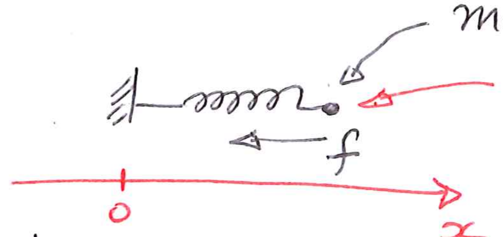
Dynamics of particle systems

Consider a dimension-less (in practice, very small) material point, with position $x(t)$ (varying over time) and velocity/acceleration $v(t) = x'(t)$, $a(t) = x''(t)$.

If we have a force $f = f(x, v)$ acting on this point, we seek to identify its trajectory over time

* Where did this force come from?
  $\Rightarrow$ Some physical law. e.g. for a Hookean spring:



one possible expression:

$$f = \underbrace{-k(x - x_0) - bv}_{f(x,v)}$$

k: stiffness coefficient

b: damping coefficient

Newton's law of motion can then be written as (m : mass of particle)

$$m \cdot a(t) = f(x(t), v(t))$$

$$\Rightarrow m \cdot x''(t) = f(x, v)$$ [Lets try to avoid 2nd derivative]

$$\Rightarrow \begin{pmatrix} x'(t) \\ v'(t) \end{pmatrix} \begin{matrix} = \\ = \end{matrix} \begin{pmatrix} v(t) \\ (1/m)\, f(x(t), v(t)) \end{pmatrix} \quad (*)$$

This is now a <u>system</u> of differential equations that only involve 1-st derivatives.

Write $Y(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$

then $(*) \Rightarrow Y'(t) = F(Y(t)) \quad (**)$

where $F(Y(t)) = \begin{pmatrix} v(t) \\ m^{-1} \cdot f(x(t), v(t)) \end{pmatrix}$

Let try to devise a solution for $(**)$.

- $\Rightarrow$ Generally, cannot do in closed form
- $\Rightarrow$ Instead, we discretize the time line : $t_0, t_1, \dots, t_k, \dots$ and seek approximations $Y_k \approx Y(t_k)$

One possible way :

$$(**) \Rightarrow \int_{t_k}^{t_{k+1}} Y'(t)\, dt = \int_{t_k}^{t_{k+1}} F(Y(t))\, dt$$

$$\Rightarrow Y(t_{k+1}) - Y(t_k) = \underbrace{\int_{t_k}^{t_{k+1}} F(Y(t))\, dt}_{(I)}$$

How we choose to approximate the integral (I) will determine the time-evolution scheme we construct. For example

$$\boxed{\int_a^b f(x)\, dx \approx (b-a) f(a)}$$

"Left" rectangle rule. Yields Forward Euler method

$$\int_a^b f(x)\, dx \approx (b-a) f(b)$$

$$\approx (b-a) f\left(\frac{a+b}{2}\right)$$

$$\approx \frac{b-a}{2} \{f(a) + f(b)\}$$

$\Rightarrow$ Vastly different (not today!)

Apply to (**) :

$$\underbrace{Y_{k+1} - Y_k}_{} = \underbrace{(t_{k+1} - t_k)}_{:= dt \text{ if uniform}} F(\underbrace{Y(t_k)}_{\text{replace with } Y_k})$$

this is what $Y(t_{k+1})$, $Y(t_k)$ were approximated by, when the rectangle rule was used to approximate the R.H.S.

$$\Rightarrow \begin{pmatrix} v_{k+1} - v_k \\ x_{k+1} - x_k \end{pmatrix} = \begin{pmatrix} (dt/m)\, f(x_k, v_k) \\ dt \cdot v_k \end{pmatrix}$$

The "Forward (or Explicit) Euler" method.

$$\Rightarrow \boxed{\begin{aligned} x_{k+1} &\leftarrow x_k + dt\, v_k \\ v_{k+1} &\leftarrow v_k + \frac{dt}{m} f(x_k, v_k) \end{aligned}}$$

⇒ Note : All remains valid if $\underline{x}$, $\underline{v}$ represent collections of particles (e.g. mesh vertices) instead of a single one.

Code samples

https://github.com/uwgraphics/PhysicsBasedModeling-Demos/Simple-Deformations