## Example

### Corotated Elasticity

$$\Psi(F) = \mu \|F-R\|_F^2 + \frac{\lambda}{2} \mathrm{tr}^2(S-I) \qquad (F=RS)$$

(see notes)

$$= \mu \|\Sigma - I\|_F^2 + \frac{\lambda}{2} \mathrm{tr}^2(\Sigma - I) = \Psi(\Sigma)$$

$$\Psi(\sigma_1,\sigma_2,\sigma_3) = \mu \sum_{i=1}^{3} (\sigma_i - 1)^2 + \frac{\lambda}{2}\left(\sum_{i=1}^{3} \sigma_i - 1\right)^2$$

$$\frac{\partial \Psi}{\partial \sigma_i} = 2\mu(\sigma_i - 1) + \lambda \, \mathrm{tr}(\Sigma - I)$$

$$\frac{\partial^2 \Psi}{\partial \sigma_i^2} = 2\mu + \lambda, \quad \frac{\partial^2 \Psi}{\partial \sigma_i \partial \sigma_j} = \lambda$$

$(i \neq j)$

$$\Rightarrow \quad A = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda \\ \lambda & 2\mu+\lambda & \lambda \\ \lambda & \lambda & 2\mu+\lambda \end{pmatrix}$$

(Note: Always positive definite)

The matrix $B_{ij}$ is defined as

$$B_{ij} = \underbrace{\frac{1}{2} \frac{\partial\psi/\partial\sigma_i - \partial\psi/\partial\sigma_j}{\sigma_i - \sigma_j}}_{\alpha_{ij}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$+ \frac{1}{2} \underbrace{\frac{\partial\psi/\partial\sigma_i + \partial\psi/\partial\sigma_j}{\sigma_i + \sigma_j}}_{\beta_{ij}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

We remember that, for ensuring definiteness of the stiffness matrix, we will need to "project" both $A$ and each of $B_{ij}$ to their positive-definite part. For $A$, specifically for corotated, as we saw we don't need to do anything (it is always positive definite). For $B_{ij}$, we observe:

$$B_{ij} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\text{"}Q\text{"}} \begin{pmatrix} 2\alpha_{ij} & \\ & 2\beta_{ij} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}}_{\text{"}Q^T\text{"}}$$

since this is an analytic eigenanalysis, the "projection" is simply ensuring (or clamping) the $\alpha_{ij}, \beta_{ij} \geq 0$.

In the case of corotated:

$$\alpha_{ij} = \frac{1}{2} \frac{2\mu(\sigma_i - \sigma_j)}{\sigma_i - \sigma_j} = \mu$$

$$\beta_{ij} = \frac{1}{2} \frac{2\mu(\sigma_i + \sigma_j) - 4\mu + 2\lambda \operatorname{tr}(\Sigma - I)}{\sigma_i + \sigma_j} =$$

$$= \underbrace{\mu + \frac{\lambda \operatorname{tr}(\Sigma - I) - 2\mu}{\sigma_i + \sigma_j}}_{\text{must be clamped to } \geq 0 \text{ for definiteness!}}$$