

On local behavior of Newton-type methods near critical solutions of constrained equations

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Abstract For constrained equations with nonisolated solutions and a certain family of Newton-type methods, it was previously shown that if the equation mapping is 2-regular at a given solution with respect to a direction which is interior feasible and which is in the null space of the Jacobian, then there is an associated large (not asymptotically thin) domain of starting points from which the iterates are well defined and converge to the specific solution in question. Under these assumptions, the constrained local Lipschitzian error bound does not hold, unlike the common settings of convergence and rate of convergence analyses. In this work, we complement those previous results by considering the case when the equation mapping is 2-regular with respect to a direction in the null space of the Jacobian which is in the tangent cone to the set, but need not be interior feasible. Under some further conditions, we still show linear convergence of order $1/2$ from a large domain around the solution (despite degeneracy, and despite that there may exist other solutions nearby). Our results apply to constrained variants of the Gauss–Newton and Levenberg–Marquardt methods, and to the LP-Newton method. An illustra-

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tion for a smooth constrained reformulation of the nonlinear complementarity problem is also provided.

Keywords Newton-type methods · constrained equations · singular solutions · critical solutions · 2-regularity · Gauss–Newton method · Levenberg–Marquardt method · LP-Newton method · nonlinear complementarity problem

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1 Introduction

The studies of local convergence of the classical Newton method for unconstrained equations in the case of singular solutions, most relevant to our developments here, date back to [14]. In that work, convergence had been shown to special singular solutions from large (not asymptotically thin) starlike domains around such solutions. In [15], those results were extended to a certain *perturbed* Newton method framework, which subsumes in addition to the pure Newton method the classical Levenberg–Marquardt method [20, 22] (see also [23, Section 10.3]) and the LP-Newton method [5]. The study in [15] also highlights the special role of certain solutions, called critical [16]. The case of constrained equations (i.e., the problem where a solution must satisfy an equation and also belong to a given set), with possibly nonisolated solutions, was considered in [9]. One principal condition for convergence in [9] is that there exists a direction in the null space of the Jacobian pointing into the interior of the constraint set. Further analysis for the case of piecewise smooth equations can be found in [7].

The purpose of this work is to complement the previous studies by considering the case when the direction in question does not necessarily point in the interior of the constraint set but can be tangential to it. Note that if, for instance, the interior of the constraint set is empty, the results from [9] are not applicable in principle.

The rest of the paper is organized as follows. In Section 2, we state the problem and the methods to be considered. We also discuss the previous results in some more detail, and the key notions involved (such as 2-regularity and error bounds). In Section 3 we provide convergence results for the constrained Gauss–Newton method. Then, in Section 4 those results are used to show convergence of the constrained Levenberg–Marquardt method and of the LP-Newton method. Section 5 contains an illustration for a smooth constrained reformulation of the nonlinear complementarity problem. We finish with some concluding remarks in Section 6.

Some words about our notation. For a set $U \subset \mathbb{R}^p$, the interior of U is denoted by $\text{int } U$, and its closure by $\text{cl } U$. By $\text{cone } U$ we mean the conic hull of U (the smallest convex cone in \mathbb{R}^p that contains U). If U is convex, then the tangent cone to U at $u \in U$ is given by $T_U(u) = \text{cl cone}(U - u)$. If U is a linear subspace of \mathbb{R}^p , then $\dim U$ stands for its dimension. Given a set of

elements $u^i \in \mathbb{R}^p$, $i = 1, \dots, r$, by $\text{span}\{u^1, \dots, u^r\}$ we denote the subspace spanned by those elements. The notation $\ker A$ and $\text{im } A$ is standard and refers to the null and range spaces of operator (or matrix) A , respectively. By $x \circ y$ we denote the Hadamard product of $x, y \in \mathbb{R}^n$, i.e., $x \circ y = (x_1 y_1, \dots, x_n y_n)$. For $x \in \mathbb{R}^n$, $\text{diag } x$ is the diagonal $n \times n$ matrix with the coordinates of x on the diagonal. By \mathcal{I} we denote the identity matrix, with the dimension being always clear from the context. The notation $|I|$ stands for the cardinality of the index set I . For an index set $I \subset \{1, \dots, p\}$, by x_I we mean the sub-vector of $x \in \mathbb{R}^n$ formed by the coordinates indexed by I . For a vector x with nonzero components, x^{-1} stands for the vector whose components are the inverses of those of x .

2 Problem setting and preliminaries

We consider the constrained equation

$$\Phi(u) = 0, \quad u \in P, \quad (1)$$

where $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a given mapping, and $P \subset \mathbb{R}^p$ is a given closed convex set. Let \bar{u} be a solution of (1), and assume that Φ is differentiable near \bar{u} . In this work, we are interested in local behavior of various Newton-type methods for solving (1) when the solution \bar{u} is singular, by which we mean that the Jacobian $\Phi'(\bar{u})$ is a singular matrix. In particular, any nonisolated solution of the equation in (1) is necessarily singular.

At any singular solution \bar{u} , it holds that $\ker \Phi'(\bar{u}) \neq \{0\}$, and the relative position of $\ker \Phi'(\bar{u})$ and of the tangent cone $T_P(\bar{u})$ to P at \bar{u} , with respect to each other, has a crucial impact on the behavior of Newton-type methods near \bar{u} . We refer to [15] for some results highlighting this issue. In this work, we restrict our attention to the list of Newton-type methods specified below.

For a current iterate $u^k \in P$, the constrained Gauss–Newton (GN) method defines $u^{k+1} = u^k + v^k$, where v^k is a solution of the convex quadratic optimization problem

$$\text{minimize } \frac{1}{2} \|\Phi(u^k) + \Phi'(u^k)v\|^2 \quad \text{subject to } u^k + v \in P. \quad (2)$$

Due to the Frank–Wolfe theorem [11], this subproblem always has a solution in case of a polyhedral P , but a solution need not be unique.

The constrained Levenberg–Marquardt (LM) method employs the regularized version of (2), i.e.,

$$\text{minimize } \frac{1}{2} \|\Phi(u^k) + \Phi'(u^k)v\|^2 + \frac{1}{2} \sigma(u^k) \|v\|^2 \quad \text{subject to } u^k + v \in P, \quad (3)$$

where $\sigma : \mathbb{R}^p \rightarrow \mathbb{R}_+$ is a function defining the values of the regularization parameter. If $\sigma(u^k) > 0$, the objective function of this subproblem is strongly convex quadratic, and the subproblem has the unique solution.

Finally, the Linear-Programming-Newton (LPN) method [5] has subproblems of the form

$$\begin{aligned} \text{minimize } \gamma \text{ subject to } & \|\Phi(u^k) + \Phi'(u^k)v\| \leq \gamma \|\Phi(u^k)\|^2, \\ & \|v\| \leq \gamma \|\Phi(u^k)\|, \\ & u^k + v \in P, \end{aligned} \quad (4)$$

with respect to $(v, \gamma) \in \mathbb{R}^p \times \mathbb{R}$. The name originates from the fact that in case of a polyhedral P , and if the l_∞ -norm is used, this subproblem is a linear programming problem. It is always solvable (perhaps nonuniquely) provided $\Phi(u^k) \neq 0$.

One principal case for consideration is when

$$\ker \Phi'(\bar{u}) \cap T_P(\bar{u}) = \{0\},$$

i.e., $\bar{u} + \ker \Phi'(\bar{u})$ intersects P “transversally” at \bar{u} (the word “transversally” is used here in a nonstrict meaning, as the opposite to “tangentially”). In this case, \bar{u} is necessarily an isolated solution of (1), and moreover, it can be easily verified that the constrained error bound condition

$$\|u - \bar{u}\| = O(\|\Phi(u)\|) \text{ as } u \in P \text{ tends to } \bar{u} \quad (5)$$

is valid. Assuming that Φ has a Lipschitz-continuous derivative near \bar{u} , condition (5) implies local quadratic convergence of the constrained LM method [4, 19], and of the LPN method [5].

Another case of interest, investigated in [9], is when

$$\ker \Phi'(\bar{u}) \cap \text{int } T_P(\bar{u}) \neq \{0\}.$$

Let Φ be twice differentiable near \bar{u} , with its second derivative Lipschitz-continuous on P near \bar{u} with respect to \bar{u} , that is,

$$\|\Phi''(u) - \Phi''(\bar{u})\| = O(\|u - \bar{u}\|) \text{ as } u \in P \text{ tends to } \bar{u}. \quad (6)$$

Assume further that Φ is 2-regular (see, e.g., [1]) at \bar{u} in some nonzero direction $\bar{v} \in \ker \Phi'(\bar{u}) \cap \text{int } T_P(\bar{u})$, that is,

$$\text{im } \Phi'(\bar{u}) + \Phi''(\bar{u})[\bar{v}, \ker \Phi'(\bar{u})] = \mathbb{R}^p.$$

For the role and nature of the notion of 2-regularity in Optimization and Nonlinear Analysis see, e.g., [3, 12, 13, 17, 18].

For any $\varepsilon > 0$ and $\delta > 0$, define the set

$$K_{\varepsilon, \delta} = K_{\varepsilon, \delta}(\bar{u}, \bar{v}) = \left\{ u \in \mathbb{R}^p \left| \begin{array}{l} \|u - \bar{u}\| \leq \varepsilon, \\ \left| \|\bar{v}\|(u - \bar{u}) - \|u - \bar{u}\|\bar{v} \| \leq \delta \|u - \bar{u}\| \|\bar{v}\| \end{array} \right. \right\}.$$

In [9] it was shown that under the stated assumptions, for every $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, there exist $\varepsilon > 0$ and $\delta > 0$ such that for any starting point $u^0 \in K_{\varepsilon, \delta}$, any of the algorithms listed above (GN, LM with certain conditions on the choice of $\sigma(\cdot)$, and LPN) uniquely defines the sequence $\{u^k\}$, this sequence

is contained in $K_{\hat{\varepsilon}, \hat{\delta}}$ and coincides with the sequence generated by the unconstrained version of the algorithm in question, obtained by removing the constraints in (2), (3), and the last constraint in (4), respectively. In other words, being initialized within $K_{\hat{\varepsilon}, \hat{\delta}}$, the unconstrained GN, LM, and LPN methods uniquely define the iterates that automatically remain feasible. This further allows to apply the analysis for the unconstrained case in [15, Theorem 1], yielding convergence of $\{u^k\}$ to \bar{u} at a linear rate, with the asymptotic common ratio being exactly equal to $1/2$.

Passing to more detail, observe that the unconstrained version of GN (2) is equivalent to the linear equation

$$(\Phi'(u^k))^\top (\Phi(u^k) + \Phi'(u^k)v) = 0,$$

and since in the specified setting this equation remains uniquely solvable along the iterations, it follows that both unconstrained and constrained GN methods perform in this case exactly the same way as the basic Newton method with the subproblem

$$\Phi(u^k) + \Phi'(u^k)v = 0. \quad (7)$$

Employing decomposition of every $u \in \mathbb{R}^p$ into the sum $u = u_1 + u_2$ with uniquely defined $u_1 \in (\ker \Phi'(\bar{u}))^\perp$ and $u_2 \in \ker \Phi'(\bar{u})$, according to [10, Lemma 1], the unique solution v^N of (7) for $u^k = u \in K_{\hat{\varepsilon}, \hat{\delta}}$ with sufficiently small $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$ satisfies

$$u_1 + v_1^N - \bar{u}_1 = O(\|u - \bar{u}\| \|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^3), \quad (8)$$

$$u_2 + v_2^N - \bar{u}_2 = \frac{1}{2}(u_2 - \bar{u}_2) + O(\|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^2) \quad (9)$$

as $u \rightarrow \bar{u}$.

As for the LM and LPN methods, they can be interpreted as the perturbed Newton method with the subproblem

$$\Phi(u^k) + \Phi'(u^k)v = \omega(u^k), \quad (10)$$

where the perturbation term $\omega : \mathbb{R}^p \rightarrow \mathbb{R}^p$ accounts for the differences in the iterates of the methods and satisfies the requirements needed for application of the analysis developed in [15], namely,

$$\omega(u) = O(\|u - \bar{u}\|^2), \quad (11)$$

$$\Pi \omega(u) = O(\|u - \bar{u}\| \|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^3) \quad (12)$$

as $u \rightarrow \bar{u}$, where Π is the orthogonal projector in \mathbb{R}^p onto $(\text{im } \Phi'(\bar{u}))^\perp$. More precisely, considerations in [7, 9] yield the estimate

$$\omega(u) = O(\|u - \bar{u}\| \|\Phi(u)\|) \quad (13)$$

as $u \in K_{\hat{\varepsilon}, \hat{\delta}}$ tends to \bar{u} , provided $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$ are taken small enough. Then (13) evidently implies the estimate

$$\omega(u) = O(\|u - \bar{u}\| \|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^3), \quad (14)$$

further implying both (11) and (12).

The focus of this work is on the intermediate case when it might be that $\ker \Phi'(\bar{u}) \cap \text{int } T_P(\bar{u}) = \emptyset$, but

$$\ker \Phi'(\bar{u}) \cap T_P(\bar{u}) \neq \{0\},$$

and there exists a nonzero $\bar{v} \in \ker \Phi'(\bar{u}) \cap T_P(\bar{u})$ such that Φ is 2-regular at \bar{u} in the direction \bar{v} . As discussed in [2, p. 624], the latter condition may only hold when the constrained error bound (generalizing (5) to the case of a possibly nonisolated solution)

$$\text{dist}(u, \Phi^{-1}(0) \cap P) = O(\|\Phi(u)\|) \text{ as } u \in P \text{ tends to } \bar{u} \quad (15)$$

is violated, actually no matter whether $\bar{v} \in \text{int } T_P(\bar{u})$ or not.

We emphasize that the specified case is not covered by any Newtonian theories developed so far. In particular, if $\text{int } P = \emptyset$, the results in [9] can never be applied. At the same time, $T_P(\bar{u}) \neq \{0\}$ unless P is a singleton, and the constructions and results of this work have their chance to be applicable even when $\text{int } P = \emptyset$.

It is clear that in the current case under consideration, one cannot expect, say, the constrained GN method initialized within $K_{\varepsilon, \delta}$ with arbitrarily small $\varepsilon > 0$ and $\delta > 0$ to work as the basic Newton method. The idea, however, is to interpret the constrained GN method as a *perturbed (unconstrained!)* Newton method with the subproblem (10), where the perturbation term satisfies (14). Then, the local convergence and rate of convergence result is obtained by applying [15, Theorem 1], again yielding convergence of GN sequences to \bar{u} at a linear rate, with the asymptotic ratio 1/2 (this would require some restrictions on the set P). Utilizing the obtained results for the GN method, the analysis is then further extended to LM and LPN methods.

To conclude this section, we note that violation of Lipschitzian error bounds is closely related to the notion of critical solutions. For the case of unconstrained equations, we refer the readers to [16, 15]. For the constrained case, see the discussions in [9] and [8].

3 Constrained Gauss–Newton method

For a given $u \in \mathbb{R}^P$, set

$$\rho_{GN}(u) = \inf_{v \in P-u} \|\Phi(u) + \Phi'(u)v\|. \quad (16)$$

In particular, if the constrained GN subproblem

$$\text{minimize } \frac{1}{2} \|\Phi(u) + \Phi'(u)v\|^2 \text{ subject to } u + v \in P \quad (17)$$

has a solution v^{GN} , then the perturbed Newton relation (10) holds with $u^k = u$, and with $\omega(u) = \omega^{GN}(u)$, where

$$\omega^{GN}(u) = \Phi(u) + \Phi'(u)v^{GN} \quad (18)$$

satisfies

$$\|\omega^{GN}(u)\| = \rho_{GN}(u).$$

We start this section with two examples demonstrating some peculiarities of the convergence issues under consideration. The first example shows that without further requirements, the constrained GN method may not fit the perturbed unconstrained Newton method framework with the needed estimates of the perturbation term.

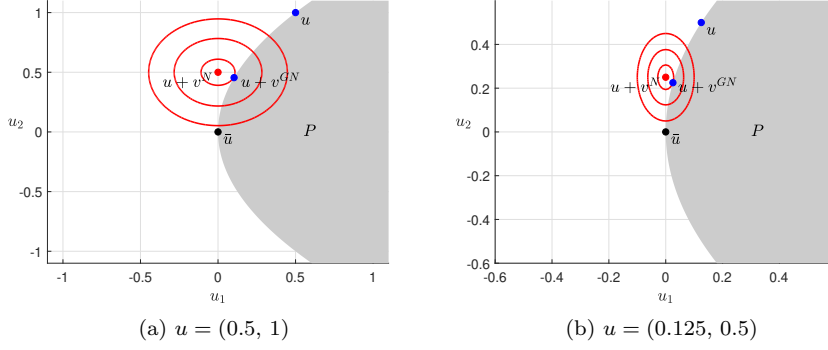


Fig. 1: Example 3.1 with $a = 1$.

Example 3.1 Let $p = 2$, $\Phi(u) = (u_1, u_2^2/2)$, $P = \{u \in \mathbb{R}^p \mid u_1 \geq au_2^2/2\}$, where $a \geq 0$ is a parameter. Then the unique solution of (1) is $\bar{u} = 0$, and for any $u, v \in \mathbb{R}^p$

$$\Phi'(u) = \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad \Phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi''(0)[v] = \begin{pmatrix} 0 & 0 \\ 0 & v_2 \end{pmatrix},$$

where Π is the orthogonal projector in \mathbb{R}^2 onto $(\text{im } \Phi'(0))^\perp$. Therefore, Φ is 2-regular at 0 in every nonzero direction in $\ker \Phi'(0) = \{0\} \times \mathbb{R}$, including the directions of interest $\bar{v} = (0, \pm 1)$.

The constrained GN subproblem (17) takes the form

$$\begin{aligned} & \text{minimize} \quad \frac{1}{2} \left((u_1 + v_1)^2 + \left(\frac{1}{2}u_2^2 + u_2v_2 \right)^2 \right) \\ & \text{subject to} \quad u_1 + v_1 \geq \frac{a}{2}(u_2 + v_2)^2. \end{aligned} \tag{19}$$

Assuming that $u_2 \neq 0$, the unique critical point of the objective function of (19) is its unique unconstrained minimizer $v^N = (-u_1, -u_2/2)$ defining the basic Newton step. Since v^N cannot satisfy the constraint in (19) as a strict inequality, the unique solution of (19) is characterized by the system

$$u_1 + v_1 - \mu = 0, \quad \left(\frac{1}{2}u_2^2 + u_2v_2 \right) u_2 + \mu a(u_2 + v_2) = 0, \quad u_1 + v_1 = \frac{a}{2}(u_2 + v_2)^2,$$

with some multiplier $\mu \in \mathbb{R}$ which is automatically nonnegative. Indeed, it must hold that

$$\mu = u_1 + v_1 = \frac{a}{2}(u_2 + v_2)^2, \quad (20)$$

and hence,

$$\left(\frac{1}{2}u_2 + v_2\right)u_2^2 + \frac{a^2}{2}(u_2 + v_2)^3 = 0.$$

Denoting $t = u_2 + v_2$, this equation can be written as $a^2t^3 + 2u_2^2t - u_2^3 = 0$. Elementary analysis shows that for any $a > 0$, this cubic equation has the unique solution $t = \tau u_2$ with some constant $\tau \in (0, 1/2)$ uniquely defined by the equation $a^2\tau^3 + 2\tau - 1 = 0$. Then, taking into account (20), the constrained GN step is given by

$$v_1^{GN} = -u_1 + \frac{a}{2}t^2 = -u_1 + \frac{a}{2}\tau^2u_2^2, \quad v_2^{GN} = -u_2 + t = -(1 - \tau)u_2. \quad (21)$$

Therefore, according to (18),

$$\omega^{GN}(u) = \left(u_1 + v_1^{GN}, \frac{1}{2}u_2^2 + u_2v_2^{GN}\right) = \left(\frac{a}{2}\tau^2u_2^2, -\left(\frac{1}{2} - \tau\right)u_2^2\right),$$

and $\omega(\cdot) = \omega^{GN}(\cdot)$ satisfies (11), but not (12), since $\tau \neq 1/2$. In particular, from (21) we further have that

$$u + v^{GN} = \left(\frac{a}{2}\tau^2u_2^2, \tau u_2\right),$$

implying the linear convergence rate, but with the asymptotic common ratio being $\tau < 1/2$. While if both (11) and (12) were to be satisfied, the ratio would have been exactly $1/2$.

Observe that $\tau \rightarrow 1/2^-$ as $a \rightarrow 0+$, and $\tau \rightarrow 0+$ as $a \rightarrow +\infty$. If $a = 0$ (in which case $P = \mathbb{R}_+ \times \mathbb{R}$ is a half-space), then $\tau = 1/2$, and both (11) and (12) are satisfied.

The observations above are illustrated for $a = 1$ in Figure 1, for two different points $u \in P$. The red lines in Figure 1a and Figure 1b are contours of the quadratic objective function in (17) (shifted by u). One can see how the direction of elongation of those ellipses changes as u becomes closer to \bar{u} .

The example above suggests that the perturbed Newton method framework can be applicable to the constrained GN method under some further requirements on the constraint set P , like that \bar{v} is a feasible direction for P at \bar{u} . Observe that this is automatic for $\bar{v} \in T_P(\bar{u})$ when P is polyhedral. In the example above, P is not polyhedral if $a > 0$, and \bar{v} is not a feasible direction for P at 0 in this case, but P is polyhedral if $a = 0$.

In the next example, P is a very simple polyhedral set, but the second derivative of Φ is not Lipschitz-continuous with respect to \bar{u} , and it turns out that this seemingly insignificant lack of smoothness can also be a reason why the constrained GN method may not fit the perturbed unconstrained Newton method framework.

Example 3.2 Let $p = 2$, $\Phi(u) = (u_1 + 3u_2^{7/3}/7, u_2^2/2)$, $P = \{u \in \mathbb{R}^p \mid u_1 \leq 0\}$. Then the unique solution of (1) is $\bar{u} = 0$, and for any $u, v \in \mathbb{R}^p$

$$\Phi'(u) = \begin{pmatrix} 1 & u_2^{4/3} \\ 0 & u_2 \end{pmatrix}, \quad \Phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi''(0)[v] = \begin{pmatrix} 0 & 0 \\ 0 & v_2 \end{pmatrix}.$$

Therefore, Φ is 2-regular at 0 in every nonzero direction in $\ker \Phi'(0) = \{0\} \times \mathbb{R}$, including the directions $\bar{v} = (0, \pm 1)$.

Assuming that $u_1 = 0$ (and hence, $u \in P$), the constrained GN subproblem (17) takes the form

$$\text{minimize } \frac{1}{2} \left(\left(\frac{3}{7}u_2^{7/3} + v_1 + u_2^{4/3}v_2 \right)^2 + \left(\frac{1}{2}u_2^2 + u_2v_2 \right)^2 \right) \text{ subject to } v_1 \leq 0.$$

The unique solution of this subproblem is

$$v^{GN} = \left(0, -\frac{1}{2}u_2 \frac{1 + 6u_2^{2/3}/7}{1 + u_2^{2/3}} \right).$$

Therefore, (18) yields

$$\omega^{GN}(u) = \left(-\frac{u_2^{7/3}}{14(1 + u_2^{2/3})}, \frac{u_2^{8/3}}{14(1 + u_2^{2/3})} \right),$$

and $\omega(\cdot) = \omega^{GN}(\cdot)$ again satisfies (11), but not (12).

In both Examples 3.1 and 3.2, the constrained GN method actually possesses local linear convergence to the solution, and moreover, in Example 3.2, the asymptotic ratio equals $1/2$ (unlike in Example 3.1 with $a > 0$). However, these convergence properties cannot be established through the perturbed Newton method framework of [15]. Observe also that in Example 3.2, the pure Newton step has the form $v^N = (-u_1 + u_2^{7/3}/14, -u_2/2)$, and it satisfies (9), but not (8).

In what follows, for $u \in \mathbb{R}^p$, we use the further decomposition of u_2 into the sum $u_2 = u_{21} + u_{22}$ with the uniquely defined $u_{21} \in \text{span}\{\bar{v}\}$ and $u_{22} \in \ker \Phi'(\bar{u}) \cap (\text{span}\{\bar{v}\})^\perp$.

Proposition 3.1 *Let $P \subset \mathbb{R}^p$ be closed and convex, let $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be twice differentiable near a solution \bar{u} of (1), and assume that (6) holds. Assume further that Φ is 2-regular at \bar{u} in a direction $\bar{v} \in \ker \Phi'(\bar{u})$ which is feasible for P at \bar{u} .*

Then there exists $\hat{\delta} > 0$ such that $\rho_{GN}(\cdot)$ defined by (16) satisfies

$$\rho_{GN}(u) = O(\|u - \bar{u}\| \|u_{22} - \bar{u}_{22}\|) + O(\|u - \bar{u}\| \|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^3) \quad (22)$$

for $u \in P \cap K_{\bar{u}, \hat{\delta}}$ as $\hat{\varepsilon} \rightarrow 0+$.

Proof Choose $\widehat{\varepsilon} > 0$ and $\widehat{\delta} > 0$ according to [10, Lemma 1] so that for every $u \in P \cap K_{\widehat{\varepsilon}, \widehat{\delta}}$, there exists the unique solution v^N of the Newtonian iteration equation

$$\Phi(u) + \Phi'(u)v = 0, \quad (23)$$

and (8)–(9) hold as $u \rightarrow \bar{u}$. For any $v \in P - u$, by (16) we then obtain that

$$\begin{aligned} \rho_{GN}(u) &\leq \|\Phi(u) + \Phi'(u)v\| \\ &\leq \|\Phi(u) + \Phi'(u)v^N\| + \|\Phi'(u)(v - v^N)\| \\ &\leq \|\Phi'(u)(v_1 - v_1^N)\| + \|(\Phi'(u) - \Phi'(\bar{u}))(v_2 - v_2^N)\| \\ &= O(\|v_1 - v_1^N\|) + O(\|u - \bar{u}\|\|v_2 - v_2^N\|) \end{aligned} \quad (24)$$

as $u \rightarrow \bar{u}$.

Set $v = -(u_1 - \bar{u}_1) - (u_{21} - \bar{u}_{21})/2 - (u_{22} - \bar{u}_{22})$. Then $u + v = \bar{u} + (u_{21} - \bar{u}_{21})/2$ belongs to P provided $\widehat{\varepsilon}$ and $\widehat{\delta}$ are taken small enough, since \bar{v} is a feasible direction for P at \bar{u} . Furthermore, from (8) and (9) we obtain that

$$v_1 - v_1^N = O(\|u - \bar{u}\|\|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^3), \quad (25)$$

$$v_2 - v_2^N = -\frac{1}{2}(u_{22} - \bar{u}_{22}) + O(\|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^2).$$

Hence, (24) implies (22) as $u \rightarrow \bar{u}$. \square

Consider, for example, the special case when $\dim \ker \Phi'(\bar{u}) = 1$. Then $u_{22} - \bar{u}_{22} = 0$, and (22) readily implies the needed estimate (14).

We next discuss two different special cases when (14) can be derived without involving (22) from Proposition 3.1, but rather by employing different choices of v in (24). In all these cases, the GN method fits the perturbed Newton method framework on the entire set $P \cap K_{\widehat{\varepsilon}, \widehat{\delta}}$.

Remark 3.1 Suppose that there exist $\widehat{\varepsilon} > 0$ and $\widehat{\delta} > 0$ such that

$$(\bar{u} + \ker \Phi'(\bar{u})) \cap K_{\widehat{\varepsilon}, \widehat{\delta}} \subset P. \quad (26)$$

Then for every $u \in K_{\widehat{\varepsilon}, \widehat{\delta}}$, taking $v = -(u_1 - \bar{u}_1) - (u_2 - \bar{u}_2)/2$ yields

$$u + v = \bar{u} + (u_2 - \bar{u}_2)/2 \in (\bar{u} + \ker \Phi'(\bar{u})) \cap K_{\widehat{\varepsilon}, \widehat{\delta}},$$

and hence, by (26), $u + v \in P$. Moreover, from (8) and (9) we obtain (25) and

$$v_2 - v_2^N = O(\|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^2).$$

Hence, (24) implies the needed estimate (14).

Feasibility of the direction \bar{v} for P at \bar{u} implies that (26) is automatically satisfied with $\widehat{\varepsilon} > 0$ and $\widehat{\delta} > 0$ small enough if $\dim \ker \Phi'(\bar{u}) = 1$. In the case when P is a polyhedral set, it is also satisfied more generally when \bar{v} is a feasible direction for the relative interior of some face of P at \bar{u} , such that $\bar{u} + \ker \Phi'(\bar{u})$ is contained in the affine hull of this face.

Suppose now that for some $\widehat{\varepsilon} > 0$ and $\widehat{\delta} > 0$, instead of (26) it holds that

$$P \cap K_{\widehat{\varepsilon}, \widehat{\delta}} \subset \bar{u} + \ker \Phi'(\bar{u}). \quad (27)$$

Then for every $u \in P \cap K_{\widehat{\varepsilon}, \widehat{\delta}}$ we have $u_1 - \bar{u}_1 = 0$. Due to the convexity of P , taking $v = -(u_2 - \bar{u}_2)/2 = -(u - \bar{u})/2$ yields $u + v = (\bar{u} + u)/2 \in P$, while from (8) and (9) we obtain

$$v_1 - v_1^N = O(\|u - \bar{u}\|^3), \quad v_2 - v_2^N = O(\|u - \bar{u}\|^2).$$

Hence, (24) again implies the needed estimate (14). For example, (27) is automatic in another extreme with respect to $\dim \ker \Phi'(\bar{u}) = 1$ case of singularity, i.e., in the case when $\Phi'(\bar{u}) = 0$.

Note that if $p \leq 2$, then the only possibilities of singularity for a solution \bar{u} are the two extreme ones mentioned above: $\Phi'(\bar{u}) = 0$ or $\dim \ker \Phi'(\bar{u}) = 1$ (for $p = 1$, these two possibilities are the same). That said, once again we recall Examples 3.1 and 3.2 above, where $p = 2$ and $\dim \ker \Phi'(\bar{u}) = 1$, but the needed estimates do not hold because of violation of other assumptions.

We proceed with the following consequence of Proposition 3.1 for the case of a polyhedral set P .

Corollary 3.1 *Under the assumptions of Proposition 3.1, let P be a polyhedral set, and assume that $\bar{v} \neq 0$ and*

$$\ker \Phi'(\bar{u}) \cap (P - \bar{u}) \subset \text{span}\{\bar{v}\}. \quad (28)$$

Then there exists $\widehat{\delta} > 0$ such that $\omega(\cdot) = \omega^{GN}(\cdot)$ defined according to (18) with any solution v^{GN} of (17) satisfies the estimate (14) for $u \in P \cap K_{\widehat{\varepsilon}, \widehat{\delta}}$ as $\widehat{\varepsilon} \rightarrow 0+$.

Proof We need to prove the existence of $\widehat{\delta} > 0$ such that

$$u_{22} - \bar{u}_{22} = O(\|u_1 - \bar{u}_1\|) \quad (29)$$

holds for $u \in P \cap K_{\widehat{\varepsilon}, \widehat{\delta}}$ as $\widehat{\varepsilon} \rightarrow 0+$.

Let $P = \{u \in \mathbb{R}^p \mid Au \leq b\}$, with some $A \in \mathbb{R}^{q \times p}$ and $b \in \mathbb{R}^q$. Without loss of generality, assume for simplicity that $\bar{u} = 0$ and $\|\bar{v}\| = 1$. Condition $\bar{u} \in P$ then implies that $b \geq 0$. Define the matrix \bar{A} consisting of rows of A indexed by $i \in \{1, \dots, q\}$ such that $b_i = 0$ and $(A\bar{v})_i = 0$. It can be easily seen that if $\widehat{\varepsilon} > 0$ and $\widehat{\delta} > 0$ are fixed small enough, then $P \cap K_{\widehat{\varepsilon}, \widehat{\delta}} = \{u \in K_{\widehat{\varepsilon}, \widehat{\delta}} \mid \bar{A}u \leq 0\}$. Observe also that if $\widehat{\delta} < 1$, then for every nonzero $u \in K_{\widehat{\varepsilon}, \widehat{\delta}}$ it necessarily holds that $u_{21} \neq 0$, and $u_{21}/\|u_{21}\| = \bar{v}$.

Suppose the contrary to (29): let there exist a sequence $\{u^k\} \subset P \cap K_{\widehat{\varepsilon}, \widehat{\delta}}$ such that $\{u^k\} \rightarrow 0$, and $u_1^k = o(\|u_{22}^k\|)$ as $k \rightarrow \infty$. For every k , set

$$\widehat{u}_\tau^k = \tau u_1^k + \frac{\|u_{22}^k\|}{\|u_{21}^k\|} u_{21}^k + \tau u_{22}^k = \|u_{22}^k\| \bar{v} + \tau u_{22}^k + \tau o(\|u_{22}^k\|),$$

where by taking $\tau > 0$ small enough one can ensure that $\hat{u}_\tau^k \in K_{\hat{\varepsilon}, \hat{\delta}}$ for all k large enough. Therefore, by the definition of \bar{A} it holds that $\bar{A}\hat{u}_\tau^k = \tau \bar{A}u^k \leq 0$.

Since the sequence $\{\hat{u}^k/\|u_{22}^k\|\}$ is bounded, we can assume (passing onto a subsequence if necessary) that it converges to some $\hat{v}_\tau \in \mathbb{R}^p$ with $(\hat{v}_\tau)_1 = 0$, $\|(\hat{v}_\tau)_{21}\| = 1$, $\|(\hat{v}_\tau)_{22}\| = \tau > 0$, and $\bar{A}\hat{v}_\tau \leq 0$. Therefore, $\hat{v}_\tau \in \ker \Phi'(0)$, and $t\hat{v}_\tau \in K_{\hat{\varepsilon}, \hat{\delta}}$ for all $\tau > 0$ and $t > 0$ small enough, the latter implying that $t\hat{v} \in P$, but $\hat{v}_\tau \notin \text{span}\{\bar{v}\}$. The existence of such \hat{v}_τ contradicts (28). \square

We are now in position to state the main result of this section on the convergence of constrained GN iterates.

Theorem 3.1 *Let $P \subset \mathbb{R}^p$ be closed and convex, let $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be twice differentiable near a solution \bar{u} of (1), and assume that (6) holds. Assume further that Φ is 2-regular at \bar{u} in a nonzero direction $\bar{v} \in \ker \Phi'(\bar{u})$ that is feasible for P at \bar{u} . Let there exist $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$ such that either (26) holds (which is automatic when $\dim \ker \Phi'(\bar{u}) = 1$), or (27) holds (which is automatic when $\Phi'(\bar{u}) = 0$), or assume that P is a polyhedral set and (28) holds.*

Then there exist $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ such that for every $u \in P \cap K_{\bar{\varepsilon}, \bar{\delta}}$, the GN subproblem (17) has the unique solution v^{GN} , and

$$u_1 + v_1^{GN} - \bar{u}_1 = O(\|u - \bar{u}\|^2), \quad (30)$$

$$u_2 + v_2^{GN} - \bar{u}_2 = \frac{1}{2}(u_2 - \bar{u}_2) + O(\|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^2) \quad (31)$$

as $u \rightarrow \bar{u}$.

Furthermore, for every $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, there exist $\varepsilon > 0$ and $\delta > 0$ such that for any starting point $u^0 \in K_{\varepsilon, \delta}$, the constrained GN method uniquely defines the sequence $\{u^k\}$, this sequence is contained in $K_{\hat{\varepsilon}, \hat{\delta}}$, for each k it holds that $u_2^k \neq \bar{u}_2$, $\{u^k\}$ converges to \bar{u} , $\{\|u^k - \bar{u}\|\}$ converges to zero monotonically,

$$\frac{\|u_1^{k+1} - \bar{u}_1\|}{\|u_2^{k+1} - \bar{u}_2\|} = O(\|u^k - \bar{u}\|) \quad (32)$$

as $k \rightarrow \infty$, and

$$\lim_{k \rightarrow \infty} \frac{\|u_2^{k+1} - \bar{u}_2\|}{\|u_2^k - \bar{u}_2\|} = \frac{1}{2}. \quad (33)$$

Proof The needed conclusions follow by combining Remark 3.1 and Corollary 3.1 with [15, Lemma 1, Theorem 1]. The only needed additional observation is that by [15, Remark 1], if $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ are taken small enough, then $\Phi'(u)$ is invertible for all $u \in K_{\bar{\varepsilon}, \bar{\delta}}$, implying that the quadratic objective function of the GN subproblem (17) is strongly convex, and hence, this subproblem has the unique solution v^{GN} . \square

Remark 3.2 Consider the case when $\dim \ker \Phi'(\bar{u}) = 2$. (Observe that if $p = 3$, this is the only remaining possibility for singularity, in addition to the two considered in Remark 3.1, i.e., $\dim \ker \Phi'(\bar{u}) = 1$ and $\Phi'(\bar{u}) = 0$). If (28) does

not hold, there exists a direction $\hat{v} \in \ker \Phi'(\bar{u})$ feasible for P at \bar{u} , and such that \bar{v} and \hat{v} are linearly independent. Therefore, $\text{span}\{\bar{v}, \hat{v}\} = \ker \Phi'(\bar{u})$, and due to the convexity of P and feasibility of the direction \bar{v} for P at \bar{u} , we have that $v = t(\bar{v} + \tau(\hat{v} - \bar{v})) \in P - \bar{u}$ for all $t > 0$ small enough, and all $\tau \in [0, 1]$. This implies (26) for sufficiently small $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, with \bar{v} substituted by the specified v , further implying the needed estimate (14) as $u \in K_{\varepsilon, \delta}(\bar{u}, v)$ tends to \bar{u} . Moreover, since the 2-regularity property of Φ at \bar{u} in the direction \bar{v} is stable subject to small perturbations of \bar{v} , it holds that Φ is 2-regular at \bar{u} in such direction v provided $\tau > 0$ is taken small enough, and hence, Theorem 3.1 is applicable with \bar{v} replaced by v .

Somehow surprisingly, the considerations and conclusions above do not extend to the projected Newton method defining the next iterate as $\pi_P(u + v^N)$, where π_P is the metric projection operator onto P . Specifically, and unlike for the constrained GN method, it can be seen that (12) is necessarily satisfied for the projected Newton method, while the problematic requirement is (11).

Example 3.3 Let $p = 3$, $\Phi(u) = (u_1, u_2^2/2, u_1^2/2 + u_2 u_3)$, $P = \{u \in \mathbb{R}^p \mid u_1 \geq u_3, u_1 \geq -u_3\}$. Then $\bar{u} = 0$ is the unique solution of (1), and for any $u, v \in \mathbb{R}^p$

$$\Phi'(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_2 & 0 \\ u_1 & u_3 & u_2 \end{pmatrix}, \quad \Phi'(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Phi''(0)[v] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_2 & 0 \\ v_1 & v_3 & v_2 \end{pmatrix}.$$

Therefore, Φ is 2-regular at 0 in every direction $v \in \ker \Phi'(0) = \{0\} \times \mathbb{R}^2$ with $v_2 \neq 0$, including the directions of interest $\bar{v} = (0, \pm 1, 0)$ satisfying (28).

If $u_2 \neq 0$, $u_3 = 0$, then

$$v^N = \left(-u_1, -\frac{1}{2}u_2, \frac{1}{2}\frac{u_1^2}{u_2}\right),$$

implying that

$$u + v^N = \left(0, \frac{1}{2}u_2, \frac{1}{2}\frac{u_1^2}{u_2}\right) \notin P,$$

and it can be easily seen that if, say, $u_2 > 0$, then

$$\pi_P(u + v^N) = \left(\frac{1}{4}\frac{u_1^2}{u_2}, \frac{1}{2}u_2, \frac{1}{4}\frac{u_1^2}{u_2}\right).$$

Therefore, setting

$$\omega(u) = \Phi(u) + \Phi'(u)(\pi_P(u + v^N) - u),$$

we have

$$\omega(u) = \Phi'(u)(\pi_P(u + v^N) - (u + v^N)) = \left(\frac{1}{4}\frac{u_1^2}{u_2}, 0, \frac{1}{4}u_1^2\left(\frac{u_1}{u_2} - 1\right)\right).$$

This perturbation term satisfies (12), but violates (11), and hence, (14).

At the same time, the constrained GN subproblem (17) has the unique solution

$$v^{GN} = \left(\frac{1}{2} \frac{u_1^2(u_1 + u_2)}{1 + (u_1 + u_2)^2} - u_1, -\frac{1}{2}u_2, \frac{1}{2} \frac{u_1^2(u_1 + u_2)}{1 + (u_1 + u_2)^2} \right),$$

and (18) yields

$$\omega^{GN}(u) = \left(\frac{1}{2} \frac{u_1^2(u_1 + u_2)}{1 + (u_1 + u_2)^2}, 0, -\frac{1}{2} \frac{u_1^2}{1 + (u_1 + u_2)^2} \right),$$

implying (14) for $\omega(\cdot) = \omega^{GN}(\cdot)$.

4 Constrained Levenberg–Marquardt and LP-Newton methods

For a given $u \in \mathbb{R}^p$ satisfying $\sigma(u) > 0$, the constrained LM subproblem

$$\text{minimize } \frac{1}{2} \|\Phi(u) + \Phi'(u)v\|^2 + \frac{1}{2} \sigma(u) \|v\|^2 \text{ subject to } u + v \in P, \quad (34)$$

has the unique solution v^{LM} , and we need to estimate

$$\omega^{LM}(u) = \Phi(u) + \Phi'(u)v^{LM}. \quad (35)$$

Under the assumptions of Theorem 3.1, let $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ be chosen accordingly, so that for $u \in P \cap K_{\bar{\varepsilon}, \bar{\delta}}$, the GN subproblem (17) has a solution v^{GN} , and any such solution satisfies the estimates (30)–(31). From (34)–(35), taking into account that v^{GN} is feasible in problem (17), and hence also in (34), we then obtain that

$$\begin{aligned} \|\omega^{LM}(u)\|^2 &\leq \|\Phi(u) + \Phi'(u)v^{LM}\|^2 + \sigma(u) \|v^{LM}\|^2 \\ &\leq \|\Phi(u) + \Phi'(u)v^{GN}\|^2 + \sigma(u) \|v^{GN}\|^2 \\ &= \|\omega^{GN}(u)\|^2 + \sigma(u) \|v^{GN}\|^2 \\ &= \|\omega^{GN}(u)\|^2 + O(\sigma(u) \|u - \bar{u}\|^2), \end{aligned} \quad (36)$$

where the next-to-the-last equality is by (18), while the last one is by (30)–(31).

We note that typical choices of the function $\sigma(\cdot)$ satisfy $\sigma(u) = O(\|\Phi(u)\|^\tau)$ with some exponent $\tau > 0$. Observe that

$$\Phi(u) = O(\|u_1 - \bar{u}_1\|) + O(\|u - \bar{u}\|^2). \quad (37)$$

Observe also that 2-regularity of Φ in the direction \bar{v} implies that by taking $\bar{\varepsilon} > 0$ and $\bar{\delta} > 0$ small enough, one can ensure that \bar{u} is the only solution of the equation in (1) within $K_{\bar{\varepsilon}, \bar{\delta}}$. Indeed, if there exists a sequence $\{u^k\} \subset \mathbb{R}^p \setminus \{\bar{u}\}$ such that $\{u^k\} \rightarrow \bar{u}$, $\{(u^k - \bar{u})/\|u^k - \bar{u}\|\} \rightarrow \bar{v}/\|\bar{v}\|$, and $\Phi(u^k) = 0$ for every k , by the argument in [16, p. 497] we obtain a contradiction with the assumptions

that Φ is 2-regular at \bar{u} in the direction \bar{v} . Therefore, $\sigma(u) > 0$ holds for all $u \in K_{\varepsilon, \delta}$, $u \neq \bar{u}$.

Furthermore, according to Corollary 3.1, if $\bar{\delta} > 0$ is taken small enough, then (14) holds for $\omega(\cdot) = \omega^{GN}(\cdot)$ as $u \rightarrow \bar{u}$, and hence, (36) implies that it also holds for $\omega(\cdot) = \omega^{LM}(\cdot)$ if $\tau \geq 2$. We thus arrive at the following statement.

Corollary 4.1 *Under the assumptions of Theorem 3.1, for every $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, and every $\tau \geq 2$, there exist $\varepsilon > 0$ and $\delta > 0$ such that for any starting point $u^0 \in K_{\varepsilon, \delta}$, the constrained LM method uniquely defines the sequence $\{u^k\}$, this sequence is contained in $K_{\hat{\varepsilon}, \hat{\delta}}$, for each k it holds that $u_2^k \neq \bar{u}_2$, $\{u^k\}$ converges to \bar{u} , $\{\|u^k - \bar{u}\|\}$ converges to zero monotonically, and (32)–(33) hold.*

Consider now the LPN subproblem

$$\begin{aligned} &\text{minimize } \gamma \text{ subject to } \|\Phi(u) + \Phi'(u)v\| \leq \gamma\|\Phi(u)\|^2, \\ &\quad \|v\| \leq \gamma\|\Phi(u)\|, \\ &\quad u + v \in P. \end{aligned} \tag{38}$$

This subproblem is evidently solvable provided $\Phi(u) \neq 0$, and for its arbitrary solution v^{LPN} , we need to estimate the perturbation term

$$\omega^{LPN}(u) = \Phi(u) + \Phi'(u)v^{LPN}. \tag{39}$$

From (38) (and specifically, from its first constraint), and from (39), we have that

$$\|\omega^{LPN}(u)\| \leq \gamma(u)\|\Phi(u)\|^2, \tag{40}$$

where $\gamma(u)$ is the optimal value of (38). Taking again into account that v^{GN} is feasible in problem (17), and hence, satisfies the last constraint in (38), and defining

$$\gamma_{GN} = \max \left\{ \|\Phi(u) + \Phi'(u)v^{GN}\|/\|\Phi(u)\|^2, \|v^{GN}\|/\|\Phi(u)\| \right\},$$

we observe that the pair $(v, \gamma) = (v^{GN}, \gamma_{GN})$ is feasible in (38). Therefore,

$$\gamma(u) \leq \gamma_{GN}.$$

Then, by (40), we conclude that

$$\begin{aligned} \|\omega^{LPN}(u)\| &\leq \gamma_{GN}\|\Phi(u)\|^2 \\ &= \max\{\|\Phi(u) + \Phi'(u)v^{GN}\|, \|\Phi(u)\|\|v^{GN}\|\} \\ &\leq \max\{\|\omega^{GN}(u)\|, O(\|\Phi(u)\|\|u - \bar{u}\|)\} \end{aligned}$$

as $u \rightarrow \bar{u}$, where the last inequality is by (18) and (30)–(31). Employing (14) with $\omega(\cdot) = \omega^{GN}(\cdot)$, and (37), we then conclude that (14) holds with $\omega(\cdot) = \omega^{LPN}(\cdot)$ as well. This establishes the following statement.

Corollary 4.2 *Under the assumptions of Theorem 3.1, for every $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, there exist $\varepsilon > 0$ and $\delta > 0$ such that for any starting point $u^0 \in K_{\varepsilon, \delta}$, the LPN method defines a sequence $\{u^k\}$, any such sequence is contained in $K_{\hat{\varepsilon}, \hat{\delta}}$, for each k it holds that $u_2^k \neq \bar{u}_2$, $\{u^k\}$ converges to \bar{u} , $\{\|u^k - \bar{u}\|\}$ converges to zero monotonically, and (32)–(33) hold.*

An interesting observation, which in the unconstrained case goes back to [15], is that unlike Theorem 3.1 and Corollary 4.1, in Corollary 4.2 we do not claim that the iterative sequence $\{u^k\}$ is unique. The reason is that unlike for the LM method, the LPN subproblem may have multiple solutions v^{LPN} , while the perturbation term $\omega^{LPN}(\cdot)$ is defined in (39) in a posteriori manner, and can be different for different v^{LPN} .

5 A smooth constrained reformulation of the nonlinear complementarity problem

In this section, we illustrate some of the results obtained above for the constrained GN method applied to a reformulation of the nonlinear complementarity problem (NCP, [6])

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0, \quad (41)$$

with a smooth mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

One of the approaches to solving NCP consists of introducing the slack variable $y \in \mathbb{R}^n$ and reformulating (41) as the constrained equation (1) with $p = 2n$, $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ defined by

$$\Phi(u) = (F(x) - y, x \circ y), \quad (42)$$

where $u = (x, y)$, and with $P = \mathbb{R}_+^p$. Evidently, any $\bar{x} \in \mathbb{R}^n$ is a solution of NCP (41) if and only if $\bar{u} = (\bar{x}, F(\bar{x}))$ is a solution of (1) with Φ and P specified above.

Illustrations and interpretations for NCP that follow, complement those in [9, Section 4], where the direction of 2-regularity of Φ at \bar{u} was required to be in $\text{int } T_P(\bar{u})$. Here, we focus on the situations where there exist no 2-regularity directions in $\text{int } T_P(\bar{u})$, and so [9] is not applicable, but there exist such directions in $T_P(\bar{u})$.

For a given NCP solution \bar{x} define the partition $(I_>, I_=>, I_<)$ of the index set $\{1, \dots, n\}$ as follows:

$$\begin{aligned} I_> &= I_>(\bar{u}) = \{i \in \{1, \dots, n\} \mid \bar{x}_i > F_i(\bar{x}) = 0\}, \\ I_=> &= I_=(\bar{u}) = \{i \in \{1, \dots, n\} \mid \bar{x}_i = 0 = F_i(\bar{x})\}, \\ I_< &= I_<(\bar{u}) = \{i \in \{1, \dots, n\} \mid 0 = \bar{x}_i < F_i(\bar{x})\}. \end{aligned}$$

From (42) we have that

$$\Phi'(\bar{u}) = \begin{pmatrix} F'(\bar{x}) & -\mathcal{I} \\ \text{diag } F(\bar{x}) & \text{diag } \bar{x} \end{pmatrix}, \quad (43)$$

and hence,

$$\ker \Phi'(\bar{u}) = \left\{ v = (\xi, \eta) \left| \begin{array}{l} \frac{\partial F_{I_>}}{\partial x_{I_> \cup I_=}}(\bar{x}) \xi_{I_> \cup I_=} = 0, \\ \frac{\partial F_{I_= \cup I_<}}{\partial x_{I_> \cup I_=}}(\bar{x}) \xi_{I_> \cup I_=} = \eta_{I_= \cup I_<}, \\ \xi_{I_<} = 0, \eta_{I_>} = 0 \end{array} \right. \right\}. \quad (44)$$

This implies that a solution \bar{u} is singular if and only if $I_= \neq \emptyset$, or $I_> \neq \emptyset$ and $\frac{\partial F_{I_>}}{\partial x_{I_>}}(\bar{x})$ is a singular matrix. In particular, $\dim \ker \Phi'(\bar{u}) = 1$ holds if $I_= = \emptyset$, $I_> \neq \emptyset$, and $\text{rank } \frac{\partial F_{I_>}}{\partial x_{I_>}}(\bar{x}) = |I_>| - 1$, or if $|I_=| = 1$ and either $I_> = \emptyset$ or $\text{rank } \frac{\partial F_{I_>}}{\partial x_{I_> \cup I_=}}(\bar{x}) = |I_>|$.

According to (44), $\bar{v} = (\bar{\xi}, \bar{\eta})$ belongs to $\ker \Phi'(\bar{u})$ if and only if $\bar{\xi}$ satisfies

$$\frac{\partial F_{I_>}}{\partial x_{I_> \cup I_=}}(\bar{x}) \bar{\xi}_{I_> \cup I_=} = 0, \quad \bar{\xi}_{I_<} = 0, \quad (45)$$

while $\bar{\eta}$ is defined by

$$\bar{\eta}_{I_>} = 0, \quad \bar{\eta}_{I_= \cup I_<} = \frac{\partial F_{I_= \cup I_<}}{\partial x_{I_> \cup I_=}}(\bar{x}) \bar{\xi}_{I_> \cup I_=}. \quad (46)$$

As demonstrated in [9], for any such \bar{v} , 2-regularity of Φ at \bar{u} in the direction \bar{v} means that there exists no nonzero $v = (\xi, \eta) \in \ker \Phi'(\bar{u})$ satisfying

$$\frac{\partial^2 F_{I_>}}{\partial x_{I_> \cup I_=}^2}(\bar{x}) [\bar{\xi}_{I_> \cup I_=}, \xi_{I_> \cup I_=}] \in \text{im } \frac{\partial F_{I_>}}{\partial x_{I_> \cup I_=}}(\bar{x}), \quad (47)$$

$$\bar{\eta}_{I_=} \circ \xi_{I_=} + \bar{\xi}_{I_=} \circ \eta_{I_=} = 0.$$

Taking into account (44) and (46), this is equivalent to saying that there exists no $\xi \neq 0$ satisfying

$$\frac{\partial F_{I_>}}{\partial x_{I_> \cup I_=}}(\bar{x}) \xi_{I_> \cup I_=} = 0, \quad \xi_{I_<} = 0, \quad (48)$$

and such that (47) holds, and

$$\frac{\partial F_{I_=}}{\partial x_{I_> \cup I_=}}(\bar{x}) \bar{\xi}_{I_> \cup I_=} \circ \xi_{I_=} + \bar{\xi}_{I_=} \circ \frac{\partial F_{I_=}}{\partial x_{I_> \cup I_=}}(\bar{x}) \xi_{I_> \cup I_=} = 0. \quad (49)$$

Furthermore, from the definition of P , and from (46), it evidently follows that the feasibility of the direction \bar{v} for P at \bar{u} means that

$$\bar{\xi}_{I_= \cup I_<} \geq 0, \quad \frac{\partial F_{I_=}}{\partial x_{I_> \cup I_=}}(\bar{x}) \bar{\xi}_{I_> \cup I_=} \geq 0. \quad (50)$$

In particular, if $\dim \ker \Phi'(\bar{u}) = 1$, then the 2-regularity condition in a direction $\bar{v} = (\bar{\xi}, \bar{\eta})$ spanning $\ker \Phi'(\bar{u})$ means violation of at least one of the relations

$$\begin{aligned} \frac{\partial^2 F_{I_{>}}}{\partial x_{I_{> \cup I_{=}}}^2}(\bar{x})[\bar{\xi}_{I_{> \cup I_{=}}}, \bar{\xi}_{I_{> \cup I_{=}}}] &\in \text{im } \frac{\partial F_{I_{>}}}{\partial x_{I_{> \cup I_{=}}}}(\bar{x}) \\ \text{or} \\ \bar{\xi}_{I_{=}} \circ \frac{\partial F_{I_{=}}}{\partial x_{I_{> \cup I_{=}}}}(\bar{x})\bar{\xi}_{I_{> \cup I_{=}}} &= 0, \end{aligned} \quad (51)$$

coming from (47)–(49) with ξ substituted with $\bar{\xi}$.

To go beyond the case when $\dim \ker \Phi'(\bar{u}) = 1$, observe first that here $\text{int}(P \cap K_{\hat{\varepsilon}, \hat{\delta}}) \neq \emptyset$ for any $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$, and hence, (27) may only hold if $\Phi'(\bar{u}) = 0$. However, the latter never holds, according to (43), and hence, condition (27) may never hold in this context.

Furthermore, condition (26) holds with $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$ small enough if and only if for any $\xi \in \mathbb{R}^n$ close enough to $\bar{\xi}$ the equalities in (48) imply the inequalities

$$\xi_{I_{= \cup I_{<}}} \geq 0, \quad \frac{\partial F_{I_{=}}}{\partial x_{I_{> \cup I_{=}}}}(\bar{x})\xi_{I_{> \cup I_{=}}} \geq 0, \quad (52)$$

that is, (50) with ξ substituted for $\bar{\xi}$. Finally, condition (28) consists of saying that there exists no $\xi \in \mathbb{R}^n$ linearly independent with $\bar{\xi}$, and satisfying (48) and (52).

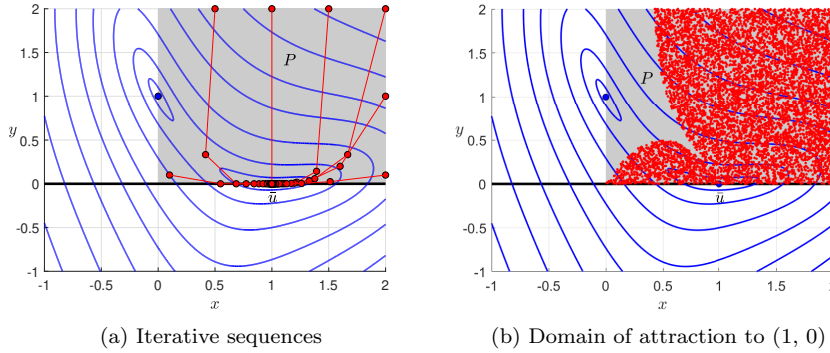


Fig. 2: The constrained GN method for NCP with $F(x) = (x - 1)^2$.

Example 5.1 Let $n = 1$. If $I_{>} \neq \emptyset$, then there are two possibilities: if $F'(\bar{x}) \neq 0$, solution \bar{u} is nonsingular, while if $F'(\bar{x}) = 0$, it holds that $\dim \ker \Phi'(\bar{u}) = 1$, and from (45)–(46) it follows that $\ker \Phi'(\bar{u})$ is spanned by $\bar{v} = (\bar{\xi}, 0)$ with any $\bar{\xi} \neq 0$. Since (50) is vacuous, any such \bar{v} is feasible for P at \bar{u} , but $\bar{v} \notin \text{int } T_P(\bar{u})$ whatever is taken as $\bar{\xi}$. Moreover, condition (47) reduces to $F''(\bar{x})\bar{\xi}\bar{\xi} = 0$, while (48)–(49) are vacuous, and hence, Φ is 2-regular at \bar{u} in such directions \bar{v} if

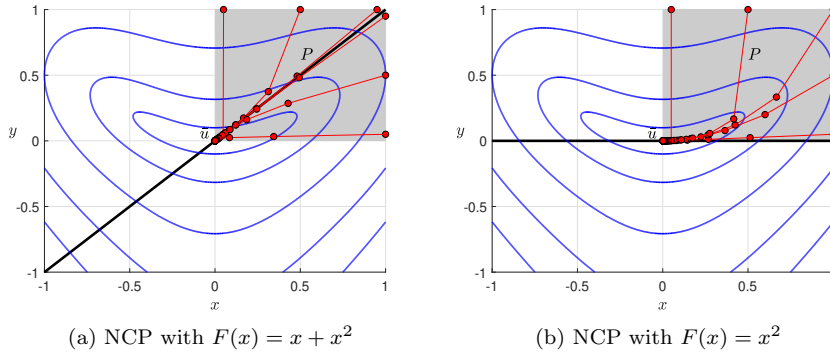


Fig. 3: Iterative sequences of the constrained GN method.

and only if $F''(\bar{x}) \neq 0$. Therefore, Theorem 3.1 and Corollaries 4.1–4.2 are applicable in this case with the specified \bar{v} provided $F'(\bar{x}) = 0$, $F''(\bar{x}) \neq 0$.

The case in question is illustrated in Figure 2 for a solution $\bar{x} = 1$ of NCP (41) with $F(x) = (x - 1)^2$. Horizontal thick straight line corresponds to $\ker \Phi'(\bar{u})$ with Φ defined in (42), where $\bar{u} = (1, 0)$. We also show the contours of the residual of the equation in (1). Figure 2a demonstrates some iterative sequences of the constrained GN method, while Figure 2b presents the collection of starting points from where convergence to the solution \bar{u} has been detected.

If $I_+ \neq \emptyset$, then solution $\bar{u} = 0$ is necessarily singular, $\dim \ker \Phi'(0) = 1$, and from (45)–(46) it follows that $\ker \Phi'(0)$ is spanned by $\bar{v} = (\bar{\xi}, F'(0)\bar{\xi})$ with any $\bar{\xi} \neq 0$. Conditions (47)–(48) are vacuous, while (49) reduces to $F'(0)\bar{\xi} \geq 0$, and hence, Φ is 2-regular at \bar{u} in such directions \bar{v} if and only if $F'(0) \neq 0$. Conditions (50) characterizing feasibility of \bar{v} for P at 0 reduce to $\bar{\xi} \geq 0$, $F'(0) \geq 0$. In particular if $F'(0) > 0$, then $\bar{v} \in \text{int } P$ provided $\bar{\xi} > 0$, the case studied in [9]. If $F'(0) = 0$, then $\bar{v} \notin \text{int } P$ whatever is taken as $\bar{\xi}$. However, as figured out above, Φ cannot be 2-regular at 0 in such directions \bar{v} in this case.

The case when $I_- \neq \emptyset$ is illustrated in Figure 3 delivering the same kind of information as in Figure 2a, for the unique solution $\bar{x} = 0$ of NCP (41) with $F(x) = x + x^2$ in Figure 3a, and with $F(x) = x^2$ in Figure 3b. The meaning of thick straight lines and the contours is the same as in Figure 2, for the solution $\bar{u} = (0, 0)$ of the equation in (1) with Φ defined in (42). In Figure 3b, convergence to the solution in question is still observed, it appears to be along the direction spanning the null space of the Jacobian, and the rate of convergence is linear but with the asymptotic ratio greater than $1/2$.

In the remaining case when $I_< \neq \emptyset$ solution \bar{u} is nonsingular, like point $(0, 1)$ in Figure 2.

Example 5.2 Let $n = 2$. Then there are essentially six possibilities (up to reordering of components).

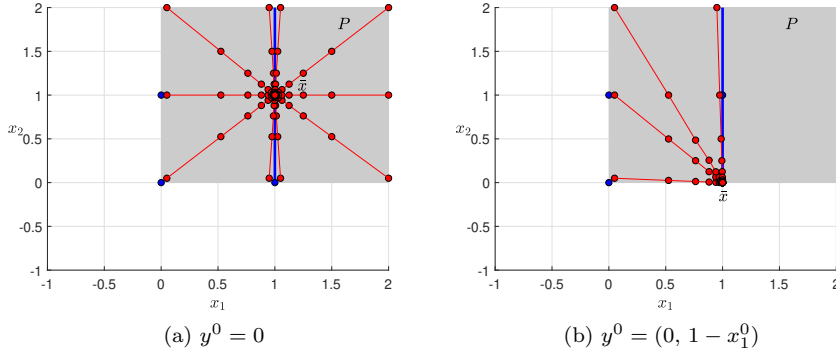


Fig. 4: Iterative sequences of the constrained GN method for NCP with $F(x) = ((x_1 - 1)^2, (x_1 - 1)(x_2 - 1))$.

If $I_> = \{1, 2\}$, then solution \bar{u} is singular if and only if $F'(\bar{x})$ is singular. In this case, it holds that $\dim \ker \Phi'(\bar{u}) = 1$ provided $\text{rank } F'(\bar{x}) = 1$, and from (45)–(46) it follows that $\ker \Phi'(\bar{u}) = \mathbb{R}^2 \times \{0\}$ provided $F'(\bar{x}) = 0$. Since (50) is vacuous, any $\bar{v} = (\bar{\xi}, 0)$ is feasible for P at \bar{u} , but $\bar{v} \notin \text{int } T_P(\bar{u})$ whatever is taken as $\bar{\xi}$. Moreover, condition (49) is vacuous, while violation of (47)–(48) for any $\xi \neq 0$ amounts to 2-regularity of F at \bar{x} in the direction $\bar{\xi}$, and therefore, the latter characterizes 2-regularity of Φ at \bar{u} in such directions \bar{v} . Finally, if $F'(\bar{x}) = 0$, then all relations in (48) and (52) are either automatic or vacuous, and hence, (28) cannot be valid, while (26) is satisfied with $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$ small enough (one can directly verify that actually, this is true with any $\hat{\delta} > 0$). Therefore, Theorem 3.1 and Corollaries 4.1–4.2 are applicable in this case with any specified \bar{v} such that F is 2-regular at \bar{x} in the direction $\bar{\xi}$.

Figure 4 demonstrates the x -parts of some iterative sequences of the constrained GN method for NCP (41) with $F(x) = ((x_1 - 1)^2, (x_1 - 1)(x_2 - 1))$. This NCP has two nonsingular solutions $(0, 0)$ and $(0, 1)$, and nonisolated solutions of the form $(1, t)$, $t \geq 0$. Among the latter, $(1, 1)$ and $(1, 0)$ play a special role since the corresponding solutions of the equation in (1) with Φ defined in (42) are critical in the sense of [16]. In Figure 4a, we use the starting value y^0 of the slack variable equal to $(0, 0)$, and this results in convergence to the solution $\bar{x} = (1, 1)$ of NCP (41), where $I_> = \{1, 2\}$, $F'(\bar{x}) = 0$, and F is 2-regular at \bar{x} in directions $\bar{\xi}$ with $\xi_1 \neq 0$.

If $I_> = \{1\}$, $I_< = \{2\}$, then $\dim \ker \Phi'(\bar{u}) \leq 1$.

If $I_< = \{1, 2\}$, then solution \bar{u} is nonsingular.

If $I_> = \{1\}$, $I_< = \{2\}$, then the solution \bar{u} is always singular, and it holds that $\dim \ker \Phi'(\bar{u}) = 1$ if and only if $F'_1(\bar{x}) \neq 0$. If $F'_1(\bar{x}) = 0$, then from (44) it follows that $\ker \Phi'(\bar{u}) = \{v = (\xi, \eta) \mid \eta_1 = 0, \eta_2 = \langle F'_2(\bar{x}), \xi \rangle\}$, and in particular, $\dim \ker \Phi'(\bar{u}) = 2$. Condition (50) characterizing feasible directions $\bar{v} \in \ker \Phi'(\bar{u})$ takes the form $\bar{\xi}_2 \geq 0$, $\langle F'_2(\bar{x}), \bar{\xi} \rangle \geq 0$, but $\bar{v} \notin \text{int } T_P(\bar{u})$ whatever

is taken as $\bar{\xi}$. Condition (48) is vacuous, while (47) and (49) amount to

$$\langle F_1''(\bar{x})\bar{\xi}, \xi \rangle = 0 \quad \text{and} \quad \langle F_2'(\bar{x}), \bar{\xi} \rangle \xi_2 + \bar{\xi}_2 \langle F_2'(\bar{x}), \xi \rangle = 0,$$

respectively. Therefore, 2-regularity of Φ at \bar{u} in such direction \bar{v} amounts to nonsingularity of the matrix

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x_1^2}(\bar{x})\bar{\xi}_1 + \frac{\partial^2 F_1}{\partial x_1 \partial x_2}(\bar{x})\bar{\xi}_2 & \frac{\partial^2 F_1}{\partial x_1 \partial x_2}(\bar{x})\bar{\xi}_1 + \frac{\partial^2 F_1}{\partial x_2^2}(\bar{x})\bar{\xi}_2 \\ \frac{\partial F_2}{\partial x_1}(\bar{x})\bar{\xi}_2 & \frac{\partial F_2}{\partial x_1}(\bar{x})\bar{\xi}_1 + 2\frac{\partial F_2}{\partial x_2}(\bar{x})\bar{\xi}_2 \end{pmatrix}. \quad (53)$$

Furthermore, in the case when $\bar{\xi}_2 > 0$ and $\langle F_2'(\bar{x}), \bar{\xi} \rangle > 0$, both inequalities in (52) do hold for ξ close enough to $\bar{\xi}$, and hence, condition (26) is satisfied with $\hat{\varepsilon} > 0$ and $\hat{\delta} > 0$ small enough. In the cases when $\bar{\xi}_2 = 0$ and $\langle F_2'(\bar{x}), \bar{\xi} \rangle > 0$, or $\bar{\xi}_2 > 0$ and $\langle F_2'(\bar{x}), \bar{\xi} \rangle = 0$, there always exists ξ arbitrarily close to $\bar{\xi}$, violating (52), while on the other hand, there always exists ξ (obtained, e.g., by arbitrarily small perturbation of $\bar{\xi}$) linearly independent with $\bar{\xi}$, satisfying (52). Therefore, neither (26) nor (28) can be satisfied in this case. Finally, in the case when $\bar{\xi}_2 = 0$ and $\langle F_2'(\bar{x}), \bar{\xi} \rangle = 0$, condition (28) may hold (it holds if $\frac{\partial F_2}{\partial x_1}(\bar{x}) = 0$ and $\frac{\partial F_2}{\partial x_2}(\bar{x}) < 0$), but the matrix in (53) is necessarily singular in this case, and hence, Φ cannot be 2-regular at \bar{u} in such directions \bar{v} .

In Figure 4b, we use the starting value y^0 of the slack variable equal to $(0, 1 - x_1^0)$, and this results in convergence to the solution $\bar{x} = (1, 0)$ of NCP (41), where $I_> = \{1\}$, $I_- = \{2\}$. Observe that we only take x^0 with $x_1^0 < 1$, as for $x_1^0 > 1$, the corresponding starting point $u^0 = (x^0, y^0)$ does not belong to P .

If $I_- = \{1\}$, $I_< = \{2\}$, then solution \bar{u} is singular, and $\dim \ker \Phi'(\bar{u}) = 1$.

In the remaining case when $I_- = \{1, 2\}$ solution $\bar{u} = 0$ is singular, and from (44) it follows that $\ker \Phi'(0) = \{v = (\xi, \eta) \mid \eta = F'(\bar{x})\xi\}$, implying, in particular, that $\dim \ker \Phi'(0) = 2$. Condition (50) characterizing feasible directions $\bar{v} \in \ker \Phi'(0)$ takes the form $\bar{\xi} \geq 0$, $F'(0)\bar{\xi} \geq 0$. If both these inequalities are strict, then $\bar{v} \in \text{int } P$ holds provided $\bar{\xi}$ is close enough to 0, the case studied in [9]. Moreover, if there exists $\hat{\xi} > 0$ with $F'(0)\hat{\xi} > 0$ (which according to Gordan's theorem of the alternatives [21] is equivalent to saying that there exists no nonzero $\eta \geq 0$ such that $(F'(0))^\top \eta \leq 0$), then neither (26) nor (28) can hold for $\bar{\xi} \geq 0$ with $F'(0)\bar{\xi} \geq 0$ provided any of these inequalities holds as equality for at least one component. Therefore, we now consider the case when such $\hat{\xi}$ does not exist. Observe that this also evidently excludes the cases when $\bar{\xi} \geq 0$, $F'(0)\bar{\xi} > 0$, and when $\bar{\xi} > 0$, $F'(0)\bar{\xi} \geq 0$, and $F'(0)\bar{\xi} > 0$ for some $\bar{\xi}$ (where the latter is equivalent to saying that there exists no nonzero $\eta \geq 0$ such that $(F'(0))^\top \eta = 0$).

Conditions (47)–(48) are vacuous, while (49) amounts to

$$\langle F_1'(0), \bar{\xi} \rangle \xi_1 + \bar{\xi}_1 \langle F_1'(0), \xi \rangle = 0, \quad \langle F_2'(0), \bar{\xi} \rangle \xi_2 + \bar{\xi}_2 \langle F_2'(0), \xi \rangle = 0.$$

Therefore, 2-regularity of Φ at \bar{u} in such direction \bar{v} means nonsingularity of the matrix

$$\begin{pmatrix} 2\frac{\partial F_1}{\partial x_1}(0)\bar{\xi}_1 + \frac{\partial F_1}{\partial x_2}(0)\bar{\xi}_2 & \frac{\partial F_1}{\partial x_2}(0)\bar{\xi}_1 \\ \frac{\partial F_2}{\partial x_1}(0)\bar{\xi}_2 & \frac{\partial F_2}{\partial x_1}(0)\bar{\xi}_1 + 2\frac{\partial F_2}{\partial x_2}(0)\bar{\xi}_2 \end{pmatrix},$$

and the latter requirement can be written as

$$\frac{\partial F_1}{\partial x_1}(0)\bar{\xi}_1 \langle F'_2(0), \bar{\xi} \rangle + \langle F'_1(0), \bar{\xi} \rangle \frac{\partial F_2}{\partial x_2}(0)\bar{\xi}_2 \neq 0. \quad (54)$$

If $\bar{\xi}_1 = 0$, $\bar{\xi}_2 > 0$, then conditions $F'(0)\bar{\xi} \geq 0$ and (54) reduce to $\frac{\partial F_1}{\partial x_2}(0) > 0$, $\frac{\partial F_2}{\partial x_2}(0) > 0$. This implies that $F'(0)\bar{\xi} > 0$, an excluded case. If $\bar{\xi} > 0$, $\langle F'_1(0), \bar{\xi} \rangle = 0$, then conditions $F'(0)\bar{\xi} \geq 0$ and (54) reduce to $\frac{\partial F_1}{\partial x_1}(0) \neq 0$, $\langle F'_2(0), \bar{\xi} \rangle > 0$. Then setting $\tilde{\xi} = (\bar{\xi}_1 + t\frac{\partial F_1}{\partial x_1}(0), \bar{\xi}_2)$, for any $t > 0$ small enough we have that $F'(0)\tilde{\xi} > 0$, which again leads to an excluded case.

6 Concluding remarks

For constrained nonlinear equations with singular/nonisolated solutions, we investigated convergence of some Newton-type methods in the situation not covered by any of the previous theories. Specifically, we considered the case when there exists a direction in the null space of the Jacobian at a solution for which the equation mapping is 2-regular at this solution, and this direction being in the tangent cone to the constraint set but not necessarily being interior feasible. Convergence and linear rate of convergence from large (not asymptotically thin) domains was shown for the constrained Gauss-Newton and Levenberg–Marquardt methods, and for the LP-Newton method, under some additional assumptions.

We conclude this discussion by mentioning that this work does not provide a full understanding about whether some of those additional assumptions cannot be avoided. In particular, it remains an open question whether the needed conclusion can be derived in case of a polyhedral P without assuming (28). Observe that according to Remarks 3.1 and 3.2, a counterexample for the estimate (14) being valid with some (and hence, almost every) \bar{v} possessing the needed properties can only be possible if $p \geq 4$ and $3 \leq \dim \ker \Phi'(\bar{u}) \leq p - 1$. Full clarification of those issues should be a subject of future research.

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