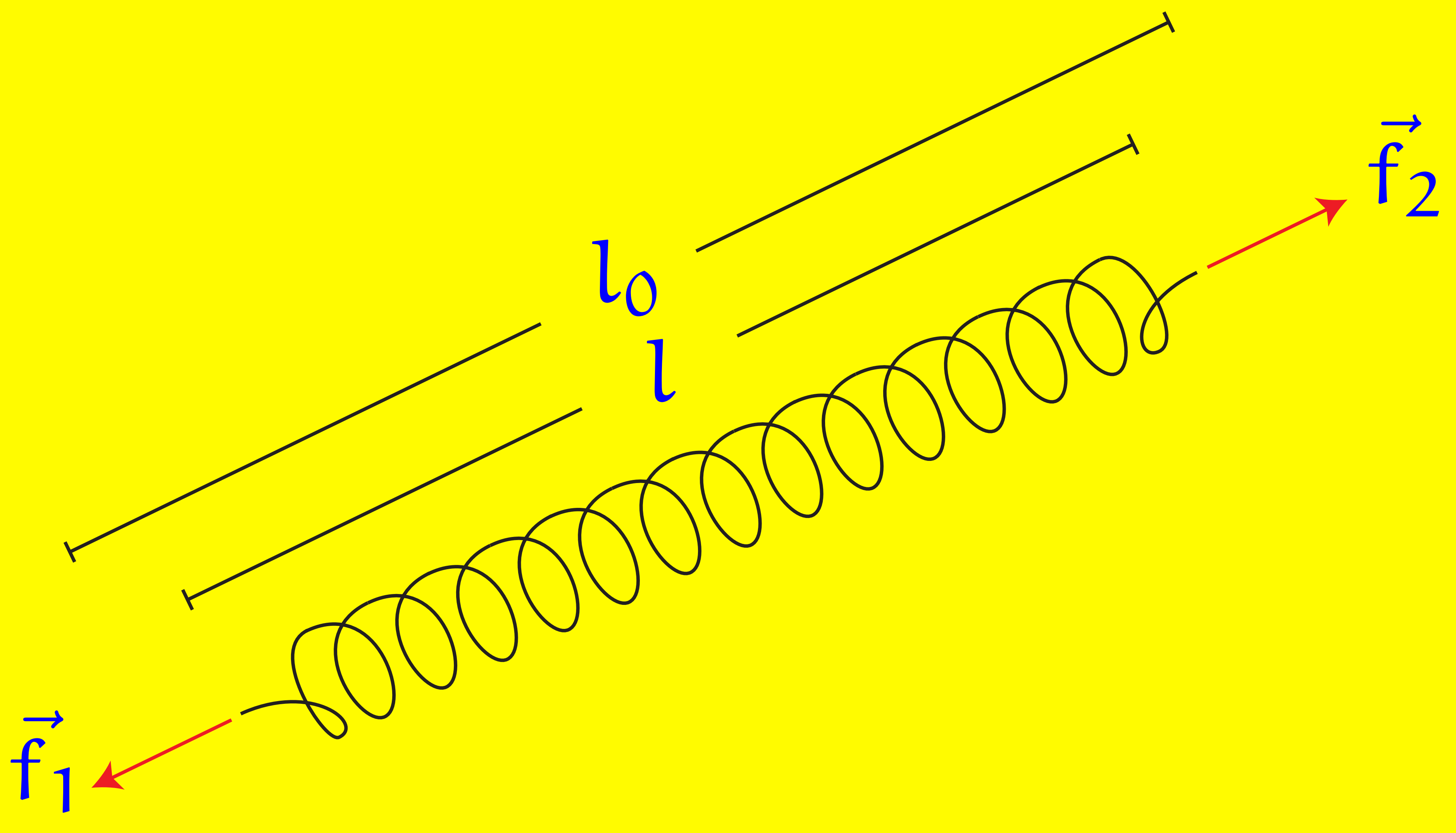

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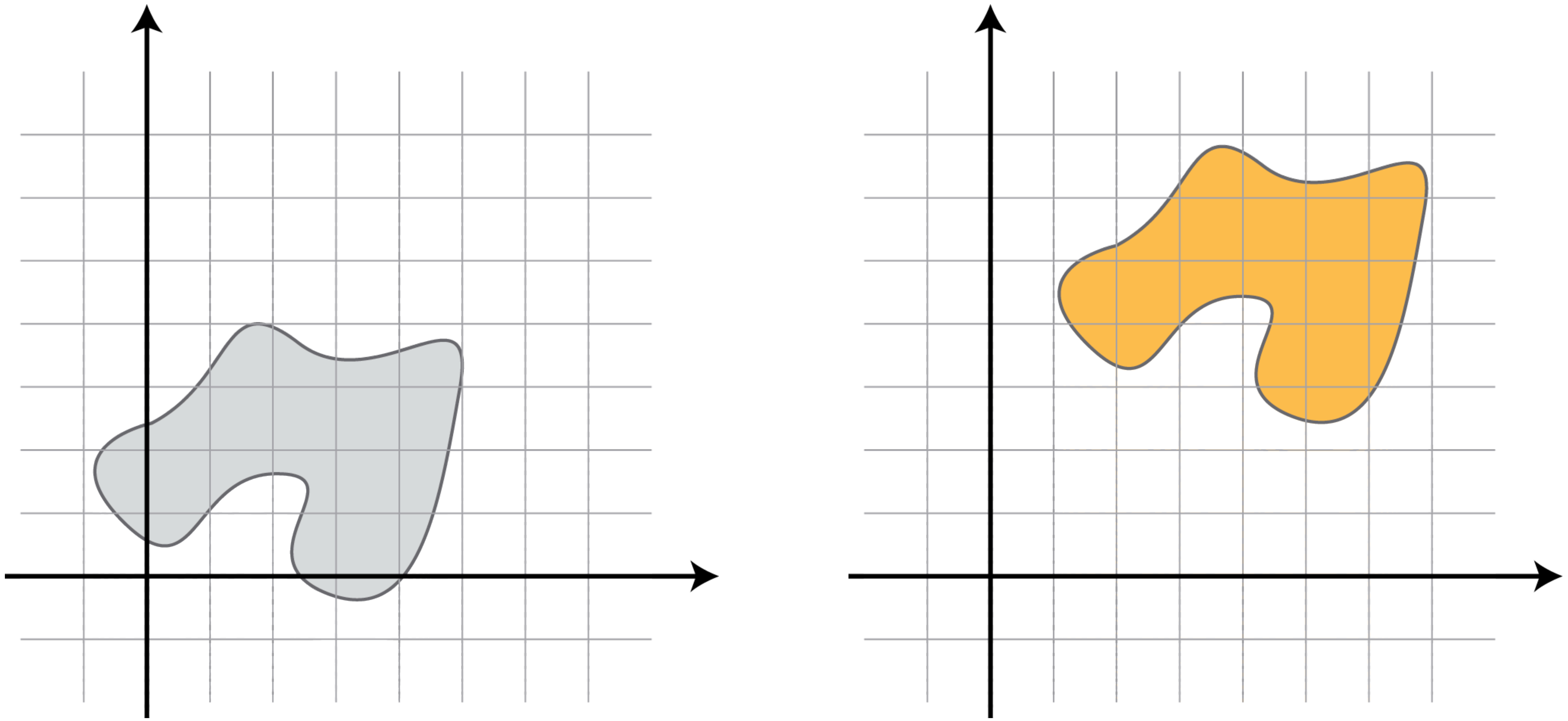
# **2D/3D Elasticity** - Deformation examples

## Simple translation

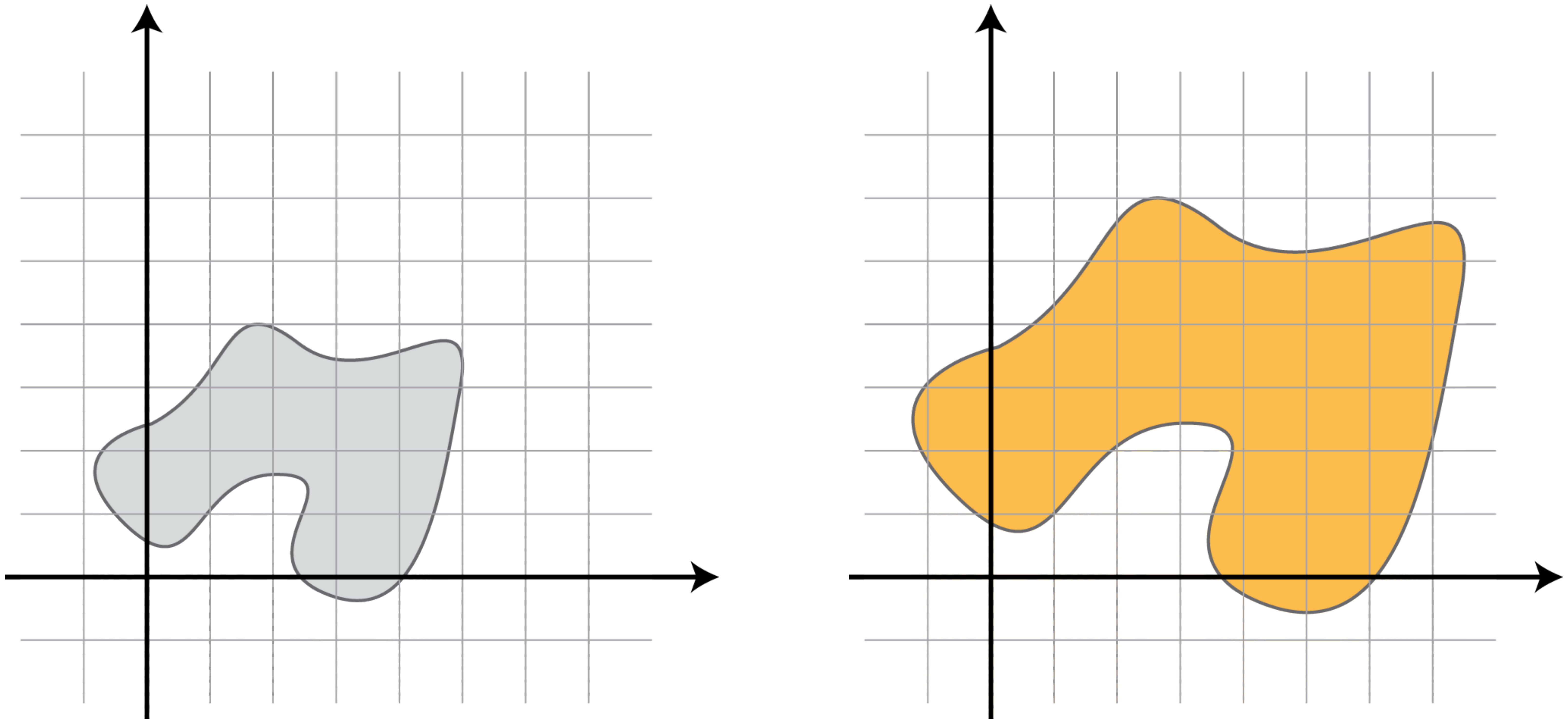


$$\vec{x} = \phi(\vec{X}) = \vec{X} + \vec{t}$$

$$\mathbf{F} = \mathbf{I}$$

# **2D/3D Elasticity** - Deformation examples

## Uniform Scaling



$$\vec{x} = \phi(\vec{X}) = \gamma\vec{X}$$

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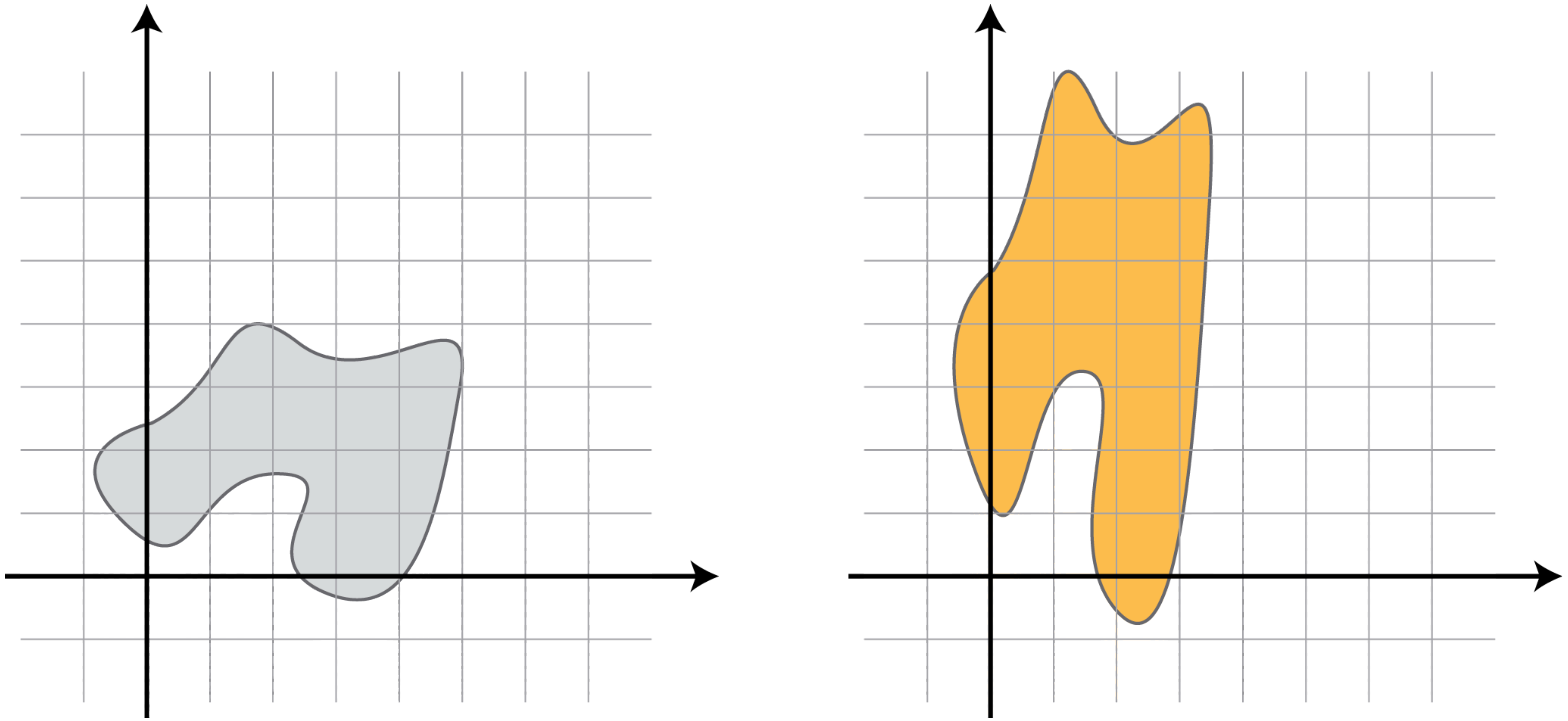
# **2D/3D Elasticity -** Deformation examples

## Anisotropic scaling

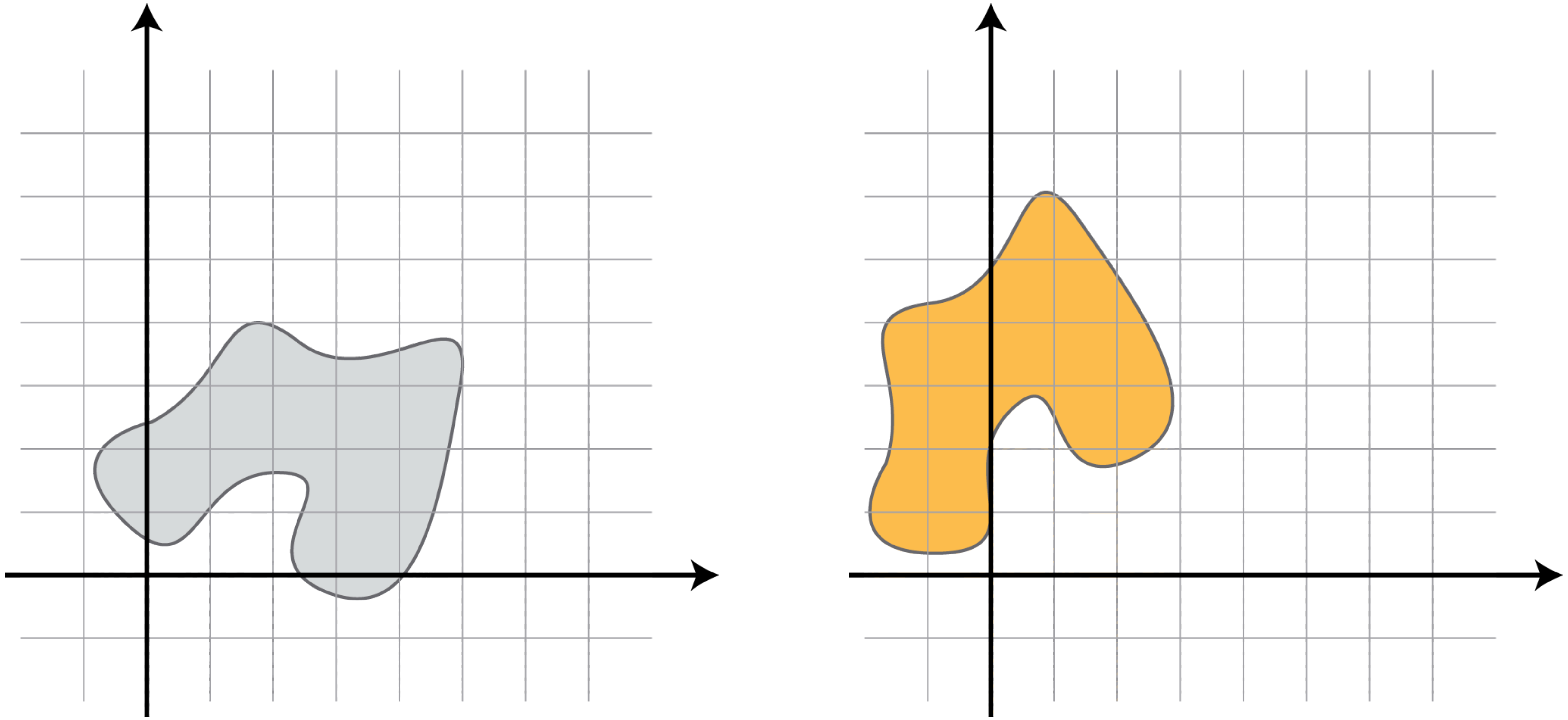


$$\vec{x} = \phi\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0.7X \\ 2Y \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 0.7 & 0 \\ 0 & 2 \end{pmatrix}$$

# 2D/3D Elasticity - Deformation examples

Rotation only



$$\vec{x} = \phi\begin{pmatrix}X\\Y\end{pmatrix} = \mathbf{R}_{45^0}\begin{pmatrix}X\\Y\end{pmatrix}$$

$$\mathbf{F} = \mathbf{R}_{45^0}$$

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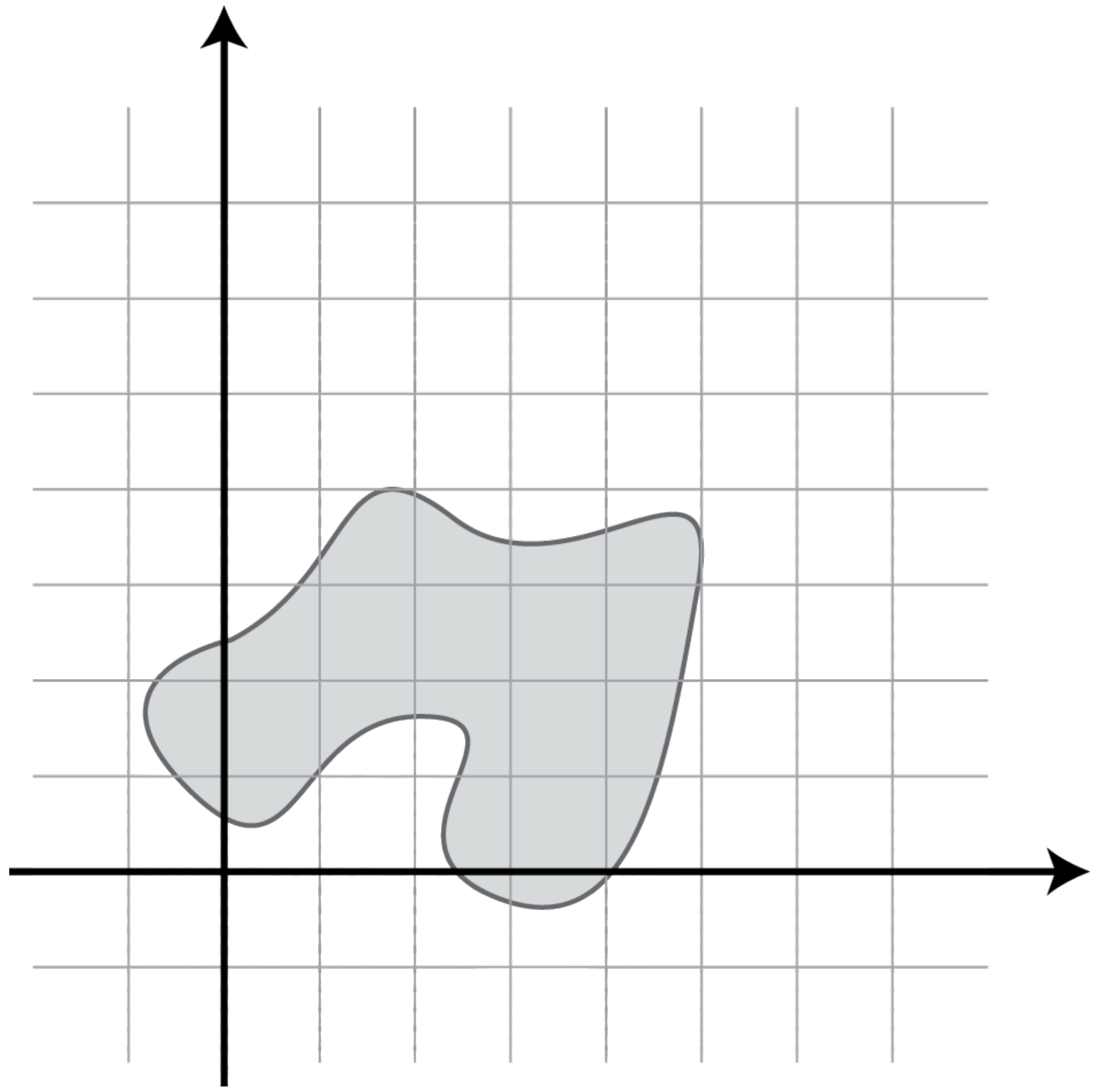

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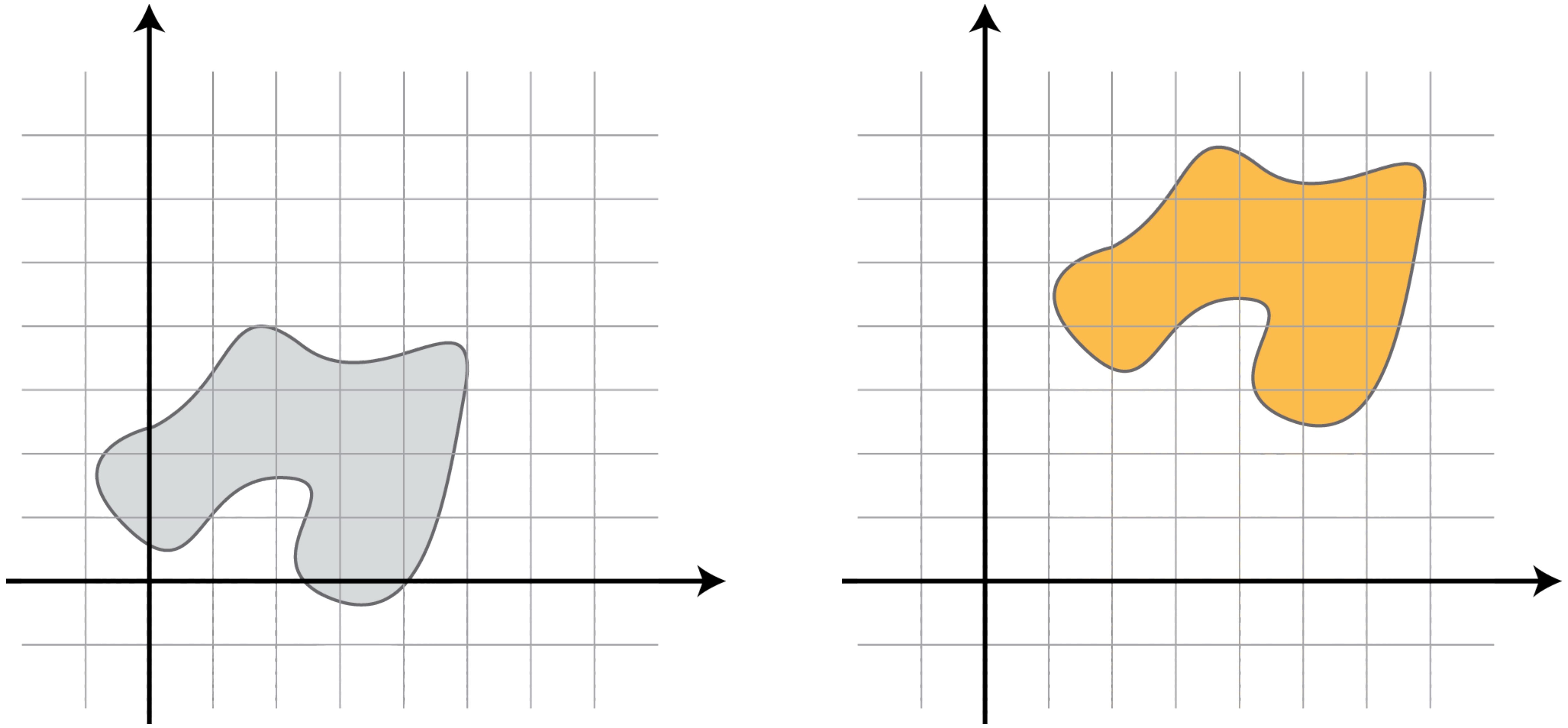

Translation

$$\mathbf{F} = \mathbf{I}$$

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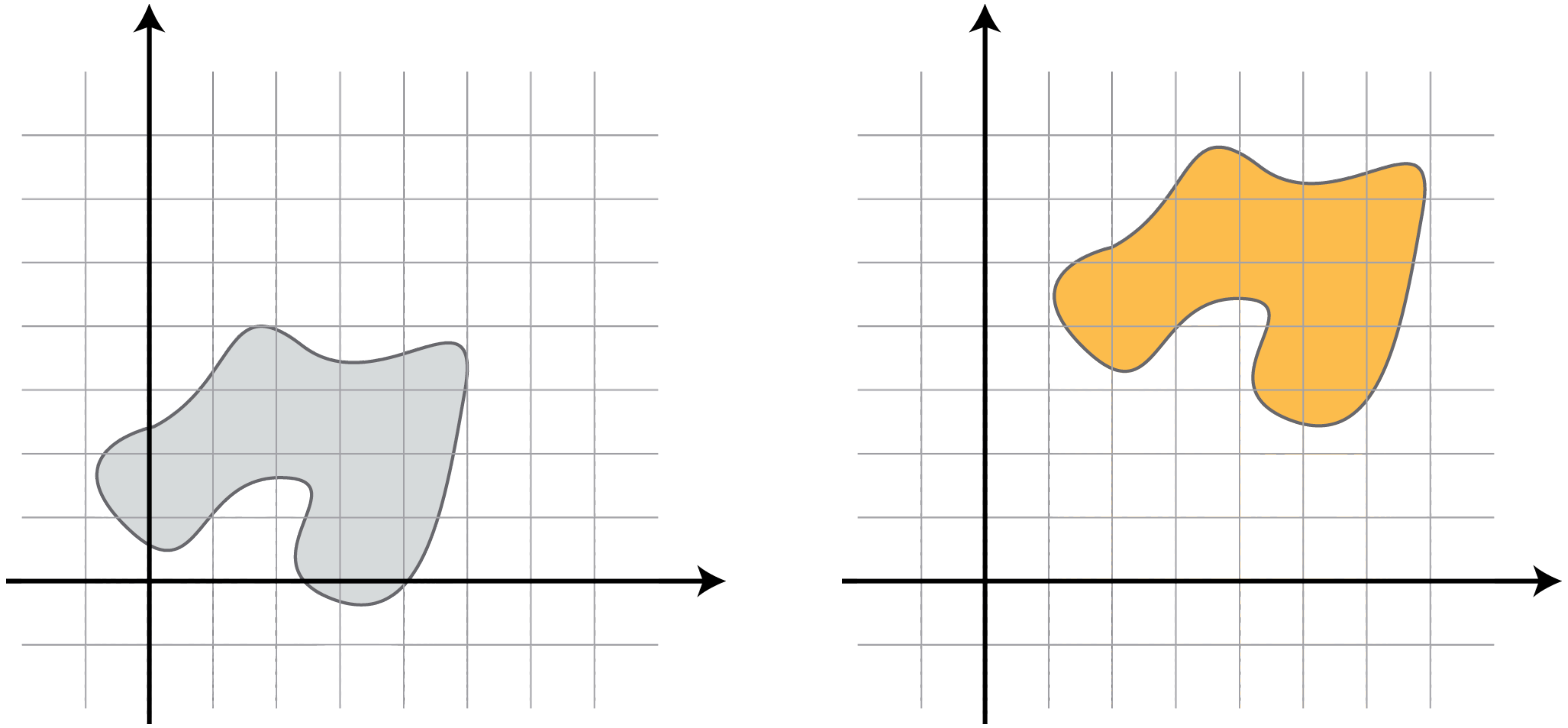

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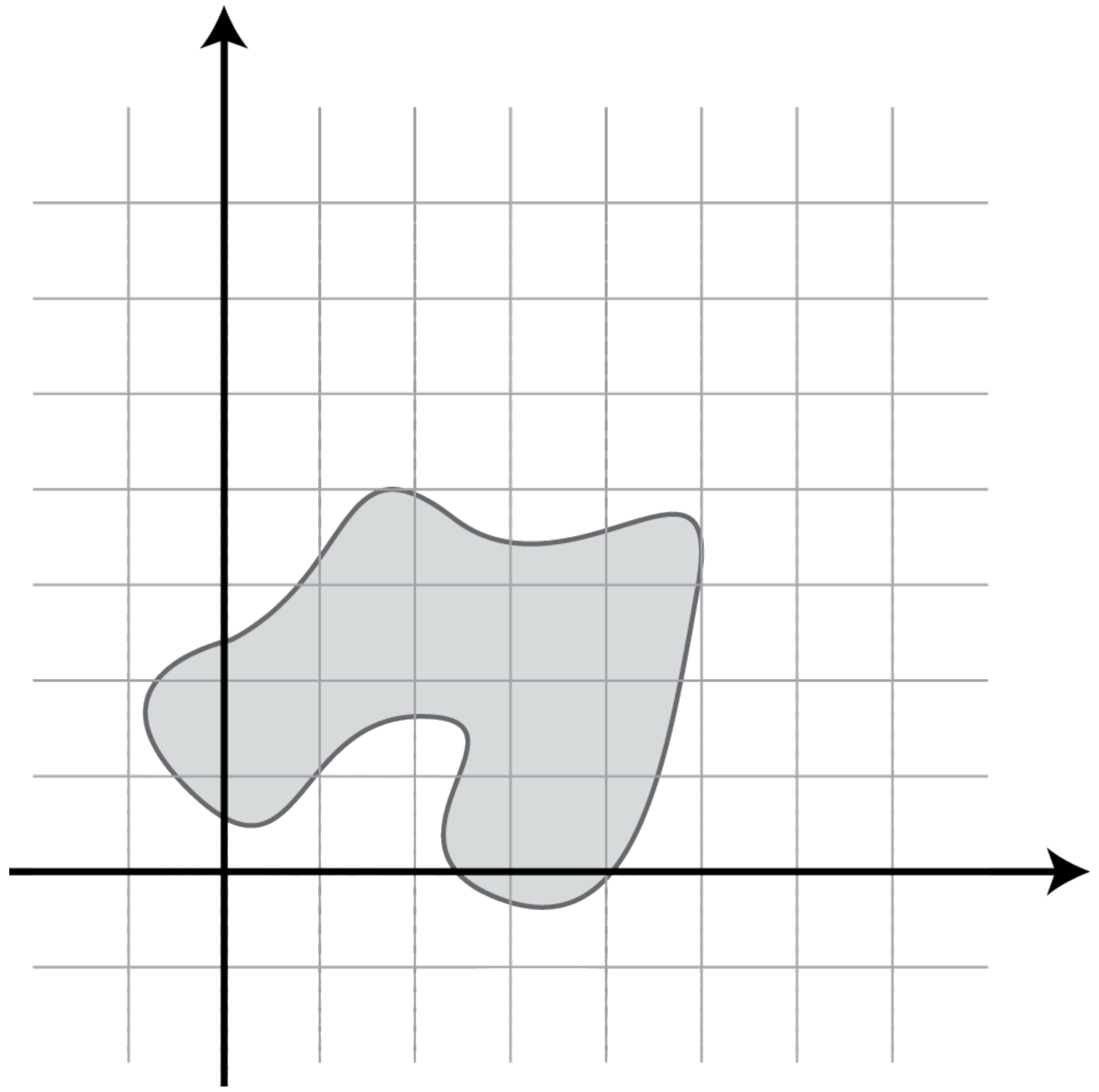

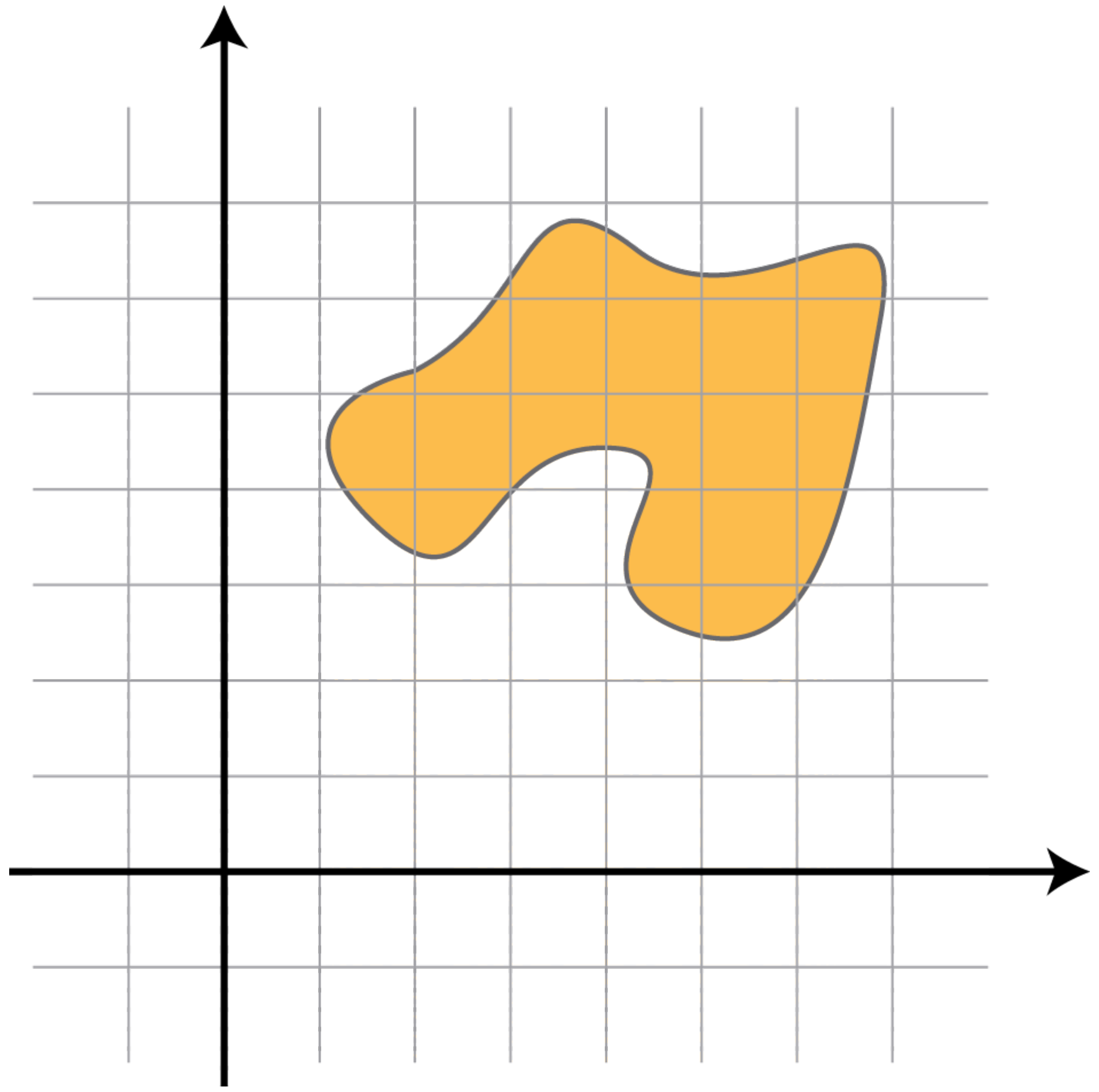

Translation

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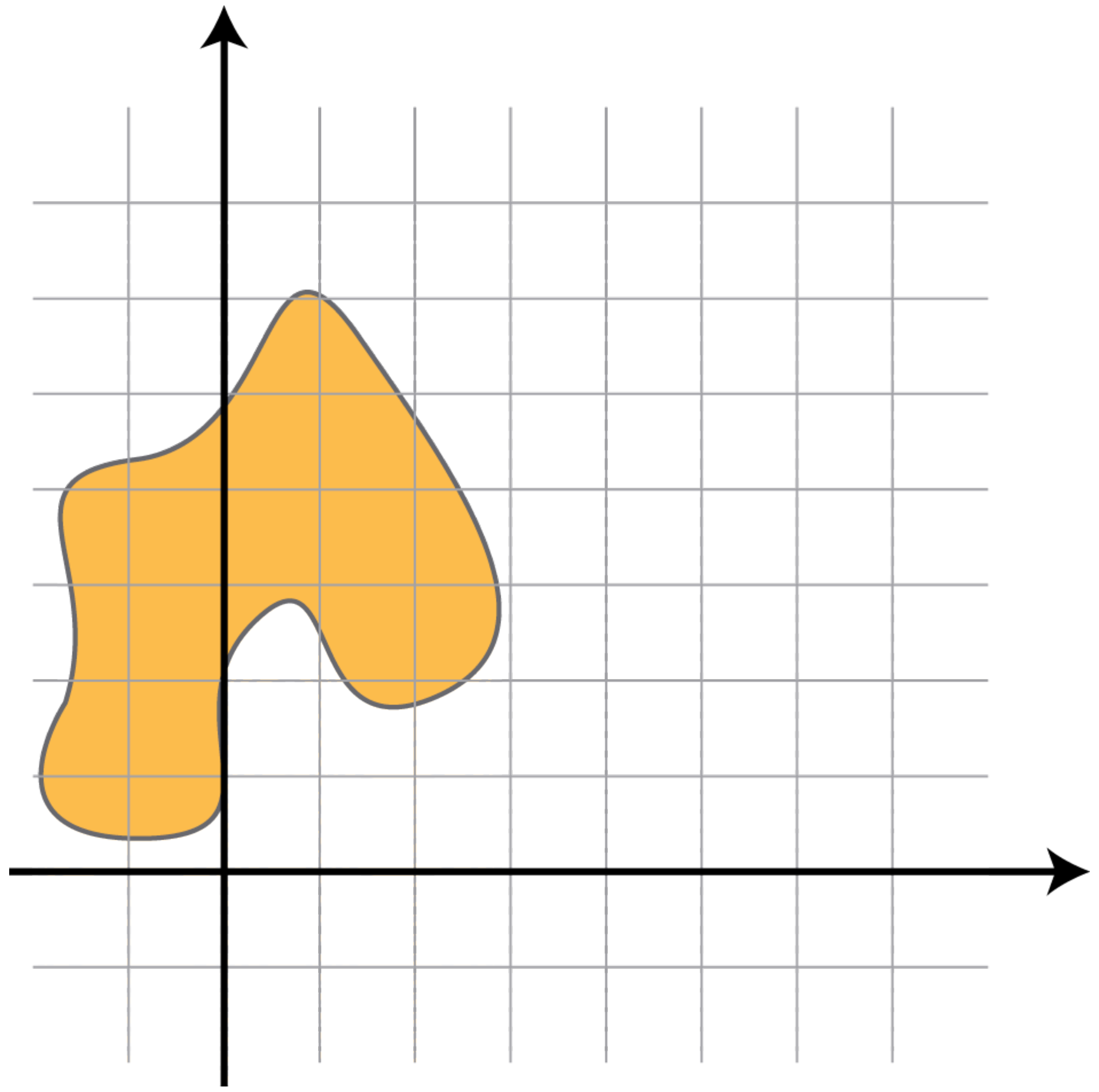

Rotation

$\mathbf{F}=\mathbf{R}$

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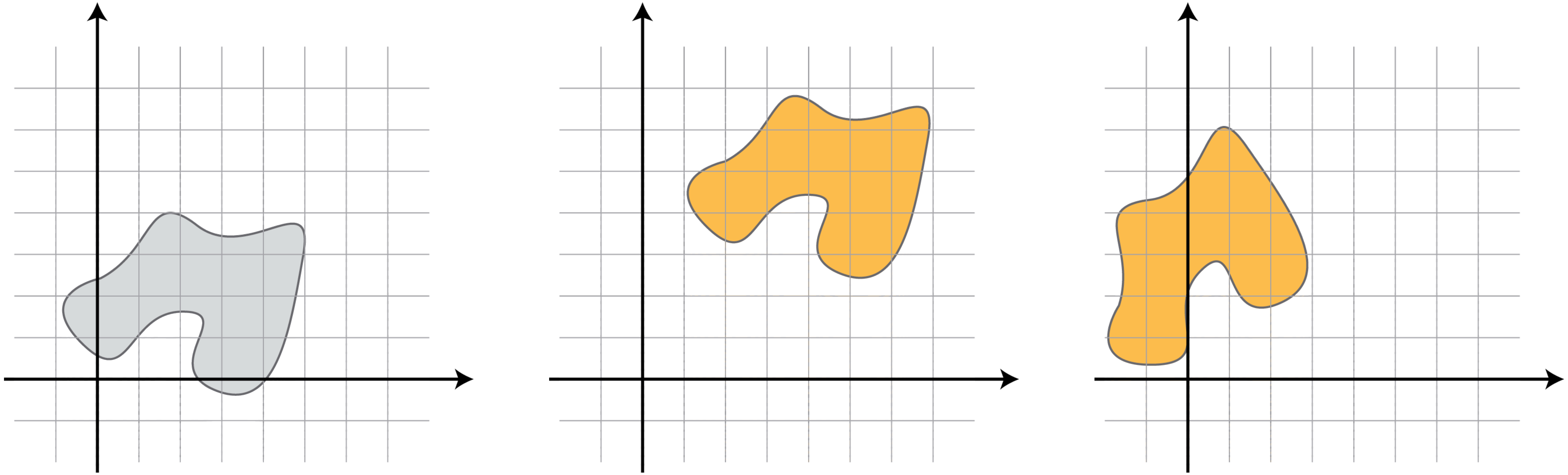

Translation

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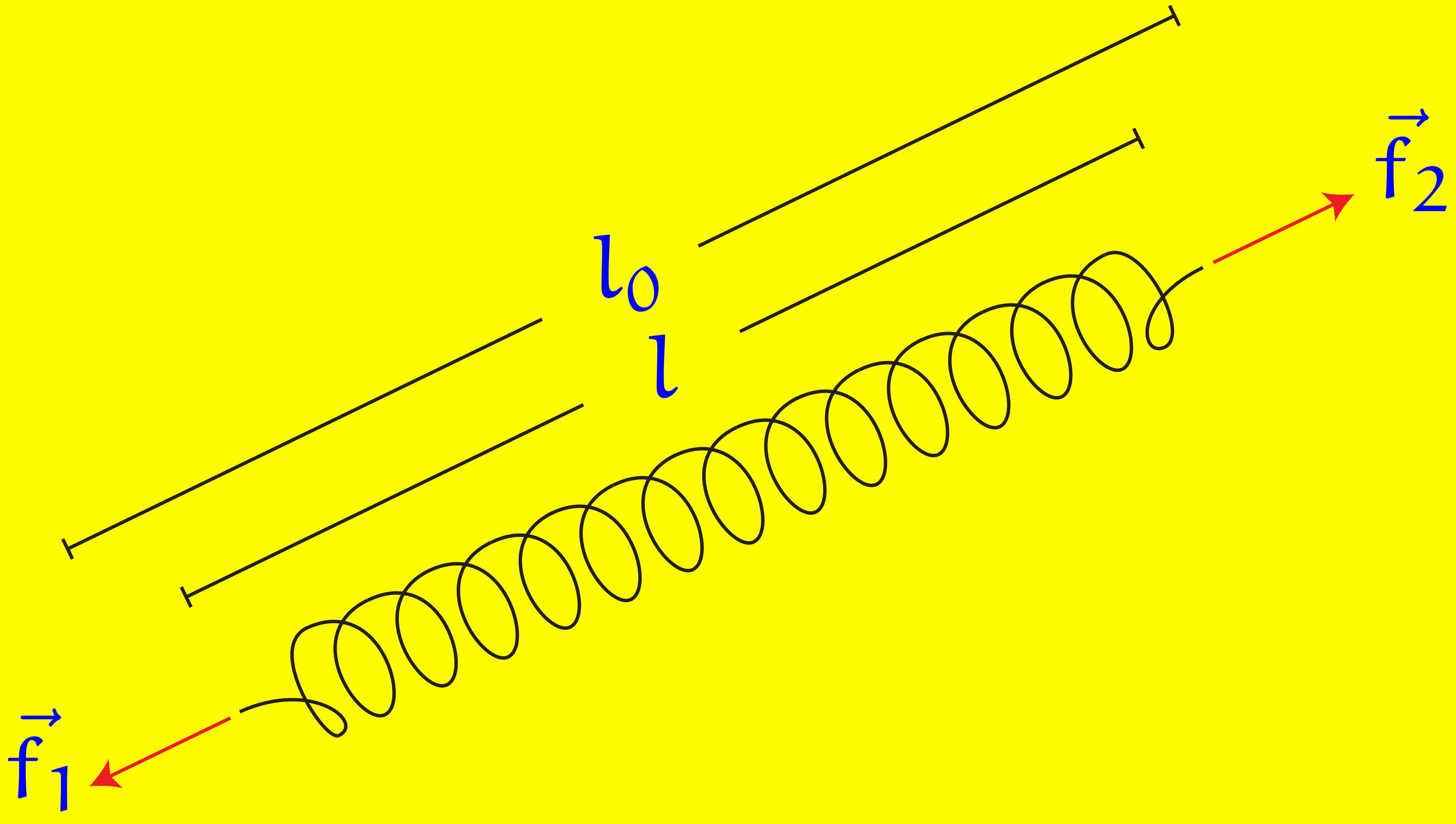

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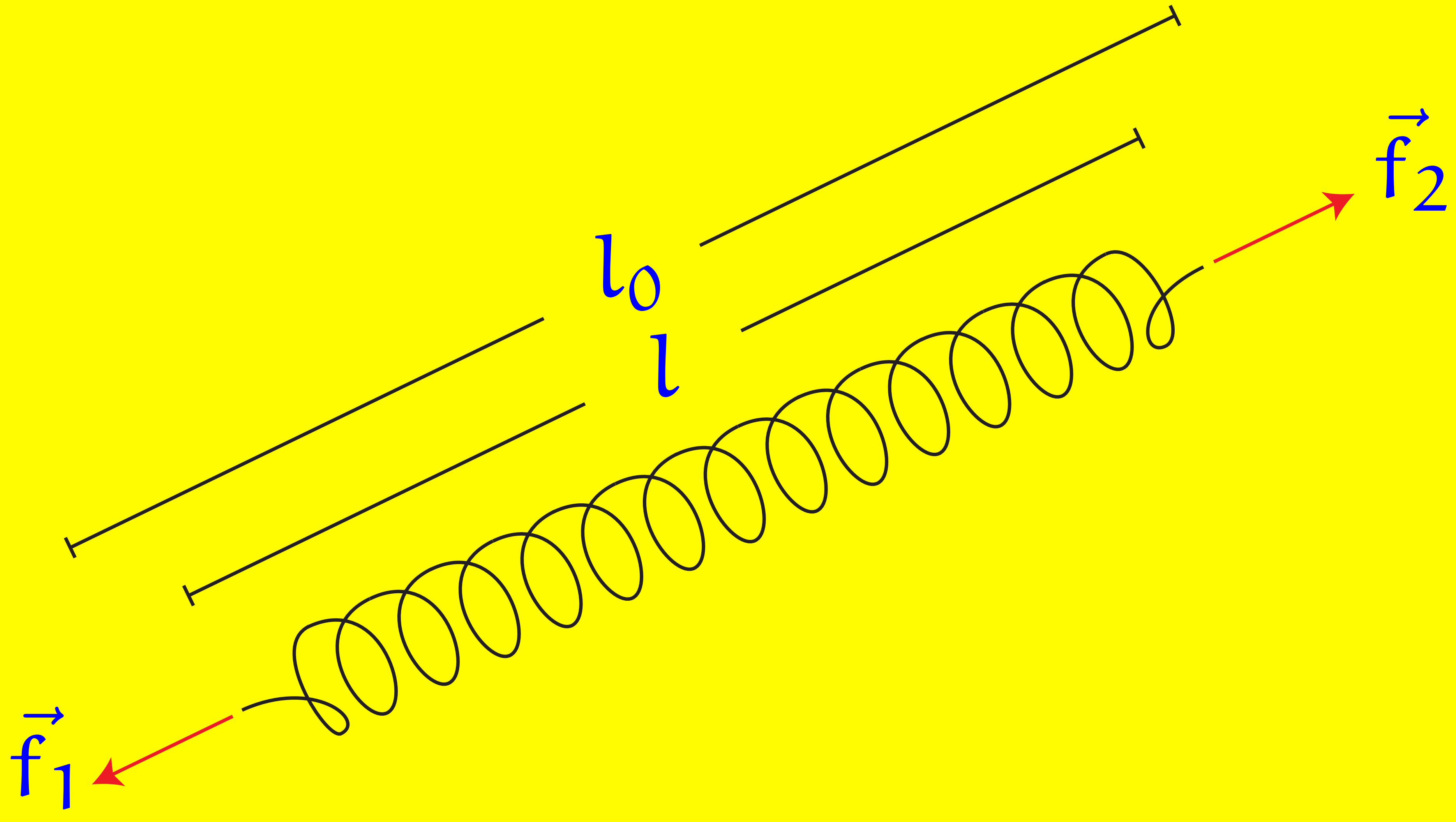

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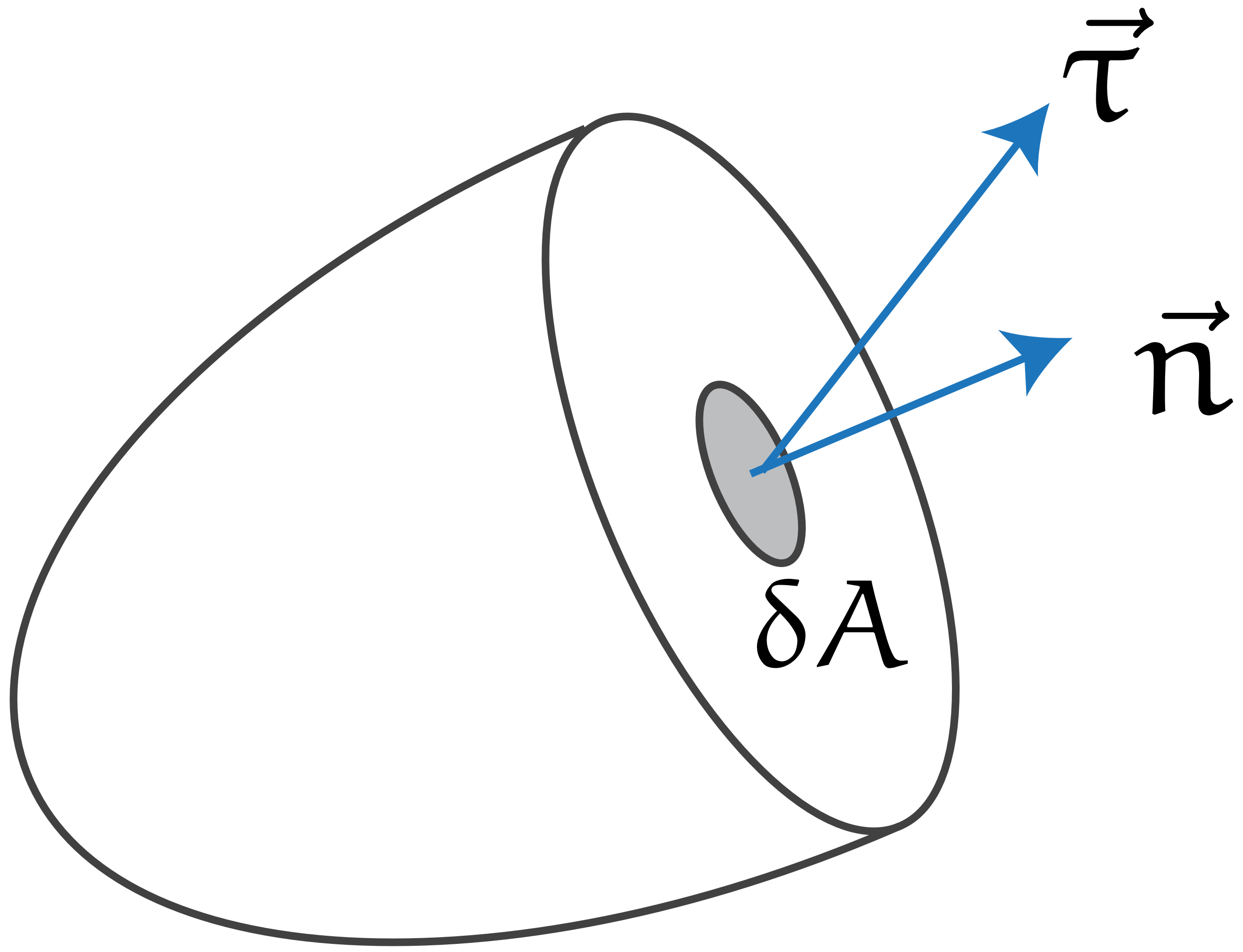

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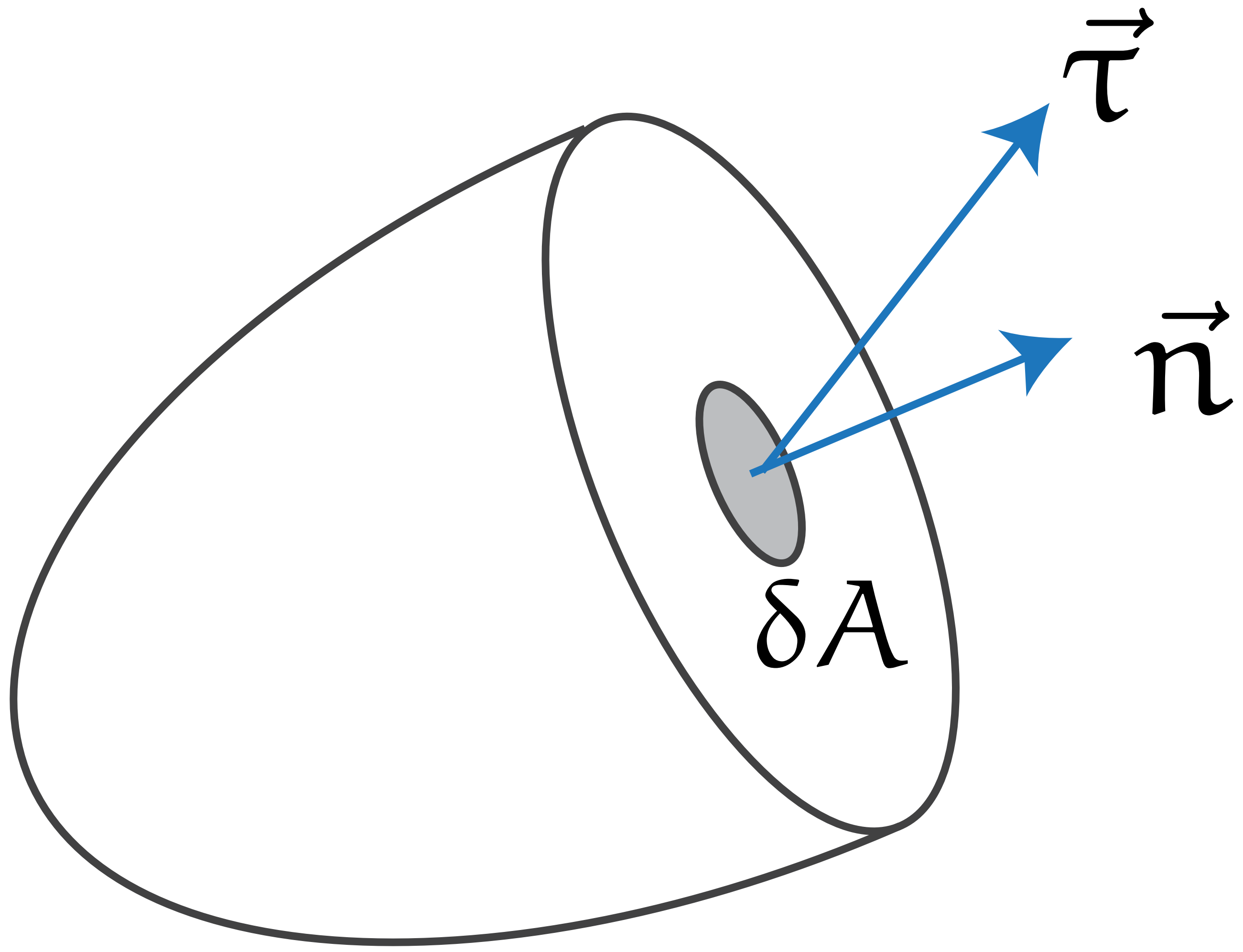

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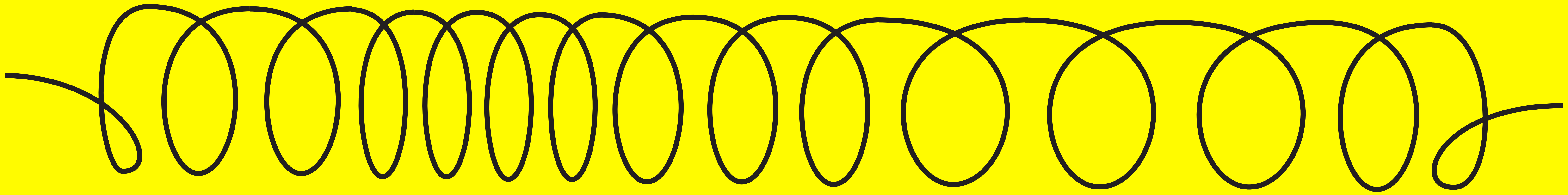

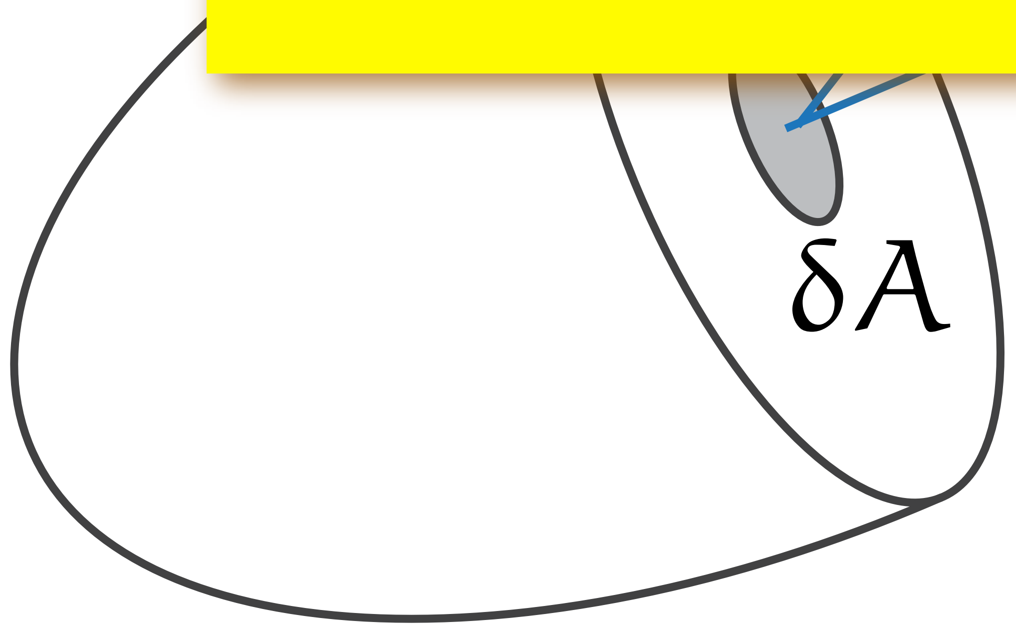

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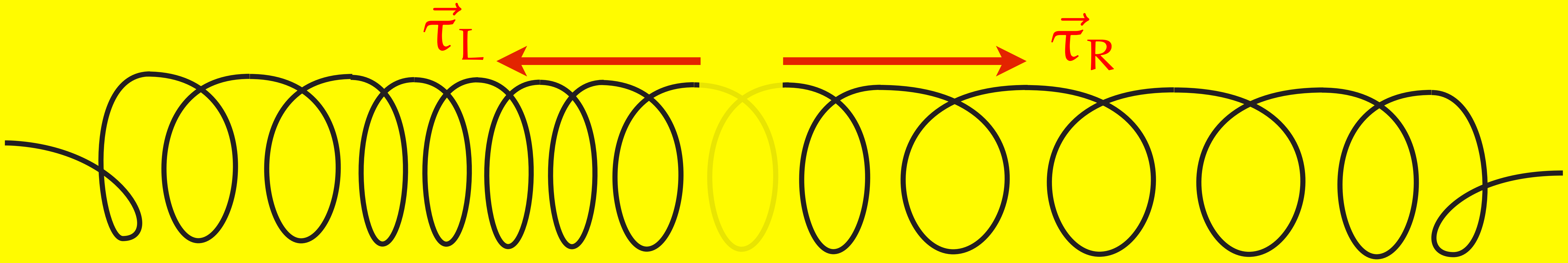

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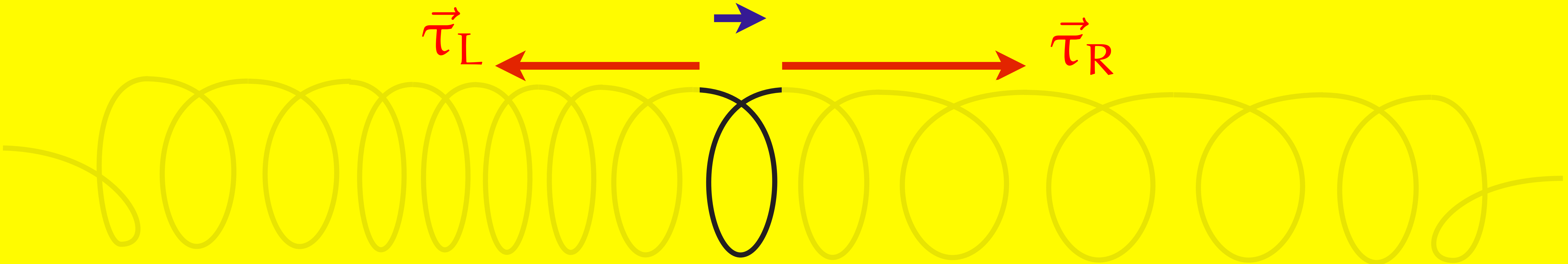
$$\vec{f} = \vec{\tau}_L + \vec{\tau}_R$$



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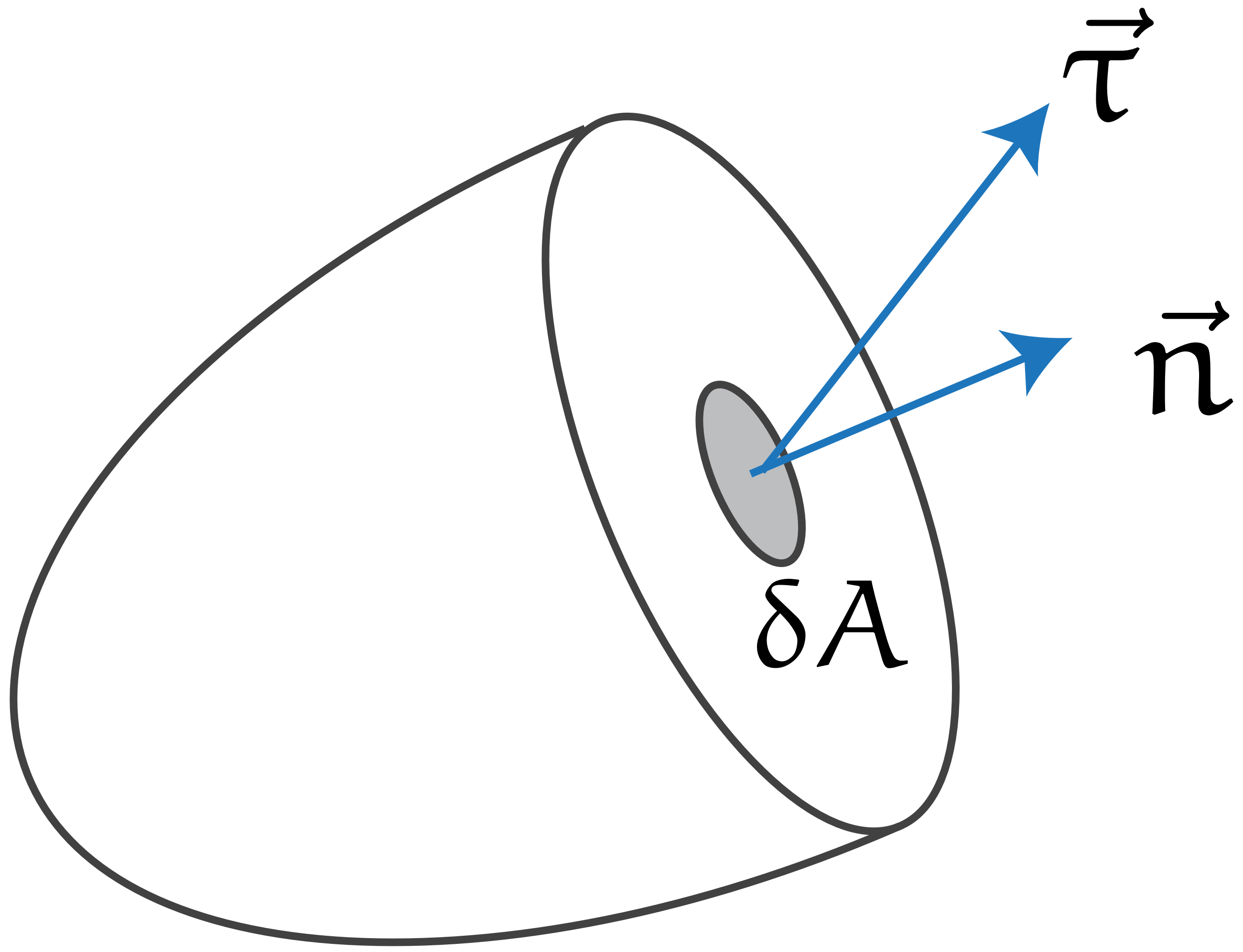

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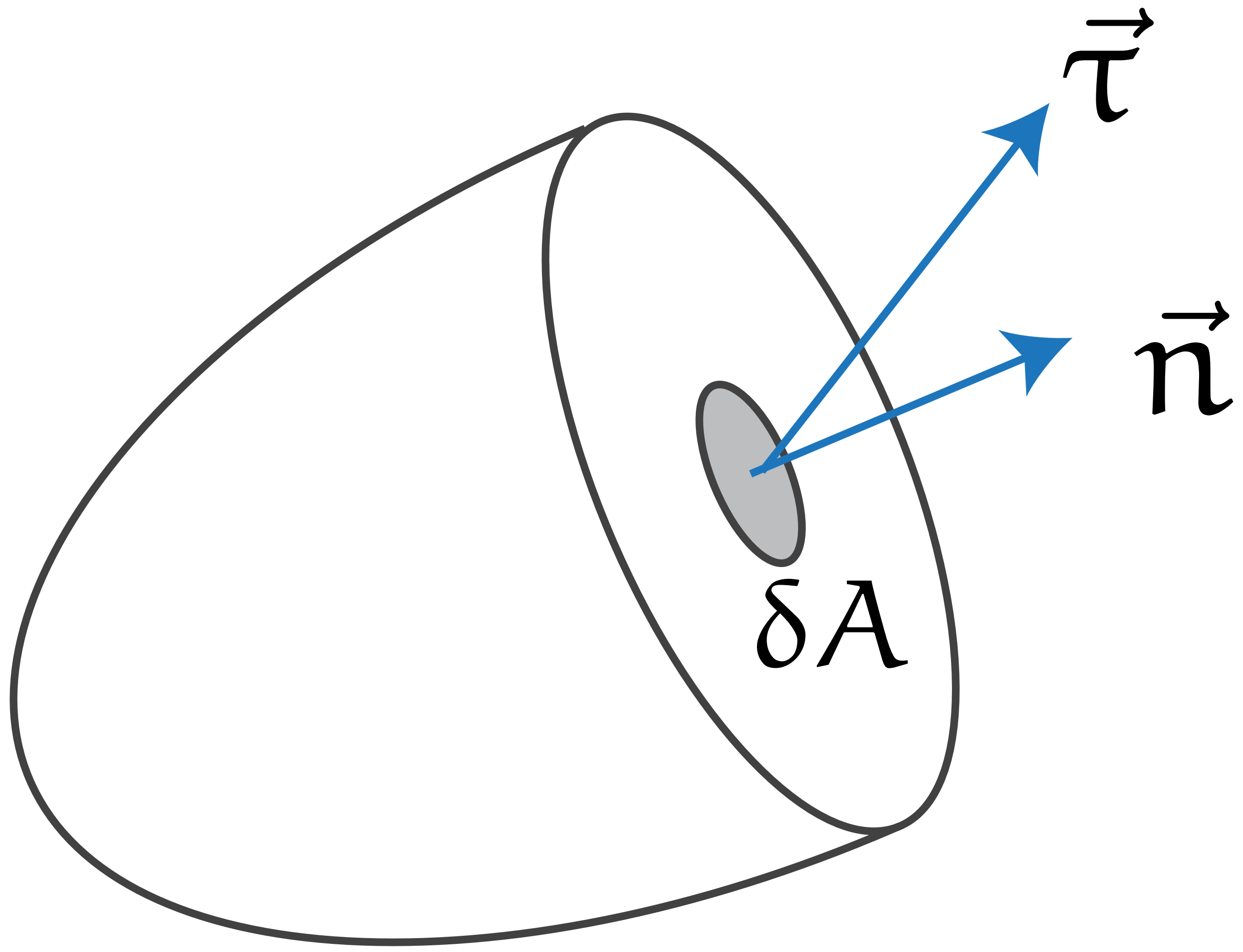

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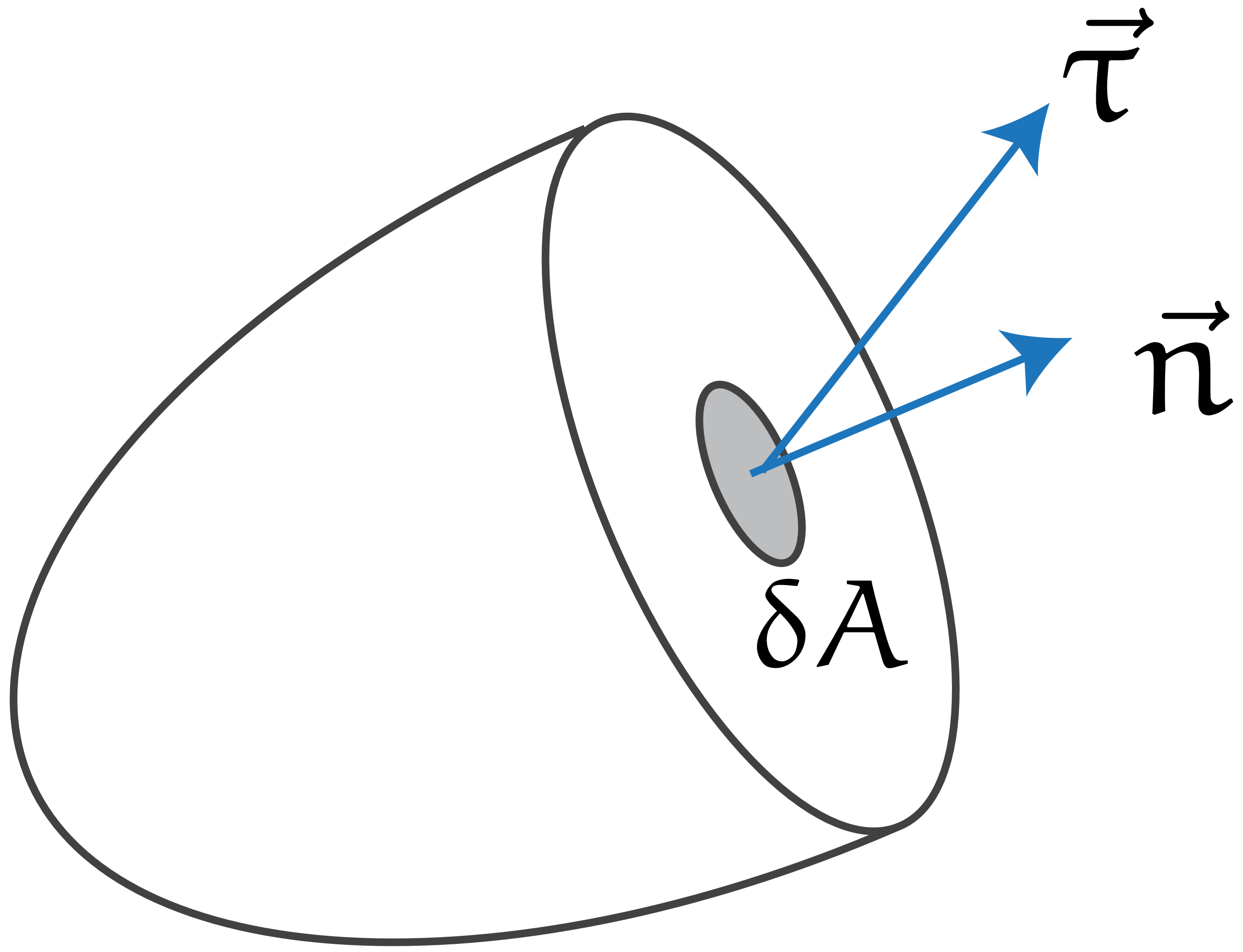

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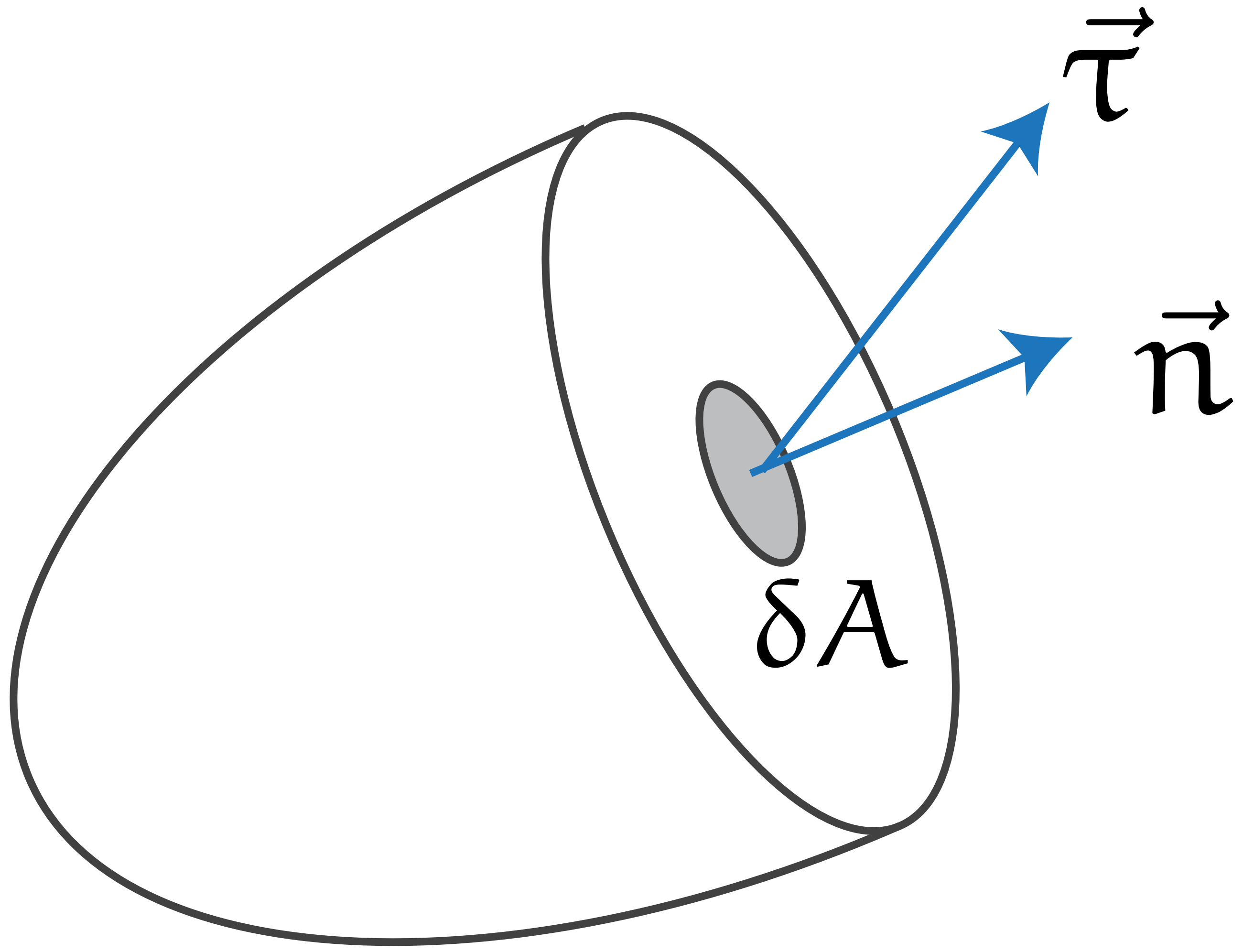

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**(Piola) Stress tensor** $( P )$ **:**

A matrix that describes force response along different orientations

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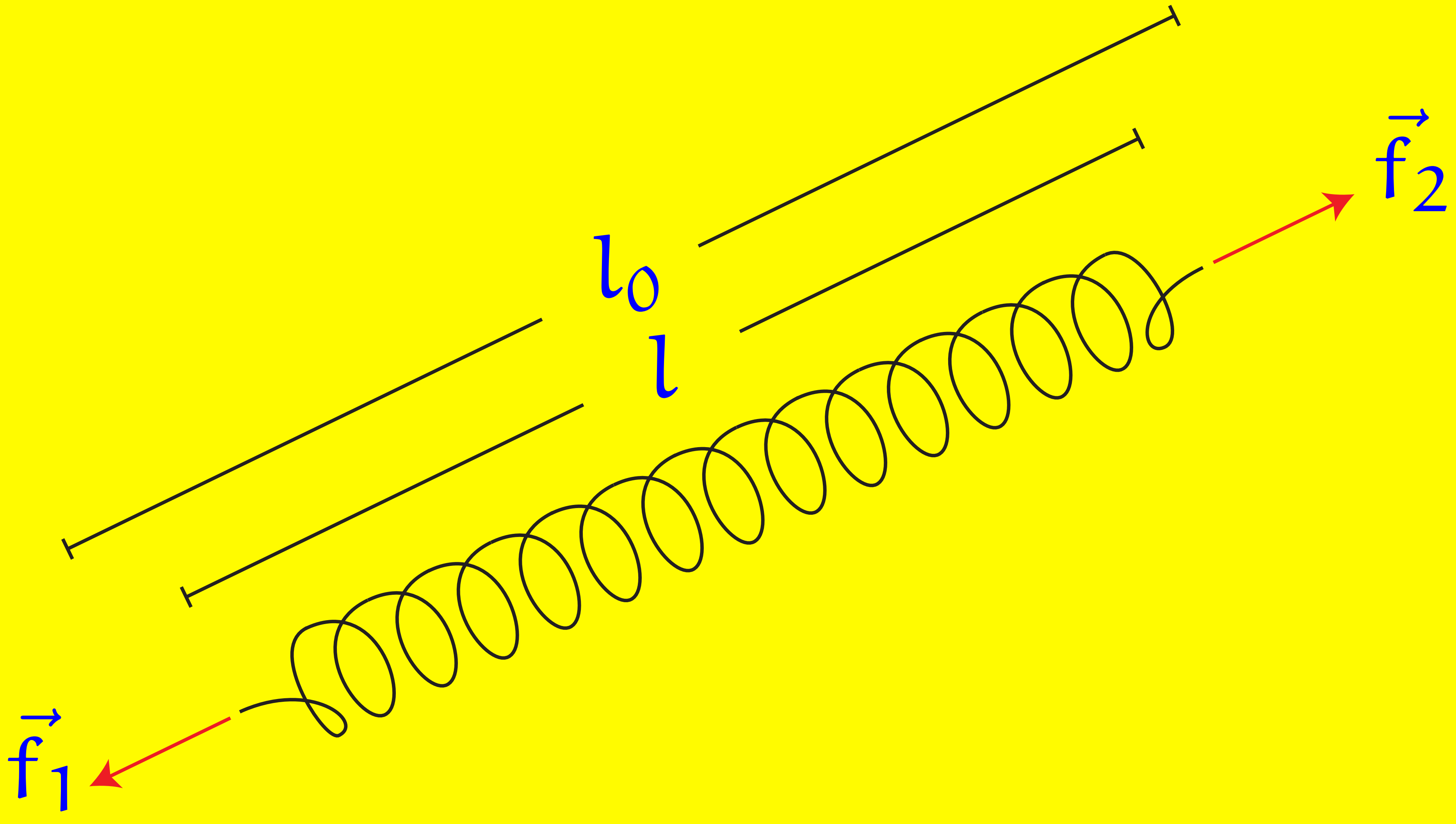

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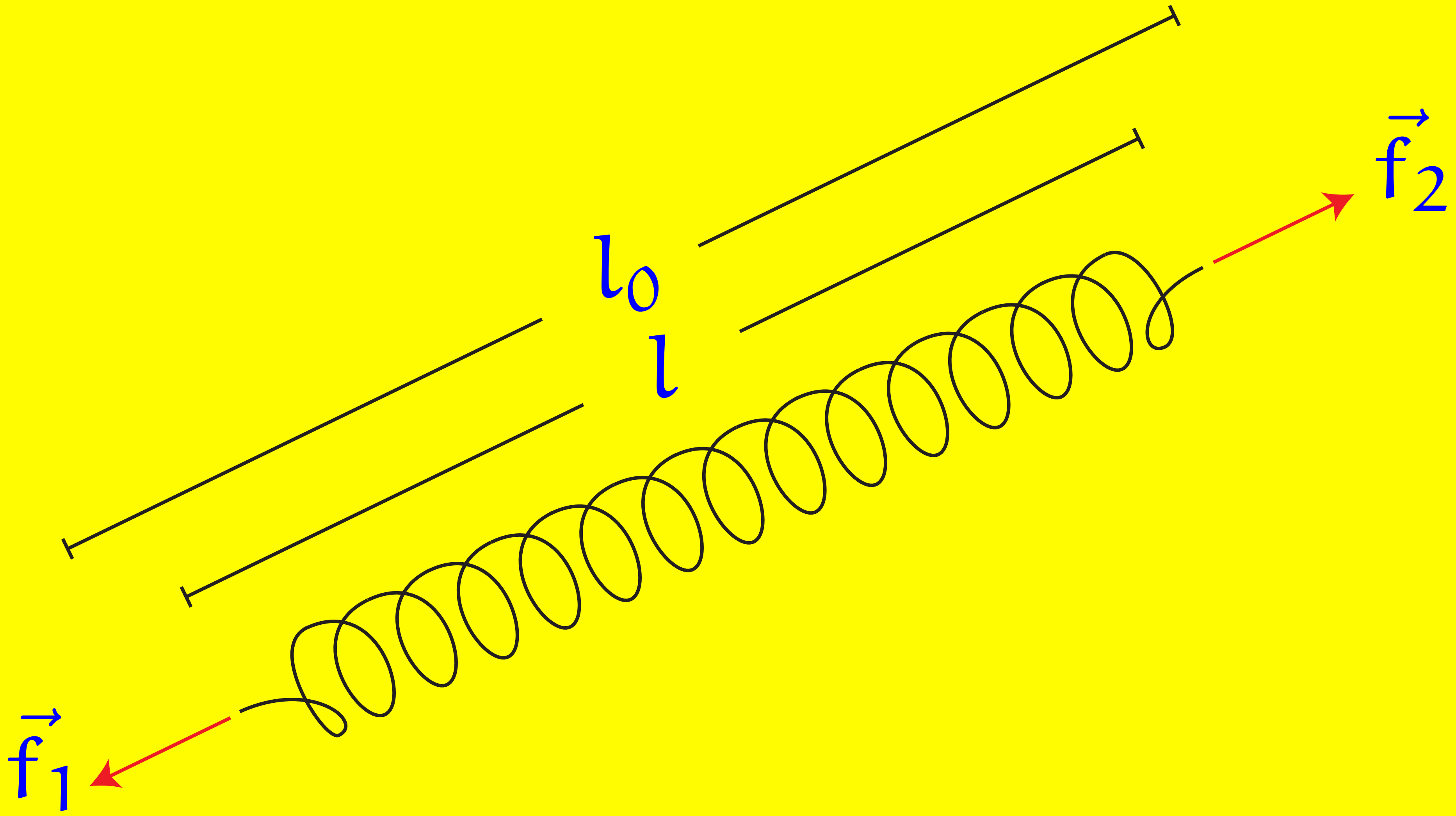

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Stress-energy relation

$$\mathbf{P} := \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}$$

# 2D/3D Elasticity - Strain energy

Linear elasticity

$$\epsilon = \frac{1}{2}(\mathbf{F} + \mathbf{F}^\top) - \mathbf{I}$$

$$\Psi = \mu \|\epsilon\|_F^2 + \frac{\lambda}{2}\mathrm{tr}^2(\epsilon)$$

$$\mathbf{P} = 2\mu\epsilon + \lambda \mathrm{tr}(\epsilon)\mathbf{I}$$

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✓ Linear force-position relation

✓ Computationally inexpensive

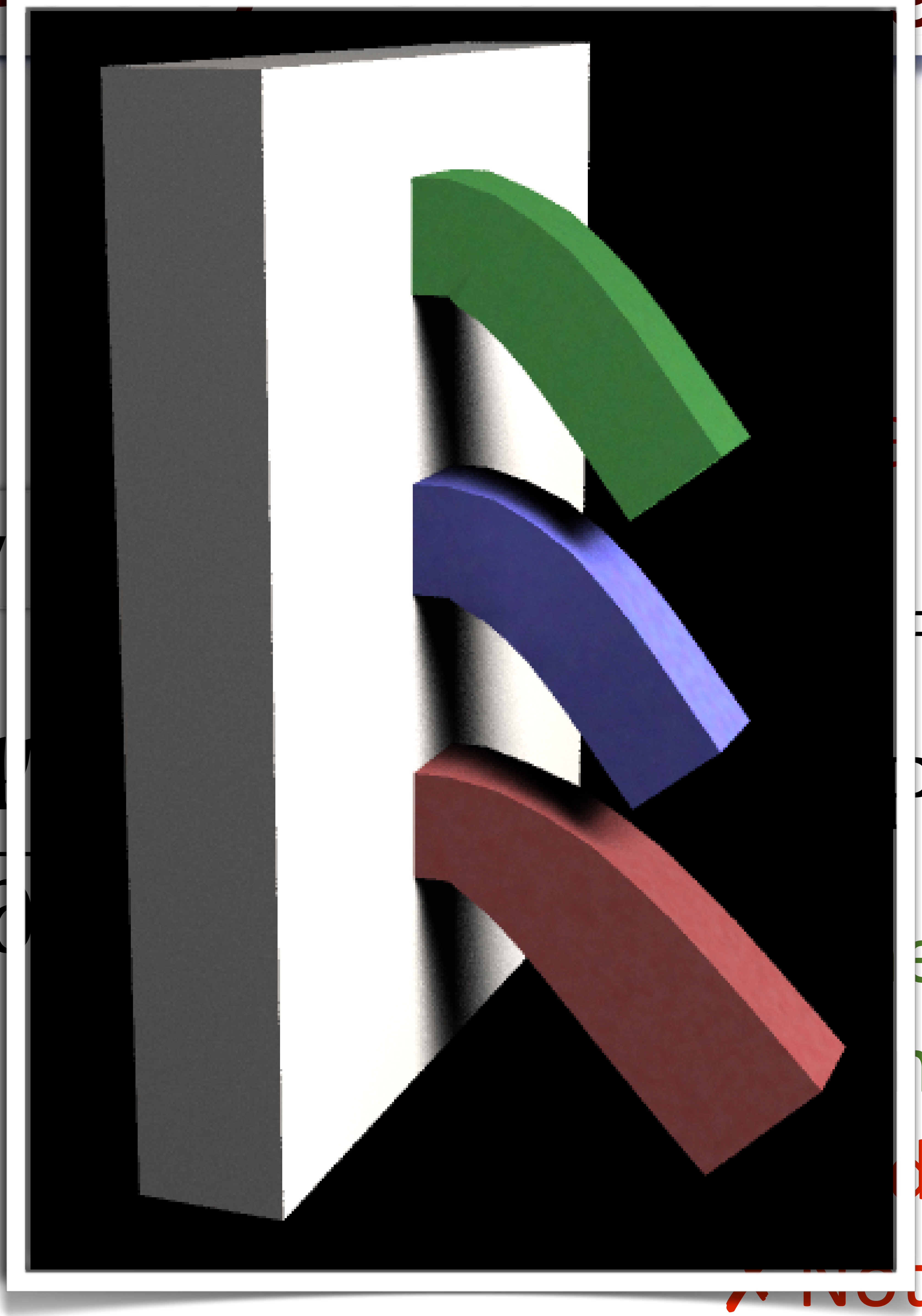
✗ Bad for large deformations

✗ Not rotationally invariant

# 2D/3D Elasticity - Strain energy

Stress-energy

$$\mathbf{P} := \frac{\partial \Psi}{\partial}$$



*Linear elasticity*

$$= \frac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathsf{T}}) - \mathbf{I}$$

$$= \mu\|\epsilon\|_F^2 + \frac{\lambda}{2}\mathrm{tr}^2(\epsilon)$$

$$\mathbf{P} = 2\mu\epsilon + \lambda\mathrm{tr}(\epsilon)\mathbf{I}$$

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✗ No rotationally invariant

*[Source: Müller et al, "Stable real-time deformations", 2002]*

# 2D/3D Elasticity - Strain energy

Stress-energy relation

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# **2D/3D Elasticity** - Material models

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- ✓ Linear force-position relation
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## *Corotated linear elasticity*

$$\mathbf{E} = \mathbf{S} - \mathbf{I} \quad [\mathbf{F} = \mathbf{R}\mathbf{S}]$$

$$\Psi = \mu \|\mathbf{E_r}\|_{\mathrm{F}}^{2} + \frac{\lambda}{2} \mathrm{tr}^{2}(\mathbf{E_r})$$

$$\mathbf{P} = \mathbf{R}\left[2\mu\mathbf{E_r} + \lambda \mathrm{tr}(\mathbf{E_r})\mathbf{I}\right]$$

- ✓ Rotationally invariant
- ✓ Survives collapse & inversion
- ✗ Polar decomposition overhead
- ✗ Inaccurate volume preservation

# 2D/3D Elasticity - Material models

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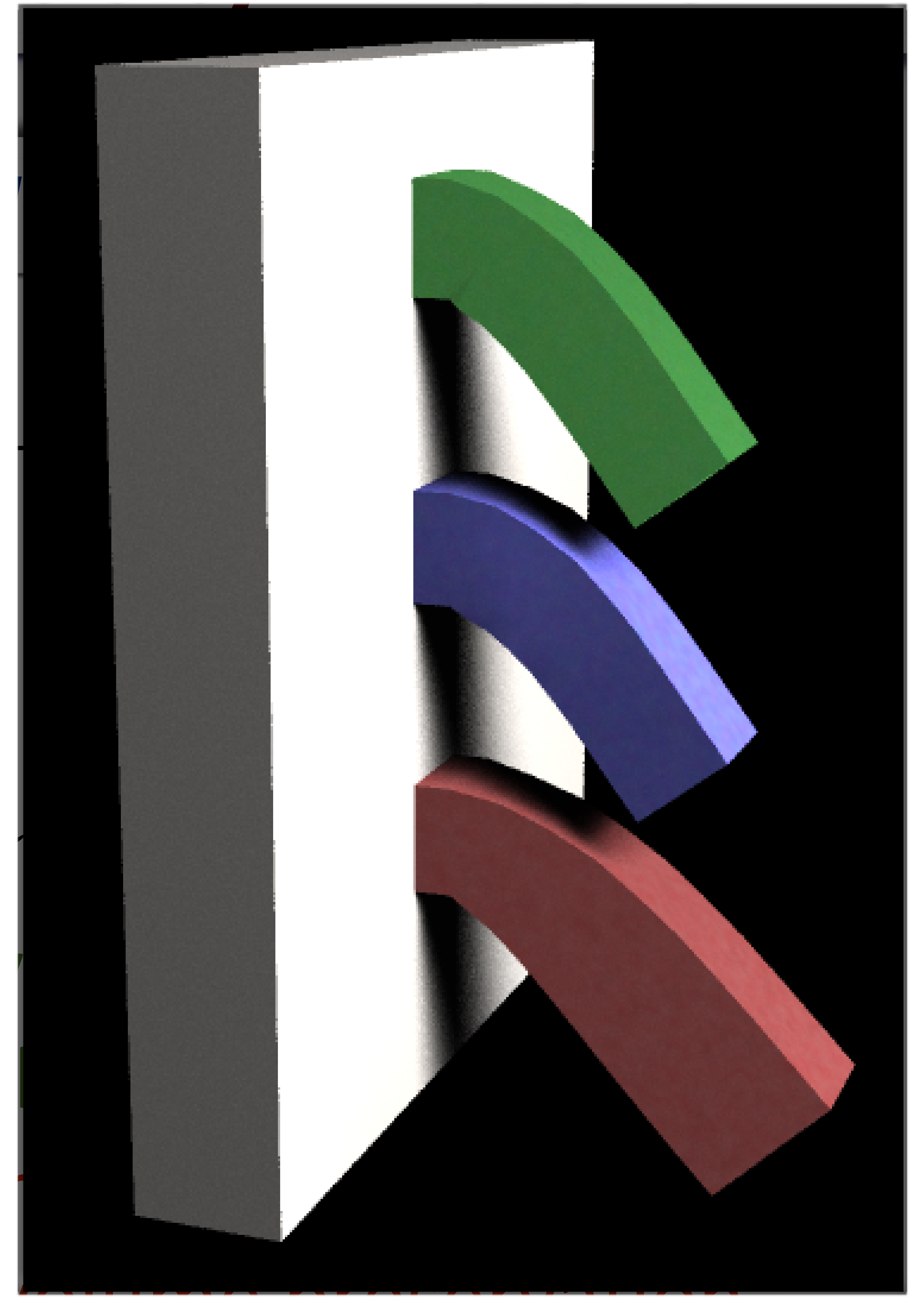
# 2D/3D Elasticity - Material models



## Corotated

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## *Neohookean elasticity*

$$I_1 = \|\mathbf{F}\|_F^2, \quad J = \det \mathbf{F}$$

$$\Psi = \frac{\mu}{2}(I_1 - 3) - \mu\log(J) + \frac{\lambda}{2}\log^2(J)$$

$$\mathbf{P} = \mu(\mathbf{F} - \mathbf{F}^{-\mathsf{T}}) + \lambda\log(J)\mathbf{F}^{-\mathsf{T}}$$

- ✓ Accurate volume preservation
- ✓ Discourages collapse/inversion
- ✗ Undefined when inverted
- ✗ Numerically stiff w/compression

# 2D/3D Elasticity - Material models

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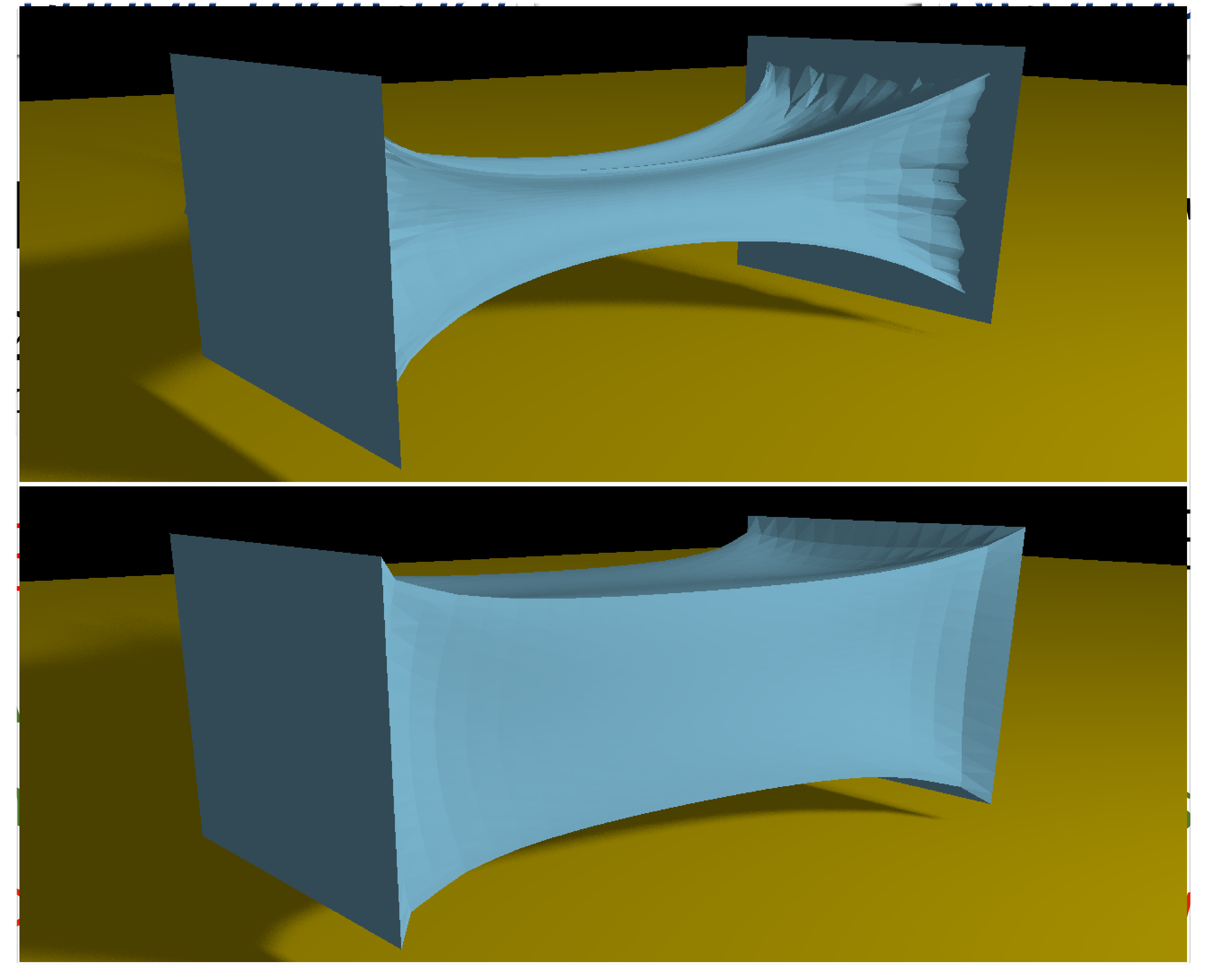

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# Additional information on lecture notes

- ✓ Extended discussion of rotational invariance, isotropy and the common isotropic invariants
- ✓ PDE form of elasticity equations
- ✓ Stress formulas for general isotropic materials
- ✓ Benefits and drawbacks of individual material models