Nonlinear Elasticity

We have seen some examples of nonlinear materials e.g:

Neohookean : $P(F) = \mu(F - F^{-T}) + \lambda \log J \cdot F^{-T}$

Co-rotated : $P(F) = 2\mu(F - R) + \lambda\, tr(R^T F - I) R$
$(F = RS)$

for those materials the elastic force function $f_{el}(\underline{x})$ is not an affine function of positions ($\underline{x}$). This creates several challenges we'll need to address.

Challenge #1: How do we survive (and possibly recover from in-valid shape configurations?

e.g. In Neohookean elasticity, we need $J = \det(F)$ to appear inside a logarithm. If any element in the mesh inverts, computation of forces fails.

The other key challenges presented by nonlinearity originate from our desire to use implicit time integration schemes.

Consider the case of backward Euler

$$(1):\ \underline{x}^{n+1} = \underline{x}^{n} + h\,\underline{v}^{n+1}$$

$$(2):\ \underline{v}^{n+1} = \underline{v}^{n} + hM^{-1}\left\{ f_{el}(\underline{x}^{n+1}) + f_d(\underline{x}^{n+1}, \underline{v}^{n+1}) \right\}$$

Note that, here, $f_{el}(\underline{x})$ is a nonlinear function, and $f_d(\underline{x},\underline{v})$ is, in principle, dependent on both position and velocity. As we did before, we will use Rayleigh damping due to its simplicity of implementation and realistic physical behavior. The definition (in the case of nonlinear materials) is:

$$f_d(\underline{x}^*, \underline{v}^*) := \gamma \cdot \left.\frac{\partial f_{el}}{\partial \underline{x}}\right|_{\underline{x}=x^*} \cdot \underline{v}^*$$

or, using the "stiffness matrix" $K(\underline{x}) := -\dfrac{\partial f_{el}}{\partial \underline{x}}$

$$f_d(\underline{x},\underline{v}) = -\gamma \cdot K(\underline{x})\,\underline{v}$$

<u>Note</u> For linear Elasticity, the stiffness matrix K was <u>constant</u>; here, it is a function of $\underline{x}$!

System (1,2) is a <u>nonlinear</u> system of equations. To solve it, we use Newton's method, similar to how we would use it for a (scalar) nonlinear equation $f(x)=0$

→ Use Taylor's Formula to approximate

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

→ After substitution, solve:

$$f(x_0) + f'(x_0)(x-x_0) \cong 0$$

$$\Rightarrow x \approx x_0 - f(x_0)/f'(x_0)$$

$$\Rightarrow x_{k+1} \leftarrow x_k - f(x_k)/f'(x_k)$$

until convergence!

We will — carefully — repeat the same operation on the Backward Euler system (1,2). As we did before, assume we have an initial guess $\underline{x}_{(0)}^{n+1}, \underline{v}_{(0)}^{n+1}$ that satisfies (1), e.g.

$$\underline{x}_{(0)}^{n+1} = \underline{x}^{n} + h\,\underline{v}_{(0)}^{n+1} \quad (3)$$

As we did before (lin. elasticity) define

$$\delta \underline{x} := \underline{x}^{n+1} - \underline{x}_{(0)}^{n+1}, \quad \delta \underline{v} := \underline{v}^{n+1} - \underline{v}_{(0)}^{n+1}$$

Then, from (1 & 3):

$$\delta \underline{x} = h \, \delta \underline{v} \quad (4)$$

The next step is to cope with the nonlinearities in (2) we do so as follows:

→ We use the Taylor approximation:

$$f_{el}(\underline{x}^{n+1}) = f_{el}(x_{(0)}^{n+1} + \delta \underline{x})$$

$$\approx f_{el}(x_{(0)}^{n+1}) + \left.\frac{\partial f_{el}}{\partial \underline{x}}\right|_{\underline{x} = \underline{x}_0^{n+1}} \cdot \delta \underline{x}$$

$$= f_{el}(\underline{x}_{(0)}^{n+1}) - K(\underline{x}_{(0)}^{n+1}) \, \delta \underline{x}$$

→ We substitute the formula for Rayleigh damping

$$f_d(\underline{x}^{n+1}, \underline{v}^{n+1}) = -\gamma K(\underline{x}^{n+1}) \, \underline{v}^{n+1}$$

$$\approx -\gamma K(\underline{x}_{(0)}^{n+1}) \, \underline{v}^{n+1}$$

(*) We will return to discuss this approximation later

$$= -\gamma K(\underline{x}_{(0)}^{n+1}) \left[\underline{v}_{(0)}^{n+1} + \delta \underline{v}\right]$$

Using these definitions/approximations equation (2) becomes:

$$\underline{v}_{(0)}^{n+1} + \delta \underline{v} = \underline{v}^{n} + hM^{-1}\Big\{ f_{el}(\underline{x}_{(0)}^{n+1}) - K(\underline{x}_{(0)}^{n+1})\delta\underline{x} - \gamma K(\underline{x}_{(0)})\,\underline{v}_{(0)}^{n+1} - \gamma K(\underline{x}_{(0)}^{n+1})\delta\underline{v}\Big\}$$

$$\overset{(4)}{\Longrightarrow} \left[\left(1+\frac{\gamma}{h}\right)K(\underline{x}_{(0)}^{n+1}) + \frac{1}{h^2}M\right]\delta\underline{x} = \frac{1}{h}M\left[\underline{v}^{n} - \underline{v}_{(0)}^{n+1}\right] + f_{el}(x_{(0)}^{n+1}) + f_d(x_{(0)}^{n+1}, v_{(0)}^{n+1})$$

Applying the correction(s) $\delta\underline{x}, \delta\underline{v}$ gets us <u>closer</u> to the solution of (1,2), but since the system is nonlinear, we will have to <u>iterate</u> to convergence as <u>follows</u>:

1. Produce initial guesses $\underline{x}_{(0)}^{n+1}, v_{(0)}^{n+1}$ that satisfy (1), start with $k \leftarrow 0$

2. Solve:

$$\left[\left(1+\frac{\gamma}{h}\right)K(x_{(k)}^{n+1}) + \frac{1}{h^2}M\right]\delta x = \frac{1}{h}M\left[\underline{v}^{n} - v_{(k)}^{n+1}\right] + f_{el}(x_{(k)}^{n+1}) + f_d(x_{(k)}^{n+1}, v_{(k)}^{n+1}) \quad (5)$$

3. Update

$$\underline{x}_{(k+1)}^{n+1} \leftarrow x_{(k)}^{n+1} + \delta\underline{x}$$

$$\underline{v}_{(k+1)}^{n+1} \leftarrow v_{(k)}^{n+1} + \frac{1}{h}\delta\underline{x}$$

4. Increase k, repeat from Step #2 until convergence.

Notes

1. To check for convergence, we may evaluate a norm of the RHS of equation (5). If small, we can terminate the Newton process and set

$$\underline{x}^{n+1} \leftarrow x_{(k)}^{n+1}, \quad \underline{v}^{n+1} \leftarrow v_{(k)}^{n+1}$$

2. We made an approximation of convenience in our use of Rayleigh damping by replacing $K(\underline{x}^{n+1})$ with $K(\underline{x}_{(k)}^{n+1})$ (*) Although this makes our method a <u>modified</u> Newton algorithm, upon convergence we do reach the same, correct result

3. If we set $h \rightarrow 0$ we obtain a simulation where
   - → Inertial effects are ignored
   - → Same effect as if boundaries are handled <u>very slowly</u>
   - → Effectively zero velocities (only need to store $\underline{x}$)
   - → Called Quasi-static simulation
   - → Update rule $\quad -K(x_{(k)}^{n+1})\,\delta\underline{x} = f_{el}(x_{(k)}^{n+1})$
   - → <u>Will revisit!</u>

We have also exposed two new challenges:

Challenge #2: How do we multiply (without constructing) with the new CG matrix

$$A = \left(1+\frac{\gamma}{h}\right) K\left(\underset{\sim}{x}^{n+1}_{(k)}\right) + \frac{1}{h^2} M \quad ? \qquad (6)$$

$\Rightarrow$ Topic of next class — read section 4.3 (&4.4) of FEM notes.

Challenge #3: How do we ensure matrix A is positive definite?

$\Rightarrow$ Next week. Pre-read:

Teran et al "Robust quasistatics & flesh simulation" Symposium on Computer Animation 2005

All these are also related to challenge #1.

We will start discussing this, but do review:

Irving et al "Invertible Finite Elements for Robust Simulation of Large Deformation" SIGGRAPH 2004.