Stability

Previously, we considered the time-discretization of an ODE $y' = f(y,t)$, but did not address the question of stability: i.e. do we run the risk of the numerical approximation exploding

For the purposes of this study, lets focus on a prototypical, very simple "model" ODE

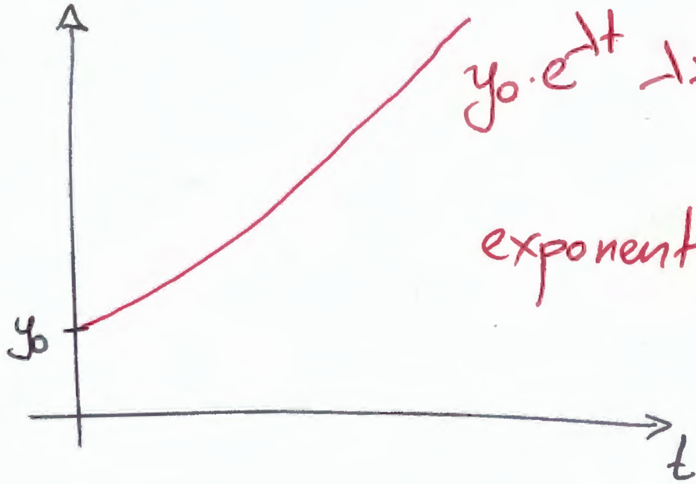
$$\boxed{y' = \lambda y}$$

This, in fact, has an analytic solution (as an initial value problem with $y(0) = y_0$)

$$y(t) = y_0 \cdot e^{\lambda t}$$

Depending on value of λ, we distinguish 3 cases:

$\lambda > 0$

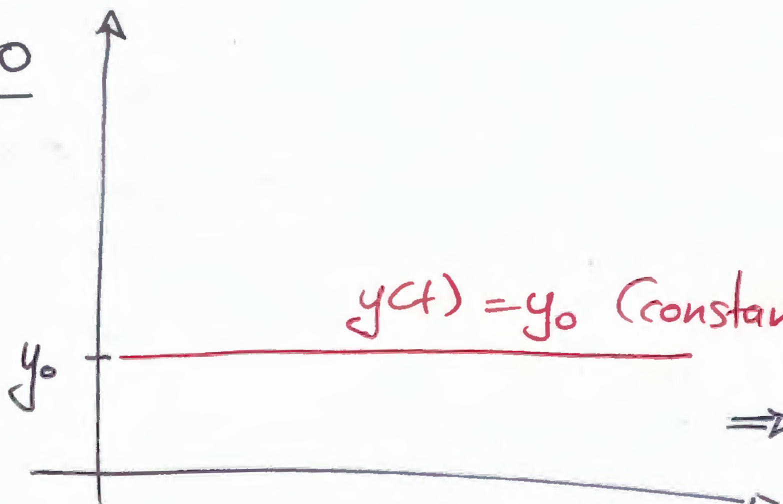


$y_0 \cdot e^{\lambda t}$ $\lambda > 0$

exponential growth!

$\Rightarrow$ ODE is unconditionally unstable

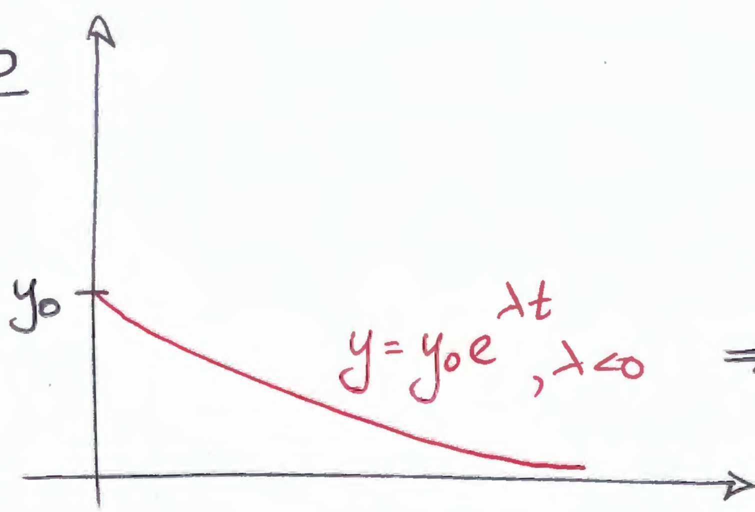
$\lambda = 0$



$y(t) = y_0$ (constant)

$\Rightarrow$ ODE is critically stable (or meta-stable)

$\lambda < 0$



$y = y_0 e^{\lambda t}$, $\lambda < 0$

$\Rightarrow$ ODE is stable

$\Rightarrow$ This is the only case we care to simulate (generally...)

$\Rightarrow$ No point to discuss "stable" methods for unstable ODE...

What about <u>systems</u> of ODEs?

e.g. for a Hookean spring we wrote:

$$f(x,v) = -kx - bv = mx''$$

$$\Rightarrow \underbrace{\begin{pmatrix} x' \\ v' \end{pmatrix}}_{Y'(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{Y(t)} \quad (*)$$

Here, the stability condition becomes:

$Re\{\lambda_i\} < 0$, where $\{\lambda_i\}_{i=1}^{N}$ are the eigenvalues of A

$\Rightarrow$ If $k>0$ & $b>0$ $\quad Re\{\lambda_i\} < 0$ ✓

$\Rightarrow$ If $k>0$ but $\underline{b=0}$ $\quad Re\{\lambda_i\} = 0$

Specifically for the system $(*)$, the eigenvalues of A are

$$\lambda_i = \frac{-b \pm \sqrt{b^2-4mk}}{2m}, \text{ if } b^2>4mk$$

$$\lambda_i = \frac{-b \pm i\sqrt{|b^2-4mk|}}{2m}, \text{ if } b^2<4mk$$

in both cases $\underline{b>0}$ (i.e. some damping) yields a stable ODE

The same way we needed to be careful when assessing stability of a system of ODEs, we need to be cautious when designing damping schemes for an ODE system governing a multi-particle body ...

For example, Newton's law of motion for a body discretized into a mesh with particles $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$ (assume with equal mass $m = m_1 = m_2 = \dots = m_N$ +) is written as:

$$m \underline{x}'' = \underline{f}(\underline{x}, \underline{x}') \quad (**) \qquad \left(\underline{x} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_N \end{pmatrix} \right)$$

Rayleigh Damping

We will propose a model for damping forces that simplifies the stability analysis of $(**)$ — and also happens to be very effective, and realistic. The definition of the damping force (f_d) follows that of the elastic force (f_{el} – as in our FEM lecture slides) as:

$$\underline{f}_d(\underline{x}_*, \underline{v}_*) = \left(\frac{\partial \underline{f}_{el}}{\partial \underline{x}} \bigg|_{\underline{x}_*} \right) \cdot \underline{v} \cdot \gamma$$

$\gamma \rightarrow$ Rayleigh Coefficient

+ The findings are similar (but proof a bit longer) even with non-equal masses

Parsing this definition is a bit easier in the case of linear forces (i.e. an affine relationship between $\underline{f}_{el}$ and $\underline{x}$) as in linear elasticity

$$\underline{f}_{el}(\underline{x}) = A\underline{x} + \underline{b}$$

(Notation: we typically denote with K the matrix $K := -\frac{\partial \underline{f}_{el}}{\partial \underline{x}}$, and call this the stiffness matrix; an extension of the "spring stiffness" of a 1D-spring) Thus, the expression above becomes

$$\underline{f}_{el}(\underline{x}) = -K(\underline{x} - \underline{X})$$

Undeformed positions

Hence, Rayleigh damping yields:

$$\underline{f}_d(\underline{v}) = -\gamma K \underline{v}$$

(Here, the damping "matrix" γK does not depend on $\underline{x}$ ... we might not be so lucky outside of linear elasticity)

Putting everything back in (**) yields:

$$m\underline{\underline{x}}'' + \gamma K \underline{\underline{x}}' + K(\underline{x} - \underline{X}) = 0$$

K is a symmetric, positive definite matix (proof soon...)
so, it admits a symmetric eigenanalysis $K = Q \Lambda Q^T$

Λ : diagonal, all-positive

$$\Rightarrow m \cancel{Q} Q^T \underline{\underline{x}}'' + \gamma \cancel{Q} \Lambda Q^T \underline{\underline{x}}' + \cancel{Q} \Lambda Q^T (\underline{x} - \underline{X}) = 0$$

(Define : $\hat{\underline{x}} \leftarrow Q^T \underline{x}$, $\hat{\underline{X}} \leftarrow Q^T \underline{X}$)

$$\Rightarrow m \hat{\underline{\underline{x}}}'' + \gamma \Lambda \hat{\underline{\underline{x}}}' + \Lambda (\hat{\underline{x}} - \hat{\underline{X}}) = 0$$

$\Rightarrow$ N decoupled equations of the form:

$$m \hat{x}_i'' + \gamma \lambda_i \hat{x}_i' + \lambda_i (\hat{x}_i - \hat{X}_i) = 0$$

For which the stability analysis applies, as long as $\gamma > 0$.

Implementation (linear elasticity)

We observe that

$$f_d(v) = -\gamma K v = \gamma \cdot \frac{d}{dt} f_{el}(x) \quad !$$

Thus, we use the "dot" notation for time derivatives, and can write

$$\dot{D}_s = \left[\dot{x}_1 - \dot{x}_4 \,\middle|\, \dot{x}_2 - \dot{x}_4 \,\middle|\, \dot{x}_3 - \dot{x}_4 \right]$$

$$\left(\text{really} = \left[\vec{v}_1 - \vec{v}_4 \,\middle|\, \vec{v}_2 - \vec{v}_4 \,\middle|\, \vec{v}_3 - \vec{v}_4 \right] \right)$$

$$\dot{F} = \dot{D}_s D_m^{-1}$$

$$\dot{\epsilon} = \frac{d}{dt} \epsilon = \frac{d}{dt} \left\{ \frac{1}{2} (F + F^T) - I \right\} = \frac{1}{2} \left(\dot{F} + \dot{F}^T \right)$$

$$P_d = \left[2\mu \dot{\epsilon} + \lambda \operatorname{tr}(\dot{\epsilon}) I \right] \cdot \gamma$$

Rayleigh coeff.

$$H_d = \left[f_{1,d} \,\middle|\, f_{2,d} \,\middle|\, f_{3,d} \right] = -\mathrm{Vol}(\det) P_d D_m^{-T}$$

$$f_{4,d} = -\left(f_{1,d} + f_{2,d} + f_{3,d} \right) \quad \text{as before...}$$

(You can verify this is <u>really</u> $\gamma \cdot \frac{d}{dt} f_{el}(x)$ using the chain rule and careful derivation)