Analysis of Energy Markets Modeled as Equilibrium Problems with Equilibrium Constraints

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July 16, 2020

Abstract

Equilibrium problems with equilibrium constraints are challenging both theoretically and computationally. However, they are suitable/adequate modeling formulations in a number of important areas, such as energy markets, transportation planning, and logistics. Typically, these problems are characterized as bilevel Nash-Cournot games. For instance, determining the equilibrium price in an energy market involves top-level decisions of the generators that feed the bottom-level problem of an independent system operator. Taking the Karush-Kuhn-Tucker conditions of the lower-level optimization problems and inserting them into each upper-level player's problem is one popular approach, but it has both numerical and theoretical difficulties. To tackle the resulting highly nonlinear model we propose a primal-dual regularization with the remarkable property of yielding equilibrium prices of minimal norm. This theoretical feature can be seen as a stabilizing mechanism for price signals. It proves also useful in guiding the solution process when solving such problems computationally, via the mixed complementarity formulations. For a general energy market model we prove existence theorems for a specific equilibrium, and convergence of the proposed regularization scheme. Our numerical results using the PATH solver illustrate the proposal. In particular, we exhibit that, in the given context, PATH with the regularization approach computes a genuine equilibrium almost always, while without regularization the outcome is quite the opposite.

1 Introduction

The issue of obtaining clearing prices in energy markets is a topic of intensive research in many levels, from economic modeling to actual calculations in practice. The excellent surveys [Jos08] and [Bub+19], covering more than thirty years of work on the subject, confirm that generating adequate and clear-to-interpret price signals is fundamental for the energy sector, to ensure the financing of operational and investment costs; see also [Sto02]. For a recent computational proposal in this direction (in one specific setting), see [LSS19]. Both [New17] and [BLB18] point out the need to continue to evolve in the design of energy markets, to achieve cost-effective policies for resource adequacy. This task is by no means straightforward, as illustrated by the many different mechanisms that have been proposed to provide capacity remuneration and ensure future investment [CF07], [Wil10], [COS13]. In the past years, traditional concerns mentioned in [Ore05], such as market welfare, generation adequacy, and security of supply, have been complemented with targets of renewables integration and sustainability; see [Hog17]. Furthermore, the world-wide trend in incentivizing flexibility added another complicating layer to the problem, particularly regarding the operation of the network.

One goal of energy markets is to provide market-based price signals for energy operations and investments– important for market flexibility. Typically, prices can be realistically modeled as the output of an equilibrium problem involving the interaction between the system operator and the agents. A resulting equilibrium price later on may be compounded with compensations that the regulator determines to correct market distortions, protecting consumers from abuses of market power [Sto02] and agents from missing-money problems [New16].

From a modeling point of view, initial mathematical programming models [GM85] evolved to more sophisticated equilibrium problems, cast as Nash games or complementarity formulations [HMP00], [Gab+12]; see also [MPS19] and [FMP01]. The research on equilibrium models for energy systems is very rich. We mention a few works that show the large spectrum of related literature. Market-power issues and bidding strategies for the Brazilian power system were investigated in [KBP01], and [Cru+16]. Bilevel game-theoretic models for electricity markets are discussed in [HR07], particularly regarding the effect of congestion and network transmission on the existence of equilibria; see also [DHP02]. In [GL10], a bilevel approach with potentially discrete variables is presented with an application to the stylized Benelux power network using a mathematical program with equilibrium constraints formulation varying market leaders. Other theoretical studies are [Ehr04] and [EJ06], analyzing multi-leader Cournot equilibria, and Walrasian and non-cooperative Nash games, respectively. Economic implications with respect to market design can be found in [EN09] and [Hu+04]. In [LSS13] and [LSS16] stochastic games and relations with complementarity models are explored, for risk-neutral and risk-averse agents. For more studies on the impact of risk models on mathematical optimization problems with equilibrium constraints, we refer to [PFW16] and [FP18]. Finally, [BGB19] considers bilevel optimization for analyzing investments in power markets and develops several algorithms to efficiently solve the resulting equilibrium problem.

As commented in [Bub+19], given the unusual characteristics of electricity generation and transmission, the efficient allocation of resources in an energy market is a highly challenging task. Our work confirms this assertion by pinpointing pitfalls that arise at the very basis of the pricing mechanism, namely when solving equilibrium problems to define a price signal. For such a market we consider a Nash equilibrium problem with equilibrium constraints (EPEC) that arises when agents optimize their revenue in a bilevel setting. Specifically, in the upper level each agent decides both the bidding price and the corresponding generation, subject to operational constraints and a shared constraint ensuring the dispatch clears the market. The optimal dispatch is defined in the lower level by the independent system operator (ISO), who solves the same problem for all the agents, taking into account demand satisfaction and other system-wide constraints. For a general EPEC model of this type, we develop a theoretical study identifying configurations leading to equilibrium prices that are larger than the maximal dispatched bid. This undesirable situation may induce agents to manipulate their offers in order to reach those critical configurations. Another problematic aspect refers to *lack of uniqueness* of the price signal: if the mathematical model does not provide a unique answer, the pricing policy is not well-defined. Under such circumstances, reproducibility becomes an issue too, as different solvers or different computational methods may produce different output, and find different prices. As stated in Proposition 1 below, even in the more simple instances, an interval of equilibrium prices is not unusual for some market configurations.

The drawbacks mentioned above are inherent to EPEC models. Yet, formulating the equilibrium problem in a bilevel setting represents the market behavior well. In order to mitigate the model downsides, we introduce a sequence of approximating EPECs, depending on certain regularization parameters. The novel approach, based on a scheme that controls the largest marginal rent, ensures that in the limit the *minimal norm* price signal will be found, see Theorem 5 below. Our analysis of the ISO lower-level problem complements a primal view with a dual perspective that provides an interesting economic interpretation. With our approach the ISO maintains controlled the agents' marginal rent by having access to certain reserve of energy; see Section 4.

The rest of this work is organized as follows. In Section 2 we introduce some notation and the considered EPEC model, a generalized Nash game with bilinear constraints, resulting from including the optimality conditions for the ISO problem in the upper-level agents' problems. Section 3 presents our new regularized formulation that discourages the undesirable situation in which, rather than choosing the minimum possible value for the marginal price, the highest one is chosen by the EPEC model. In Section 4, by means of Convex Analysis techniques, the material introduced in Section 3 is interpreted from the perspective of the ISO, adopting a primal point of view. The analysis is complemented in Section 5. with numerical tests solving a mixed complementarity system that approximates the regularized EPEC model. To illustrate the type of difficulties that must be addressed when solving EPECs, the section also includes several numerical experiments with PATH solver [DF95] using GAMS. The benchmark provides insights on how the approach helps in obtaining solutions to the complementarity system that are equilibria for the EPEC. In the last section, concluding remarks are provided.

2 Main features considered for the energy market

In what follows, we introduce the notation for the static deterministic EPECs described in this paper. We note that, because of the inherent difficulties already present in our stylized model, neither dynamic relations nor uncertainty are included in the study.

2.1 The general setting

Suppose there are N agents that generate energy-bids in the market for one time period. For the ith agent, $i \in \{1, \dots, N\}$, the *bid* $0 \leq (p_i, g_i) \in \mathbb{R}^2$ consists of a selling price and the amount of energy, respectively, that the agent is willing to generate for this price. The unit cost of generation is φ_i.

The ISO receives bids from all the agents and, taking into account the system demand, decides both the *market price* $P(p, g, l)$ that will be paid for the energy and the *dispatch* $l = (l_1, l_2, \dots, l_N)$. The latter is the output of an optimization problem solved by the ISO, noting that for some agents there may be a difference between offer and dispatch. The market price, a specific function of the bids, generation and dispatch, depends on the ISO's policy. For example, if the aim is to give a preference to less expensive bidders, it is sound to remunerate the generators with the highest price, among all the dispatched agents:

$$P(p, g, l) := \max\{p_j : l_j > 0\} \tag{1}$$

(this price function is not explicit, but results from the actions of the agents in the market). This pricing mechanism, in principle sound and intuitive, turns out to be difficult to implement for equilibrium problems; see the comments after Proposition 1 below.

The revenue for the ith agent is:

$$l_i P(p, g, l) - g_i \varphi_i \, . \tag{2}$$

In a competitive framework, each agent behaves strategically by maximizing profit, taking into account possible actions of the other agents. For convenience, below we consider the objective function for the ith agent:

$$f_i(p_i, g_i, l_i, P(p, g, l)) \, ,$$

representing a disutility, such as the negative of profit or some function to hedge against downside profit risk. The ith-agent determines bids by solving the minimization problem:

$$\left\{ \begin{array}{ll} \min\limits_{(p_i, g_i, l_i)} & f_i(p_i, g_i, l_i, P(p, g, l)) \\ \text{s.t.} & (p_i, g_i) \in S_i^{\text{OP}} \\ & g \in S_{\text{shared}}^{\text{OP}} \\ & (p, g, l) \in S_{\text{shared}}^{\text{ISO}} \, , \end{array} \right. \tag{3}$$

where the notation describing the feasible set is discussed below. First, regarding the objective function and keeping (2) in mind, the product $l_i P(p,g,l)$ introduces *bilinearity* that poses a challenge in the optimization process. Second, the feasible region in (3) is split into three sets, $S_i^{\rm OP}$, $S_{\rm shared}^{\rm OP}$, and $S_{\rm shared}^{\rm ISO}$, with different structure. Operational constraints that are specific to the technology employed by the ith agent to generate energy are included in the first set, $S_i^{\rm OP}$, that is specific to each agent. If the maximum generation capacity is $g_i^{\max}$ and there is a maximum price $p_i^{\max}$, this set could be as follows with just upper and lower bounds:

$$S_i^{\rm OP} := \{(p,g) : \varphi_i \le p \le p_i^{\max}, 0 \le g \le g_i^{\max}\} \tag{4}$$

(the marginal cost of generation is a natural lower bound for the bidding price while the upper bound could be a price cap set by the regulator, e.g., the value of lost load.) The second set, $S_{\rm shared}^{\rm OP}$, includes operational constraints that are *shared* by several agents. Typically, this situation arises for a group of agents generating hydropower from a set of cascaded reservoirs. Like the first set of constraints, this second set is explicitly described by equality and inequality constraints involving components of the generation vector g (e.g. stream-flow balance constraint, exchange limits, etc.). The third set $S_{\rm shared}^{\rm ISO}$ is also shared by all the agents but describes how the dispatch l is determined by the ISO in an *implicit* manner, by means of another optimization problem. In an abstract formulation,

$$S_{\rm shared}^{\rm ISO} := \{(p,g,l) : l \in D(p,g)\} , \tag{5}$$

where the *dispatch multifunction* $D : \mathbb{R}^N \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ returns, for a given bid (p,g), the set of $D(p,g)$ of quantities dispatched according to (1), for example.

2.2 A simple ISO problem

Consider the following simple, but illustrative, formulation representing the ISO decision process. Given the bidding prices p and quantities g, the ISO minimizes the total expense, $p^\top l$, in a manner that satisfies the demand d. The corresponding optimization problem is

$$\left\{\begin{array}{lll} \min\limits_{l} & p^\top l & \\ \text{s.t.} & 0 \le l & \\ & l \le g & (\lambda) \\ & \sum\limits_{j=1}^{N} l_j = d & (\pi)\,, \end{array}\right. \tag{6}$$

where the right-most variables in parentheses, λ and π, denote the Lagrange multipliers associated to the constraints on the left. The meaning of these multipliers (also known as dual variables) as *shadow prices* is well-known. An economic interpretation of the optimal multiplier λ^* as *marginal rent* will be discussed in Section 3 below. The optimal multiplier π^* is the *marginal price*, representing the infinitesimal change in the expense arising from an infinitesimal change in the demand.

Since π^* measures the price for producing one more unit of energy, its value can be seen by the operator as revealing the true price of energy for the electricity system as a whole. From this point of view, instead of using (1), the ISO could set $P(p,g,l) = \pi^*$ as pricing policy. Even though this definition looks simple and natural at first sight, Proposition 1 shows that in some configurations it is not a well-defined function of the decision variables.

The interest of the simple formulation (6) is that the corresponding optimal primal and dual solutions can be expressed in a closed form. To do so, the statement and its proof below use the following notation:

- the index $j_{\tt mg}$ refers to the *marginal* agent, that is, the one whose bid completes the demand:
$$\sum_{k<{\tt mg}} g_{j_k} < d \le \sum_{k\le{\tt mg}} g_{j_k} .$$
- The N-dimensional vector with all components equal to 1 is denoted by $\mathbf{1}$.
- For two vectors of matching dimensions, $u \perp v$ means that $u^\top v = 0$.

Proposition 1 (Solution to ISO problem (6)**)** *Re-ordering the indices if necessary, suppose in problem* (6) *the bidding prices satisfy the relation*

$$p_{j_1} \leq p_{j_2} \leq \cdots \leq p_{j_N} ,$$

and set $p_{j_{N+1}} := \infty$. *The following holds:*

(i) *The dispatch defined by*

$$l_j^* = \begin{cases} g_j & \text{if } j \in \{j_k : 1 \leq k < \mathtt{mg}\} \\ d - \sum_{k<\mathtt{mg}} l^*_{j_k} = d - \sum_{k<\mathtt{mg}} g_{j_k} & \text{if } j = j_{\mathtt{mg}} \\ 0 & \text{otherwise,} \end{cases}$$

solves (6) *and, hence,* $l^* \in D(p,g)$, *the set-valued mapping from* (5).

(ii) *The optimal Lagrange multipliers associated to the demand constraint in* (6) *are*

$$\Pi^* = \begin{cases} \{p_{j_{\mathtt{mg}}}\} & \text{if } l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}} \\ \left[p_{j_{\mathtt{mg}}}, \min_{k>\mathtt{mg}, g_{j_k}>0} p_{j_k}\right] & \text{if } l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}, \text{ and } j_{\mathtt{mg}} < N \\ \left[p_{j_{\mathtt{mg}}}, +\infty\right) & \text{if } l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}, \text{ and } j_{\mathtt{mg}} = N\,. \end{cases}$$

possibly non-unique.

(iii) *For each* π^* *in the set-valued mapping* Π^*, *the vector* λ^{π^*} *with components*

$$\lambda_j^{\pi^*} = [\pi^* - p_j]^+ \text{ for } 1, \ldots, N$$

is an optimal multiplier for the capacity constraint in (6).

Proof. The vector l^* is clearly feasible for (6) and $\lambda^{\pi^*} \geq 0$. Also, we have that

$$\lambda_j^{\pi^*} = \begin{cases} \pi^* - p_j & \text{if } j \in \{j_k : 1 \leq k \leq \mathtt{mg}\} \\ 0 & \text{otherwise} \end{cases}$$

and if $l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}}$, then $\pi^* = p_{j_{\mathtt{mg}}}$ and so $\lambda^{\pi^*} = 0$. Taking into account all these relations and the definition of l^*, yields the complementarity relation

$$0 \leq g - l^* \perp \lambda^{\pi^*} \geq 0.$$

On the other hand, note that

$$p + \lambda^{\pi^*} - \pi^* \mathbf{1} = ([\pi^* - p_j]^+ - (\pi^* - p_j))_{j=1}^N \geq 0$$

and, since $l^*_{j_k} = 0$ for $k > \mathtt{mg}$, we have also that

$$0 \leq l^* \perp p + \lambda^{\pi^*} - \pi^* \mathbf{1} \geq 0\,,$$

which shows that the primal-dual point $(l^*, \pi^*, \lambda^{\pi^*})$ is optimal for (6). Finally, taking another pair of multipliers $(\bar{\pi}, \bar{\lambda})$, since $l^*_{j_{\mathtt{mg}}} > 0$, from $l^*_{j_{\mathtt{mg}}}(p_{j_{\mathtt{mg}}} + \bar{\lambda}_{j_{\mathtt{mg}}} - \bar{\pi}) = 0$, we have that $0 \leq \bar{\lambda}_{j_{\mathtt{mg}}} \leq \bar{\pi} - p_{j_{\mathtt{mg}}}$. Also, when $l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}}$ it holds that $\bar{\lambda}_{j_{\mathtt{mg}}} = 0$ which implies $p_{j_{\mathtt{mg}}} = \bar{\pi}$. In case $l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}$, and $k > \mathtt{mg}$, with $g_{j_k} > 0$. Since $l^*_{j_k} = 0$ and $(g_{j_k} - l^*_{j_k})\bar{\lambda}_{j_k} = 0$ we have that $\bar{\lambda}_{j_k} = 0$ and using that $p_{j_k} + \bar{\lambda}_{j_k} - \bar{\pi} \geq 0$ yields $p_{j_k} \geq \bar{\pi}$, as stated. ■

In our development, we focus on the EPEC that results from simultaneously considering all the agents' problems, that is (3) written for $i = 1, \ldots, N$. Then, if there are no coupling operational constraints ($S^{\text{OP}}_{\text{shared}}$ is empty), and the ISO behavior is given by (5) and (6), we have:

$$\begin{array}{c}\text{Find a}\\ \text{Nash}\\ \text{equilibrium,}\\ \text{solving}\\ \text{for } i = 1, \ldots, N\\ \text{the } \textbf{bilevel}\\ \text{problems}\end{array} \left\{ \begin{array}{lll} \min_{g_i, p_i, l_i} & \varphi_i g_i - l_i P(g,p,l) & \\ \text{s.t.} & 0 \leq g_i \leq g_i^{\max} & \\ & \varphi_i \leq p_i \leq p_i^{\max} & \\ & l \in \arg\min(S^{\text{ISO}}_{\text{shared}}) & = \left\{ \begin{array}{lll} \min_l & p^\top l & \\ \text{s.t.} & 0 \leq l & \\ & l \leq g & (\lambda) \\ & \sum_{j=1}^N l_j = d & (\pi) \end{array} \right\} \end{array} \right. \tag{7}$$

This is a generalized Nash equilibrium problem with shared constraints $S^{\text{ISO}}_{\text{shared}}$, given implicitly by the lower-level problem.

Remark 2 (On lack of uniqueness) Regarding (5), notice that if there is more than one minimizer l^*, the dispatch function

$$D^1(p, g) = \{l^* \text{ solving (6)}\}$$

can be multi-valued (this is the reason for writing (5) as an inclusion). By Proposition 1, such is the case whenever the highest dispatched price is bid by more than one agent, the same $p_{j_{\mathtt{mg}}}$ is associated with different generation offers $g_{j_{\mathtt{mg},1}}, g_{j_{\mathtt{mg},2}}, \ldots$. This creates an indifference set for the ISO regarding the optimal dispatch l^*, as any distribution of the marginal dispatch $d - \sum_{k<\mathtt{mg}} g_{j_k}$ among the marginal agents yields the same output from the ISO point of view. The ISO problem (5) is a model without transmission constraints, so all the different $g_{j_{\mathtt{mg},l}}$ are the same when it comes to satisfaction of the demand. With a network representation in (5), there are indifference sets when two agents inject power in the same bus and bid the same marginal price. Since the ISO problem is part of an EPEC, we observed that this phenomenon of indifference can lead to numerical difficulties and cycling in the solution process. □

Proposition 1 also confirms that for the optimal multiplier of problem (6) to be unique, the marginal agent should be dispatched at a level that is *strictly smaller* than the marginal bid on generation : $l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}}$ if there is only one marginal agent. Suppose $j_{\mathtt{mg}} < N$, i.e., the marginal agent's bid is not the most expensive one in the market. If the offer is manipulated to force $l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}$, there will be a whole interval of marginal prices, $[p_{j_{\mathtt{mg}}}, p_{j_{\mathtt{mg}+1}}]$ (when $j_{\mathtt{mg}} = N$, the interval is the half-line $[p_{j_{\mathtt{mg}}}, \infty)$). In this setting the choice of the specific ISO's policy plays a fundamental role. With rule (1) the market price would be the left extreme of the interval, the smallest possible choice. If, instead, the policy of the ISO were to remunerate the energy at marginal price, the rule would be $P(p, g, l) = \pi^*$. Since this yields multi-valued information for the price whenever $l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}$, the revenue would not be well-defined and the EPEC model would not be well-posed. This feature has a crucial impact on practical efficiency, since often, when passing to the numerical calculations, the marginal price is taken as a proxy for the market price; as in (10) below and problem (22) in Section 5.

To handle the difficult implicit constraint, we write the dual of the dispatch problem (6):

$$\begin{cases} \max & \pi d - \lambda^\top g \\ \text{s.t.} & \pi - \lambda_j \le p_j, \ j = 1, \ldots, N \\ & \lambda \ge 0\,, \end{cases} \tag{8}$$

and replace the problem by its optimality conditions. Primal-dual feasibility and strong duality yield the equivalent formulation:

$$\begin{array}{c} \text{Find a} \\ \text{Nash} \\ \text{equilibrium,} \\ \text{solving} \\ \text{for } i = 1, \ldots, N, \\ \text{the } \textbf{bilinear} \\ \text{problems} \end{array} \quad \begin{cases} \displaystyle\min_{g_i, p_i, l_i, \pi, \lambda} & \varphi_i g_i - l_i P(g, p, l) \\ \text{s.t.} & 0 \le g_i \le g_i^{\max} \\ & \varphi_i \le p_i \le p_i^{\max} \\ & 0 \le l \le g \\ & \displaystyle\sum_{j=1}^{N} l_j = d \\ & \pi - \lambda_j \le p_j, \ j = 1, \ldots, N \\ & \lambda \ge 0 \\ & p^\top l = \pi d - \lambda^\top g \end{cases} \tag{9}$$

This rewriting eliminates the bilevel setting, at the expense of adding bilinear terms in the constraints, that are not simple to tackle, but have the merit of being explicit. By contrast, the bilinear term in the objective, $l_i P(g, p, l)$, remains implicit and is harder to deal with in an algorithm. Replacing $P(g, p, l)$ in the objective by the individual bidding price p_i leads to equilibria with high prices. A common practice is to use the dual variable as an approximation,

$$P(p, g, l) \approx \pi\,, \tag{10}$$

and look for an equilibrium for problems that replace the objective function in (9) by

$$\varphi_i g_i - l_i \pi\,.$$

However, keeping in mind the result stated in Proposition 1, in some configurations the marginal price is a full interval. Ideally, the ISO pricing preference should be the smallest possible value, as in (1). This is also reflected in the ISO objective function, which is being minimized. But with the replacement (10) what happens is exactly the opposite: since agents maximize revenue (or minimize disutility), using the proxy in the new objective function will always favor the *largest* possible choice for π, making the approximation a gross overestimation. This phenomenon is confirmed by our numerical experiments.

To address this issue, we introduce a new regularization scheme that provides a mechanism of selection in the multiplier set. The alternative approach ensures that in the limit, the minimum price signal will be found.

3 A dual view for the ISO problem

We now explore how to modify the ISO problem in (9) to prevent the EPEC model from taking the highest possible marginal price, if the proxy (10) is used. To do so, we adopt the viewpoint of a *dual* ISO, that could be thought as being more concerned with prices than with dispatch (a dispatch-oriented ISO would directly solve the primal problem). In view of the constraints in (8) (and also by the relations in Proposition 1),

$$\lambda_j = \max(\pi - p_j, 0)\,.$$

If the marginal price replaces the market price as in (10), then any dispatched agent that bids below the market price has a multiplier λ_j in (8) that represents an infra-marginal rent. The wording *rent*, sometimes termed as surplus, refers to an amount that is received by the agent without any effort (revenue, by contrast, involves some work of the agent, to generate energy that may be dispatched). The rent λ_j is positive only when $p_i < \pi$, that is, when the agent gets paid more than the bid. Generators typically rely upon such rent to cover fixed costs.

By Proposition 1, all the dispatched agents except the marginal ones are dispatched at their bidding level $l_i^* = g_i$, so for those agents it holds that $\lambda_i g_i = \lambda_i l_i^*$. If for a marginal agent the dispatch satisfies $l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}}$, the rent will be zero, because $\pi^* = p_{j_{\mathtt{mg}}}$. Moreover, since in the dual problem (8) the objective function is $\pi d - \lambda^\top g$, we conclude that the dual ISO is in fact maximizing the overall payment to the agents *net from any rent.*

On the other hand, the unfavorable situation $l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}$ leads to a positive rent for the marginal agent and an overall increase in the rent of all the other dispatched agents. In order to maintain control of the rent, instead of solving (8), we define a *regularized* problem for the dual ISO that discourages large values of marginal rent. This is done penalizing λ through a convex function $h(\lambda)$ satisfying $h(\lambda) > 0$ for $\lambda > 0$, and $h(0) = 0$. Given a penalty parameter $\beta \geq 0$, the new dual ISO problem is

$$\left\{\begin{array}{ll} \max & \pi d - \lambda^\top g - \beta h(\lambda) \\ \text{s.t.} & \pi - \lambda_j \leq p_j\,,\ \text{for } j = 1, \dots, N \\ & \lambda_j \geq 0\,,\ \text{for } j = 1, \dots, N\,. \end{array}\right. \qquad (11)_\beta$$

Notice that when taking $\beta = 0$, problem $(11)_0$ recovers the original dual linear program (8). One possibility for the penalizing function h is the ℓ_∞-norm, i.e.,

$$h(\lambda) := \|\lambda\|_\infty = \max\{|\lambda_j| : j = 1, \dots, N\}\,.$$

Among other features, the ℓ_∞-norm maintains the regularized ISO problem as a linear program and also yields more specific convergence results when β goes to zero, as shown in Corollary 12. However, this is not the only choice that might be useful, and keeping a general function h gives better insight into the properties and consequences of regularization. For this reason, to study the asymptotic behavior of $(11)_\beta$ and its relation with (8), we require the penalty to satisfy the following properties (which hold for any ℓ_r-norm $h(x) = \|x\|_r$ with $1 \leq r \leq \infty$).

Assumption 3 (Conditions on the penalizing function) *The convex function $h : \mathbb{R}^N \to \mathbb{R}$ satisfies $h \geq 0$ with $h(0) = 0$ and the conditions:*

1. *If $0 \leq x \leq y$, then $h(x) \leq h(y)$.*
2. *If $0 \leq x \leq y$ and $x_j < y_j$ whenever $y_j > 0$, then $h(x) < h(y)$.*
3. *For any $M \in \mathbb{R}$, the sublevel set $\{x : h(x) \leq M\}$ is bounded.* □

In what follows, the positive-part function of a scalar s is defined by

$$[s]^+ := \max(s, 0)\,.$$

Lemma 4 (Properties of the regularized dual ISO problem) *Let $(\pi(\beta), \lambda(\beta))$ be any solution to (11)$_\beta$, where the function h satisfies Assumption 3. Then the following holds.*

(i) *The marginal price is non-negative, $\pi(\beta) \geq 0$, and the marginal rent defined as*

$$\lambda_j^\pi(\beta) := [\pi(\beta) - p_j]^+ \text{ for } j = 1, \dots, N\,, \tag{12}$$

satisfies $\lambda^\pi(\beta) \leq \lambda(\beta)$.

(ii) *The pair $(\pi(\beta), \lambda^\pi(\beta))$ is also a solution to problem (11)$_\beta$.*

(iii) *Given any two solutions $(\pi^1(\beta), \lambda^{1,\pi}(\beta))$ and $(\pi^2(\beta), \lambda^{2,\pi}(\beta))$ to problem (11)$_\beta$,*

$$\pi^1(\beta) \leq \pi^2(\beta) \iff \lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta).$$

If, in addition, $\pi^1(\beta) < \pi^2(\beta)$ and $\lambda^{2,\pi}(\beta) \neq 0$, then $h(\lambda^{1,\pi}(\beta)) < h(\lambda^{2,\pi}(\beta))$.

Proof. Given a solution $(\pi(\beta), \lambda(\beta))$ of $(11)_\beta$, the fact that $\pi(\beta) \geq 0$ is clear. Also, it is easy to see that the pair $(\pi(\beta), \lambda^\pi(\beta))$ is feasible in $(11)_\beta$ and that, as stated in item (i), $\lambda^\pi(\beta) \leq \lambda(\beta)$.

To show item (ii), notice that by Assumption (3) we have that $h(\lambda^\pi(\beta)) \leq h(\lambda(\beta))$; and since $g \geq 0$ we have $g^\top \lambda^\pi(\beta) \leq g^\top \lambda(\beta)$. Thus $\pi(\beta) d - g^\top \lambda(\beta) - \beta h(\lambda(\beta)) \leq \pi(\beta) d - g^\top \lambda^\pi(\beta) - \beta h(\lambda^\pi(\beta))$, which shows that $(\pi(\beta), \lambda^\pi(\beta))$ is also a solution.

For item (iii), first note that $\pi^1(\beta) \leq \pi^2(\beta)$, combined with the facts that $[\cdot]^+$ is monotonically non-decreasing and (12), implies that $\lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta)$. The converse statement assumes that $\lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta)$. Now, let us consider the following two cases. Suppose first that $\lambda^{2,\pi}(\beta) = 0$. Then $\lambda^{1,\pi}(\beta) = 0$ and, by $(11)_\beta$, this forces

$$\pi^1(\beta) = \min_j \{p_j\} = \pi^2(\beta)\,.$$

In the second case, when $\lambda^{2,\pi}(\beta) \neq 0$, from (12), there exists some index j such that

$$\pi^2(\beta) - p_j = [\pi^2(\beta) - p_j]^+ = \lambda_j^{2,\pi}(\beta) \geq \lambda_j^{1,\pi}(\beta) = [\pi^1(\beta) - p_j]^+ \geq \pi^1(\beta) - p_j\,,$$

from which the desired relation follows. Finally, assuming $\pi^1(\beta) < \pi^2(\beta)$, we have that $\lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta) \neq 0$, and for all j,

$$\pi^2(\beta) - p_j > \pi^1(\beta) - p_j\,.$$

As a result, for all the components j for which $\lambda_j^{2,\pi}(\beta) > 0$, we have that

$$\lambda_j^{2,\pi}(\beta) = \pi^2(\beta) - p_j > \max\{\pi^1(\beta) - p_j, 0\} = \lambda_j^{1,\pi}(\beta)\,,$$

and, by Assumption (3), we have that $h(\lambda^{2,\pi}(\beta)) > h(\lambda^{1,\pi}(\beta))$, which concludes the proof. ∎

Lemma 4 is useful when considering convergence of a sequence of approximations, as the regularization parameter tends to zero. In (iii) and (iv) below we show that the approach converges to a price with minimal norm. The statement in item (iv), in particular, states that the limit price will always be the price bid by the marginal agent.

Theorem 5 (Asymptotic behavior of regularized dual ISO problems) *Consider any sequence of solutions to (11)$_\beta$ $\{(\pi(\beta), \lambda^\pi(\beta))\}$, parameterized by β. Under the assumptions in Lemma 4, the following holds.*

(i) *As $\beta \to 0$, the sequence $\{(\pi(\beta), \lambda^\pi(\beta))\}$ converges to a point $(\bar{\pi}, \lambda^{\bar{\pi}})$.*

(ii) *The limit point $(\bar{\pi}, \lambda^{\bar{\pi}})$ solves problem $(11)_0$, that is, the original dual ISO problem (8).*

(iii) *For any other solution to (8), say (π^0, λ^0), it holds that*

$$\bar{\pi} \leq \pi^0 \quad \text{and} \quad \lambda^{\bar{\pi}} \leq \lambda^0.$$

(iv) *The limit price $\bar{\pi}$ coincides with the marginal price $p_{j_{\text{mg}}}$ in Proposition 1.*

Proof. For item (i), we start with proving that the sequence is bounded, and then show that all its accumulation points must be the same.

Note that problems $(11)_\beta$ have the same feasible set for all β. Let (π^0, λ^{π^0}) be any solution to (8) By the optimality of $(\pi(\beta), \lambda^\pi(\beta))$ in $(11)_\beta$, for all $\beta \geq 0$ it holds that

$$\begin{aligned} \pi^0 d - (\lambda^{\pi^0})^\top g - \beta h(\lambda^{\pi^0}) &\leq \pi(\beta) d - (\lambda^\pi(\beta))^\top g - \beta h(\lambda^\pi(\beta)) \\ &\leq \pi^0 d - (\lambda^{\pi^0})^\top g - \beta h(\lambda^\pi(\beta)) \,, \end{aligned} \tag{13}$$

where the last inequality follows from $\pi(\beta) d - (\lambda^\pi(\beta))^\top g \leq \pi^0 d - (\lambda^{\pi^0})^\top g$.

By Assumption 3,

$$h(\lambda^{\pi^0}) \geq h(\lambda^\pi(\beta)) \,, \tag{14}$$

and the boundedness of the sublevel sets $\{\lambda : h(\lambda) \leq h(\lambda^{\pi^0})\}$ implies that the sequence $\{\lambda^\pi(\beta)\}$ is bounded.

Next, combining (12), the constraints in $(11)_\beta$, and the fact that $\pi(\beta) \geq 0$, implies that the sequence $\{\pi(\beta)\}$, is bounded as well.

Consider any accumulation point $(\pi^{\rm acc}, \lambda^{\rm acc})$ of $\{(\pi(\beta), \lambda^\pi(\beta))\}$, i.e., let $\beta_k \to 0$ and let $\{(\pi(\beta_k), \lambda(\beta_k))\} \to (\pi^{\rm acc}, \lambda^{\rm acc})$. It is easy to check that $(\pi^{\rm acc}, \lambda^{\rm acc})$ is feasible in $(11)_0$ and $\lambda^{\rm acc} = \lambda^{\pi^{\rm acc}}$. Passing onto the limit in (13) yields

$$\lambda^{\pi^0 \top} g - \pi^0 d = \lambda^{{\rm acc}\top} g - \pi^{\rm acc} d \,,$$

from which we conclude that $(\pi^{\rm acc}, \lambda^{\rm acc})$ solves $(11)_0$.

Before continuing with item (i), we next show that item (iii) holds for $(\pi^{\rm acc}, \lambda^{\rm acc})$ in place of $(\bar\pi, \lambda^{\bar\pi})$. Passing onto the limit in (14), as $\beta_k \to 0$, it gives that $h(\lambda^{\pi^0}) \geq h(\lambda^{\rm acc})$. Suppose, for contradiction purposes, that $\pi^{\rm acc} > \pi^0$. Lemma 4 ensures that $\lambda^{\rm acc} \geq \lambda^{\pi 0} \geq 0$, which implies $h(\lambda^{\pi 0}) \leq h(\lambda^{\rm acc})$. Therefore, $h(\lambda^{\pi^0}) = h(\lambda^{\rm acc})$. At this point, we need to consider two cases. If $\lambda^{\rm acc} = 0$ then $\lambda^{\rm acc} = \lambda^{\pi^0} = 0$. By Lemma 4, we have that $\pi^{\rm acc} = \pi^0$, which contradicts our assumption. On the other hand, assuming $\lambda^{\rm acc} \neq 0$, implies, using Lemma 4(ii), that $h(\lambda^{\pi^0}) < h(\lambda^{\rm acc})$, which also contradicts the assumption $\pi^{\rm acc} > \pi^0$. Thus, $\pi^{\rm acc} \leq \pi^0$, which also yields $\lambda^{\rm acc} \leq \lambda^{\pi^0} \leq \lambda^0$.

Considering any other accumulation point $(\hat\pi^{\rm acc}, \hat\lambda^{\rm acc})$ of $\Big\{(\pi(\beta), \lambda^\pi(\beta))\Big\}$, because we have that $(\hat\pi^{\rm acc}, \hat\lambda^{\rm acc})$ solves (8), it holds that

$$\pi^{\rm acc} \leq \hat\pi^{\rm acc} \quad \text{and} \quad \lambda^{\rm acc} \leq \hat\lambda^{\rm acc}.$$

By a similar argument we can show that

$$\hat\pi^{\rm acc} \leq \pi^{\rm acc} \quad \text{and} \quad \hat\lambda^{\rm acc} \leq \lambda^{\rm acc}.$$

Hence, $(\pi^{\rm acc}, \lambda^{\rm acc}) = (\hat\pi^{\rm acc}, \hat\lambda^{\rm acc})$. We have therefore established that all accumulation points of the bounded sequence $\Big\{(\pi(\beta), \lambda^\pi(\beta))\Big\}$ coincide, i.e., the sequence converges. And since $\lambda^{\rm acc} = \lambda^{\pi^{\rm acc}}$, we have that the limit point can be written as $(\bar\pi, \lambda^{\bar\pi})$. This concludes item (i). Then, items (ii) and (iii) follow in a straightforward way.

To see the final item (iv), recall from Proposition 1 that $p_{j_{\tt mg}} \leq \bar\pi$ and since $(p_{j_{\tt mg}}, \lambda_{p_{j_{\tt mg}}})$ solves (8), from item (iii), this means that $\bar\pi \leq p_{j_{\tt mg}}$. This concludes the proof. ■

Recall from Proposition 1(iii) that when the marginal agent is dispatched up to the bid, that is when $l^*_{j_{\tt mg}} = g_{j_{\tt mg}}$, the equilibrium price lies in the interval $\Pi^* = \Big[p_{j_{\tt mg}}, \min_{k > {\tt mg}, g_{j_k} > 0} p_{j_k}\Big]$. A remarkable feature of Theorem 5(iv) is that the property $\bar\pi = p_{j_{\tt mg}}$ holds *independently* of the dispatch. This ensures that, as announced, the multiplier $\pi(\beta)$ acts as a *selection mechanism* that in the limit provides the smallest possible value for the price, among all the (infinite) choices in the multiplier set Π^*.

4 Back to the primal ISO problem

Since solving EPECs is far from straightforward, when considering successive equilibrium problems with diminishing regularization parameters, in the numerical experiments we take a small value for β and use the corresponding ISO problem $(11)_\beta$ as an approximation for (8). Corollary 12, given at the end of this section, summarizes all the results for the ℓ_∞-regularization and provides the theoretical background for our numerical assessment. In particular, solving the regularized EPEC results in equilibrium price of minimal norm.

In order to guide the choice of the bound for the regularization parameter, and determine its impact (or interference) in the bidding process, we now play the reverse game and formulate the dual of the regularized dual ISO problem $(11)_\beta$. The resulting problem is called bi-dual, because it is the dual of the dual problem. The bi-dual is a primal ISO problem that regularizes (6) and clarifies the role of the parameter β in determining certain *reserve* that the ISO can access to complete the dispatch and keep controlled both the price and the rent.

Before presenting our theoretical results we review briefly two well-known concepts from Convex Analysis: the *subgradient* and the *Fenchel conjugate*. Detailed explanations can be found in any book of Convex Analysis, for instance [BL10], [HL13].

Given a convex function $h : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$, the subgradient of h at x is the set denoted and defined by

$$\partial h(x) = \{s : h(y) \geq h(x) + s^\top (y - x) \quad \forall\, y\}\,. \tag{15}$$

The Fenchel conjugate $h^* : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty, \infty\}$ of h is defined by

$$h^*(s) = \sup_{x \in \mathbb{R}^N} \{x^\top s - h(x)\}. \tag{16}$$

The subgradient generalizes the concept of gradient from a vector to a set, when the convex function h is not differentiable (if h is differentiable, the set is a singleton – the usual derivative). On the other hand, the Fenchel conjugate h^* is a convex function with an interesting economic interpretation. Suppose $h(x)$ represents the cost of production of a good x that can be sold at a price s. Then, the Fenchel conjugate at s defined in (16) represents the optimal profit that can be achieved for the price s by choosing the quantity x to be produced and sold.

A relation between these two concepts that we use frequently below states that

$$s \in \partial h(x) \quad \Longleftrightarrow \quad h(x) + h^*(s) = x^\top s\,. \tag{17}$$

Continuing with the economic interpretation when h represents a cost, the subdifferential of h at x is the set of all the prices that ensure an optimal profit when x is the production level.

The stage is now set to present the theoretical results in this section.

Proposition 6 (Regularized primal ISO problem) *Given a convex function h satisfying $h(\lambda) \geq 0$ and $h(0) = 0$, the primal problem associated with the regularized dual $(11)_\beta$ is*

$$\left\{\begin{array}{ll} \min\limits_{l,w} & l^\top p + \beta h^*\Big(\dfrac{l + w - g}{\beta}\Big) \\ s.t. & \mathbf{1}^\top l = d \\ & l\,, w \geq 0\,, \end{array}\right. \tag{18$_\beta$}$$

where h^* denotes the conjugate function of h.

When h is the ℓ_∞-norm, the problem above reduces to the linear program

$$\left\{\begin{array}{ll} \min\limits_{l\,,s} & l^\top p \\ s.t. & \mathbf{1}^\top l = d \\ & l \leq g + \beta s \\ & \mathbf{1}^\top s \leq 1 \\ & l, s \geq 0\,. \end{array}\right. \tag{19}$$

Proof. The Lagrangian of $(11)_\beta$ is given by

$$\begin{aligned} L(\pi, \lambda, l, w) =& \pi d - \lambda^\top g - \beta h(\lambda) - l^\top (\pi \mathbf{1} - \lambda - p) + w^\top \lambda \\ =& l^\top p + \pi(d - \mathbf{1}^\top l) + \lambda^\top (-g + l + w) - \beta h(\lambda)\,, \end{aligned}$$

where $l, w \geq 0$ are the multipliers associated to the respective constraints. Then, the optimality conditions for $(11)_\beta$ state that

$$\begin{aligned}
0 =& d - \mathbf{1}^\top l \\
0 =& -g + l + w - \beta s\,, \text{ for } s \in \partial h(\lambda) \\
0 \leq& \, l \perp \pi \mathbf{1} - \lambda - p \leq 0 \\
0 \leq& \, w \perp \lambda \geq 0\,.
\end{aligned}$$

In the second equality, the subgradient $s = \frac{l+w-g}{\beta} \in \partial h(\lambda)$ satisfies relation (17) written with x replaced by λ:

$$s \in \partial h(\lambda) \iff s^\top \lambda = h(\lambda) + h^*(s)\,.$$

Therefore, multiplying by β,

$$(l + w - g)^\top \lambda = \beta h(\lambda) + \beta h^*(s)\,.$$

Next, $w^\top \lambda = 0$ implies that $l^\top \lambda = g^\top \lambda + \beta h(\lambda) + \beta h^*(s)$, and using $l^\top(\pi \mathbf{1} - \lambda - p) = 0$ we obtain that

$$l^\top(\pi \mathbf{1} - p) = l^\top \lambda = g^\top \lambda + \beta h(\lambda) + \beta h^*(s)\,.$$

Finally, the identity $l^\top(\pi \mathbf{1}) = \pi d$ yields

$$\pi d - l^\top p = \lambda^\top g + \beta h(\lambda) + \beta h^*(s)\,.$$

The expression for the bi-dual problem follows, using the expression for s.

To specialize the result to the ℓ_∞-norm note that, directly form definition of the conjugate function,

$$h(\lambda) = \|\lambda\|_\infty \iff h^*(s) = \begin{cases} 0 & \text{if } \sum_{j=1}^N |s_j| \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

As a result, $(18)_\beta$ writes down as

$$\begin{cases}
\min\limits_{l,w} & l^\top p \\
\text{s.t.} & \mathbf{1}^\top l = d \\
& \sum\limits_{j=1}^N |l_j + w_j - g_j| \leq \beta \\
& l, w \geq 0\,,
\end{cases}$$

which is equivalent to problem (19). ■

Problem (19) is a linear program because the conjugate of the ℓ_∞-norm is the ℓ_1-norm. In the family of ℓ_r-norms, the only ones yielding linear programs in $(18)_\beta$ are the *polyhedral* norms, $r = 1$ and $r = +\infty$. The latter gives the primal ISO problem (19), whereas the former can be found in Example 7 below.

We now turn our attention to the interpretation of the regularization in the primal problem. In the original ISO problem, the marginal rent is the multiplier associated to the capacity constraint $l \leq g$, that is no longer explicit in $(18)_\beta$. At first sight, this makes less clear the meaning of the variable λ in the regularized problem. In fact, such meaning depends on the penalizing function.

The choice $h(\cdot) = \frac{1}{2}\|\cdot\|^2$, which by definition of the conjugate implies that $h^* = h$, gives in $(18)_\beta$ the objective function

$$l^\top p + \beta h^*\Big(\frac{l + w - g}{\beta}\Big) = l^\top p + \frac{1}{2\beta}\|l + w - g\|^2\,.$$

This specific regularization is a quadratic penalization of the capacity constraint.

If the penalizing function is the ℓ_∞-norm, the capacity constraint appears explicitly, and λ plays the role of a genuine marginal rent, in the sense that it is the multiplier associated to the capacity constraint. The primal format (19) reveals the regularized primal ISO as disposing of a *generation reserve* equal to βs. Indeed, the capacity constraint $l_j - \beta s_j \leq g_j$ allows for a value $l_j > g_j$ to be optimal, seemingly allowing the ISO dispatch the jth agent *beyond* the bid. Of course, this is not possible. Rather, this situation, that leads to positive values for an optimal $s_j^*(\beta)$, is to be understood as the ISO having access to an additional source of energy, out of the market – a battery perhaps.

In view of Theorem 5, the ISO uses this reserve to control the marginal rent and ensure the price π will be the minimal one. This interpretation can be extended to any penalty in the family of norms.

Example 7 (Regularizing with a norm) To each norm $h(\cdot) = \|\cdot\|$ we associate a dual norm, defined by

$$h_D(x) = \max_{h(y)=1} x^\top y \,.$$

Since by definition of the conjugate,

$$h^*(x) = \begin{cases} 0 & \text{if } h_D(x) \le 1 \\ +\infty & \text{otherwise,} \end{cases}$$

the regularized primal problem $(18)_\beta$ can be expressed as

$$\begin{cases} \min\limits_{l\,,w} & l^\top p \\ \text{s.t.} & \mathbf{1}^\top l = d \\ & h_D(l + w - g) \le \beta \\ & l, w \ge 0\,. \end{cases}$$

Notice that the meaning of $\lambda^\pi(\beta)$ as marginal rent is preserved, because

$$0 \le \lambda^\pi(\beta) \perp g(\beta) + \beta s(\beta) - l(\beta) \ge 0.$$

For the ℓ_∞-norm considered in Proposition 6, the particular instance (19) uses as h_D the ℓ_1-norm, because

$$h(\cdot) = \|\cdot\|_\infty \text{ and } h_D(\cdot) = h^*(\cdot) = \|\cdot\|_1\,.$$

The other situation yielding a linear program for $(18)_\beta$ is when

$$h(\cdot) = \|\cdot\|_1 \text{ and } h_D(\cdot) = h^*(\cdot) = \|\cdot\|_\infty\,.$$

In this case, the ISO regularized primal problem is equivalent to

$$\begin{cases} \min\limits_{l\,,s} & l^\top p \\ \text{s.t.} & \mathbf{1}^\top l = d \\ & l \le g + \beta s \\ & s \le \mathbf{1} \\ & l, s \ge 0\,. \end{cases}$$

Like in (19), the capacity constraint is explicit and involves a reserve on generation. For these two polyhedral norms, $\|\beta s\| \le \beta$, so β together with the chosen norm determines the amount of reserve. For this reason, in our numerical experiments β is bounded by a small percentage of the demand. The regularized ISO can slightly alter the dispatch, passing from g_j to $g_j + \beta s_j^*(\beta) > g_j$, using the reserve to complete the generation and attend the demand (alternatively, the difference can be thought of as real-time corrections of the dispatch). □

In order to relate the solutions to the original problem (6), given in Proposition 1, with the solutions of the regularized primal ISO problem $(18)_\beta$, the penalty must verify one more relation, stated below.

Assumption 8 (Additional condition on the penalty) *For any $x \in \mathbb{R}^N$, it holds that*

$$h(x) = h(abs(x)) \text{ where } abs(x) := (|x_1|, |x_2|, \ldots, |x_N|)\,. \qquad \square$$

Once again, Assumption 8 is satisfied by any ℓ_r-norm with $1 \le r \le \infty$. We state several technical properties related with the new assumption.

Lemma 9 (Consequences of Assumption 8) *The following holds for a function h satisfying Assumption 8.*

(i) The conjugate h^* satisfies Assumption 8.

(ii) *For any* $\lambda \in \mathbb{R}^N$ *and a subgradient* $s \in \partial h(\lambda)$*,*

$$\lambda \geq 0 \textit{ with } \lambda \neq 0 \Longrightarrow s_j \geq 0 \textit{ for any component } j \textit{ for which } \lambda_j > 0.$$

(iii) *Suppose, in addition, that* h *is a norm whose dual norm* h_D *has the property that, for any* $0 \leq x \leq y$*,*

$$\exists j \in \{1, \dots, N\} \textit{ such that } x_j < y_j \implies h_D(x) < h_D(y)\,.$$

Then $s \in \partial h(\lambda)$ *in item (ii) is such that* $s_j = 0$ *whenever* $\lambda_j = 0$*.*

Proof. To show (i), denote

$$\mathrm{sign}(x) := (\mathrm{sign}(x_1), \mathrm{sign}(x_2), \dots, \mathrm{sign}(x_N)).$$

Let $\circ$ represent the Hadamard product between vectors, that is

$$\mathrm{sign}(x) \circ y := (\mathrm{sign}(x_1)y_1, \mathrm{sign}(x_2)y_2, \dots, \mathrm{sign}(x_N)y_N)\,.$$

Then the identity below holds for any $x, y \in \mathbb{R}^N$, showing the assertion:

$$\begin{aligned} x^\top y - h(y) &= \mathrm{abs}(x)^\top(\mathrm{sign}(x) \circ y) - h(y) \\ &= \mathrm{abs}(x)^\top(\mathrm{sign}(x) \circ y) - h(\mathrm{sign}(x) \circ y)\,. \end{aligned}$$

By definition of conjugate function, the subgradient $s \in \partial h(\lambda)$ solves the problem

$$h^{**}(\lambda) = \sup_z \{\lambda^\top z - h^*(z)\}\,.$$

Since $h = h^{**}$ by convexity of h, this means that

$$h(\lambda) = \lambda^\top s - h^*(s) \geq \lambda^\top z - h^*(z) \quad \text{for all } z \in \mathbb{R}^N\,. \tag{20}$$

To show item (ii), we proceed by contradiction and suppose there exists j such that $\lambda_j > 0$ and $s_j < 0$. Then we have that $\lambda^\top \mathrm{abs}(s) > \lambda^\top s$ and since $h^*(s) = h^*(\mathrm{abs}(s))$ by item (i),

$$\lambda^\top \mathrm{abs}(s) - h^*(\mathrm{abs}(s)) > \lambda^\top s - h^*(s),$$

which contradicts (20). Thus $s_j \geq 0$ whenever $\lambda_j > 0$, as stated.

We proceed with item (iii). When h is a norm, as explained in Example 7, having $\lambda \neq 0$ implies that $h_D(s) = 1$. This gives in (20) the following:

$$h(\lambda) = \lambda^\top s \geq \lambda^\top z \quad \text{for all } z \text{ with } h_D(z) = 1\,. \tag{21}$$

We proceed again by contradiction, supposing that for some component j_0 it holds that $s_{j_0} \neq 0$ and $\lambda_{j_0} = 0$. Defining $\hat{s}$ by

$$\hat{s}_j = \begin{cases} 0 & \text{if } j = j_0 \\ |s_j| & \text{otherwise,} \end{cases}$$

we have that $\hat{s} \neq 0$, $0 \leq \hat{s} \leq \mathrm{abs}(s)$ and $\hat{s}_{j_0} < \mathrm{abs}(s)_{j_0}$. Then

$$0 < h_D(\hat{s}) < h_D(\mathrm{abs}(s)) = h_D(s) = 1.$$

This contradicts (21), because

$$\frac{\lambda^\top \hat{s}}{h_D(\hat{s})} > \lambda^\top \hat{s} = \lambda^\top \mathrm{abs}(s) \geq \lambda^\top s\,.$$

This completes the proof. ■

Thanks to the properties stated in Lemma 9, we now characterize the dispatch of the regularized ISO.

Proposition 10 (Solution to regularized ISO problem (18)$_\beta$) *Suppose* h *in (11)*$_\beta$ *satisfies Assumptions 3 and 8. Let* $(\pi(\beta), \lambda^\pi(\beta))$ *be the marginal pair from Proposition 4(i), solving the regularized dual ISO problem (11)*$_\beta$*. The following holds for* $(l(\beta), w(\beta))$*, a solution to the regularized primal problem (18)*$_\beta$*.*

(i) $s = \dfrac{l(\beta) + w(\beta) - g}{\beta} \in \partial h(\lambda^\pi(\beta))$.

(ii) *For* $j = 1, 2, \dots, N$,

$$\begin{array}{llll} \text{if } p_j < \pi(\beta), & \text{then} \quad s_j \geq 0\,, & l_j(\beta) = g_j + \beta s_j\,, & w_j = 0 \\ \text{if } p_j > \pi(\beta), & \text{then} & l_j(\beta) = 0 & w_j = g_j + \beta s_j\,. \end{array}$$

(iii) *If, in addition h is a norm and $\lambda^\pi(\beta) \neq 0$, then s has length 1 in the dual norm. Furthermore, if the dual norm h_D satisfies the conditions in Lemma 9(iii), then the statement in (ii) also includes that $s_j = 0$ whenever $p_j \geq \pi(\beta)$.*

Proof. The first item is straightforward from the optimality conditions of $(18)_\beta$ and the identity

$$l_\pi(\beta) = g - w^\pi(\beta) + \beta s\,.$$

Consider $j = 1, 2, \dots, N$. If $p_j < \pi(\beta)$, then $\lambda_j^\pi(\beta) = \pi(\beta) - p_j > 0$. The complementarity condition between λ and w implies that $w_j^\pi = 0$ and, hence, $l_j(\beta) = g_j + \beta s_j$.

For the case $p_j > \pi(\beta)$, we have that $\pi(\beta) - p_j < 0 \leq \lambda_j^\pi(\beta)$. Again, the complementarity condition between this constraint and the Lagrange multiplier l implies that $l_j(\beta) = 0$ and $w_j(\beta) = g_j + \beta s_j$, showing item (ii).

Finally, when h is a norm and $\lambda^\pi(\beta) \neq 0$, from Lemma 9, we have that $h_D(s) = 1$, and this concludes the proof. ∎

It is worth noting that the property required for the norm in Lemma 9(iii) is satisfied by the ℓ_∞-norm, but *not by the ℓ_1-norm*. This is the reason why in our numerical results the regularized EPECs are defined using the former option.

Like in Theorem 5, we now consider convergence of a sequence of approximations as the parameter β tends to zero, now from the primal point of view.

Theorem 11 (Behavior of regularized primal ISO problems) *Given the marginal price $p_{j_{\text{mg}}}$ from Proposition 1, consider the index-sets*

$$J^- = \left\{j : p_j < p_{j_{\text{mg}}}\right\} \quad \text{and} \quad J^+ = \left\{j : p_j > p_{j_{\text{mg}}}\right\}.$$

Let $\{(l(\beta), w(\beta), s(\beta))\}$ be any sequence parameterized by $\beta > 0$, where $(l(\beta), w(\beta))$ solves $(18)_\beta$ and $s(\beta) = \frac{l(\beta)+w(\beta)-g}{\beta}$. Under the assumptions in Proposition 10, the following holds.

(i) *The sequence $\{(l(\beta), w(\beta), s(\beta))\}$ is bounded.*

(ii) *Any accumulation point l^{acc} of $\{l(\beta)\}$ solves the primal ISO problem* (6).

(iii) *There exists $M > 0$ such that for $\beta > 0$ sufficiently small,*

$$\begin{array}{lllll} j \in J^- & \Longrightarrow & l_j(\beta) = g_j + \beta s_j(\beta)\,, & w_j(\beta) = 0\,, & |l_j(\beta) - g_j| \leq M\beta \\ j \in J^+ & \Longrightarrow & l_j(\beta) = 0\,, & w_j(\beta) = g_j + \beta s_j(\beta)\,, & |w_j(\beta) - g_j| \leq M\beta\,. \end{array}$$

(iv) *If, in addition h is a norm satisfying the conditions in Lemma 9(iii), then the statement above can be refined by taking $M = 1$, which implies that $w_j(\beta) = g_j$ for $j \in J^+$.*

Furthermore, the sequences $\{l_j(\beta)\}$ and $\{w_j(\beta)\}$ converge for any $j \in J^- \cup J^+$.

Proof. Consider a sequence of dual solutions $\left\{(\pi(\beta), \lambda^\pi(\beta))\right\}$, shown to be convergent in Proposition 10. Then $p_{j_{\text{mg}}} = \lim_{\beta\to 0} \pi(\beta)$, with $s(\beta) \in \partial h(\lambda^\pi(\beta))$. Since the sequence $\{\lambda^\pi(\beta)\}$ is bounded and h is convex, we have that the family of subdifferentials $\{\partial h(\lambda^\pi(\beta))\}$ is uniformly bounded, and so, the sequence $\{s(\beta)\}$ is bounded: there exists $M > 0$ such that

$$\|s(\beta)\| \leq M.$$

On the other hand, it is clear that $l(\beta)$ is feasible for problem (6) and since the feasible set of this problem is bounded, we have that the sequence $\{l(\beta)\}$ is bounded. Then, from $w(\beta) = g - l(\beta) + \beta s(\beta)$, we have that the sequence $\{w(\beta)\}$ is also bounded. Item (i) is established.

In order to prove item (ii), note that $h^*(s(\beta)) = s(\beta)^\top \lambda^\pi(\beta) - h(\lambda^\pi(\beta))$ implies that the sequence $\{h^*(s(\beta))\}$ is bounded, and hence, $\beta h^*(s(\beta)) \to 0$, as $\beta \to 0$. Now, considering the strong duality condition for $(18)_\beta$ – $(11)_\beta$

$$\pi(\beta)d - \lambda^\pi(\beta)^\top g - \beta h(\lambda^\pi(\beta)) = p^\top l(\beta) + \beta h^*(s(\beta))$$

and passing to the limit as $\beta \to 0$, taking a convergent subsequence if necessary, we have that

$$p_{j_{\text{mg}}} d - \lambda_{p_{j_{\text{mg}}}}^\top g = p^\top l^{\text{acc}}.$$

This shows that the strong duality condition also holds for (6) – (8), and since l^{acc} is feasible for (6), item (ii) follows.

Finally, note that, letting $p^- = \max_{j \in J^-} p_j$ and $p^+ = \min_{j \in J^+} p_j$, we have that

$$p^- < p_{j_{\text{mg}}} < p^+ .$$

Therefore, for $\beta > 0$ small enough,

$$p^- < \pi(\beta) < p^+ .$$

The final statement is straightforward from Proposition 10, Lemma 9, and the fact that $0 \leq s_j(\beta) \leq h_D(s_j(\beta) = 1$, for $j \in J^-$. ■

Most of the items in the theorem above are of asymptotic nature. A remarkable exception is item (iii), that characterizes the optimal dispatch for all small β. The characterization does not involve the marginal agents because, similarly to the situation pointed out in Remark 2 for the original primal problem ($\beta = 0$), there is an ambiguity created by the ISO's indifference that arises when more than one agent bids the same marginal price.

We conclude our theoretical analysis gathering the results specific for the ℓ_∞-norm, which is polyhedral and satisfies not only Assumptions 3 and 8, but also the condition given in Lemma 9(iii).

Corollary 12 (Summary of theory for the ℓ_∞-regularization) *With the notation and assumptions in Proposition 1, Theorems 5 and 11, consider the regularized dual ISO problem obtained with $h = \|\cdot\|_\infty$. The sequence $\{(\pi(\beta), \lambda^\pi(\beta))\}$ of solutions to the dual version (11)$_\beta$ satisfies the following:*

$$\left.\begin{array}{l} \lim_{\beta \to 0} \pi(\beta) = p_{j_{\text{mg}}} \\ \lim_{\beta \to 0} \|\lambda^\pi(\beta)\|_\infty = \max(p_{j_{\text{mg}}} - p_{j_1}, 0) \end{array}\right\} \textit{provide minimal-norm solutions to } (8).$$

In addition, for any $\beta > 0$ sufficiently small, the pair $(l(\beta), w(\beta))$ solving the primal version (19) *is such that*

$$l_j(\beta) = \begin{cases} g_j & \text{if } p_j > p_{\min}, j \in J^- \\ 0 & \text{if } j \in J^+, \end{cases}$$

where $p_{\min} = \min_j p_j$. Finally, The marginal dispatch completes the demand, following an arbitrary distribution among the marginal agents, if there is more than one bidding $p_{j_{\text{mg}}}$, as noted in Remark 2. □

Proof. The statements follow from the previous results. Then only exception concerns the value of $l_j(\beta)$ for indices j such that $p_j > p_{\min}$ and $j \in J^-$. For such j-indices, Theorem 11(iii) states that

$$l_j(\beta) = g_j + \beta s_j .$$

Then, from (19), it is easy to check, by optimality arguments, that $s_j = 0$ for any j such that $p_j > p_{\min}$. ■

Remark 13 In the proofs above, it is easy to see that $\sum s_j = 1$. Then, in case that there exists only one index j such that $p_j > p_{\min}$, this would force $s_j = 1$, which in turn implies that $l_j(\beta) = g_j + \beta$. When the index is not unique, there is no unique solution to (19), yet we can always choose one j that assigns a dispatch $g_j + \beta$ to one of the least expensive units.

5 Numerical Experience Based on Complementarity Formulation

Consider the EPEC model (7) with the proxy approximation (10) for the price function $P(g, p, l)$. Suppose the ISO solves the regularized problem (19), corresponding to taking the ℓ_∞-norm in the dual. This amounts to finding a Nash equilibrium, by solving for $i = 1, \dots, N$, problems of the form below:

$$\left\{\begin{array}{ll} \min\limits_{g_i, p_i, l_i} & \varphi_i g_i - l_i \pi \\ \text{s.t.} & 0 \le g_i \le g_i^{\max} \\ & \varphi_i \le p_i \le p_i^{\max} \\ & l \in \arg\min \left\{\begin{array}{ll} \min\limits_{l,s} & p^\top l \\ \text{s.t.} & \mathbf{1}^\top l = d \\ & l \le g + \beta s \\ & \mathbf{1}^\top s \le 1 \\ & l, s \ge 0\,. \end{array}\right\} = \left\{\begin{array}{ll} \max\limits_{\pi,\lambda} & \pi d - \beta \|\lambda\|_\infty \\ \text{s.t.} & \pi \mathbf{1} - \lambda \le p \\ & \lambda \ge 0\,. \end{array}\right\} \end{array}\right.$$

As before, the implicit constraint on the dispatch l is handled by writing the primal and dual optimality conditions for the lower-level problems. The max-operation in the ℓ_∞-norm of the rent is dealt with via an auxiliary scalar variable τ $(= \|\lambda\|_\infty)$, yielding a smooth reformulation for the problem solved by each agent, now written in one level, using the optimality conditions for the lower problem:

$$\left\{\begin{array}{llll} \min\limits_{\substack{g_i, p_i \\ l, s, \pi, \lambda, \tau}} & \varphi_i g_i - l_i \pi & & \\ \text{s.t.} & 0 \le g_i \le g_i^{\max} & & (\mu^1) \\ & \varphi_i \le p_i \le p_i^{\max} & & (\mu^2, \mu^3) \\ & p^\top l - \pi d + \lambda^\top g + \beta \tau = 0 & & (\gamma^1) \\ & \mathbf{1}^\top l = d & & (\gamma^2) \\ & \mathbf{1}^\top s \le 1 & & (\mu^4) \\ & l_j \le g_j + \beta s_j, & j = 1, \dots, N & (\mu_j^5) \\ & \pi - \lambda_j \le p_j, & j = 1, \dots, N & (\mu_j^6) \\ & 0 \le \lambda_j \le \tau, & j = 1, \dots, N & (\mu_j^7) \\ & 0 \le l_j,\ 0 \le s_j, & j = 1, \dots, N\,. & \end{array}\right. \tag{22}$$

The notation for each dual variable associated to the corresponding constraint is indicated between parentheses on the right, with μ^2 associated with the lower bound for the bidding price written as $-p_i + \varphi_i \le 0$.

There are a number of ways to tackle the non-convexity that appears when inserting optimality conditions of the ISO into the upper-level problem. A first direct approach leads to a computationally challenging problem with severely non-convex/disjunctive complementarity constraints, [LPR96], [IS14, Chapter 7.3].

For (22) we adopt an alternative approach that has proven successful in the literature, in which strong duality is substituted by complementarity using variables as in (23) below. We now examine how to derive a formulation that can be handled computationally, for example by the PATH solver [DF95]. It is important to keep in mind that, while being a stationary point for the complementarity system residual, the solution provided by PATH may not be an equilibrium (the bilinear objective and constraint in (22) make the resulting mixed complementarity problem non-monotone). One of our conclusions is that the output is very sensitive to the initial input. That said, experimentation and the resulting appropriate tuning of the regularization parameter leads to useful, optimal/equilibrium solutions. The results below are meant to illustrate these conclusions for the given problem. In our experiments, the complementarity system of the same market configuration is solved with 2500 different values of the generators' bid as starting points for each fixed value of β. Thanks to the theory presented above, we can make a thorough assessment of the output and gauge its quality in terms of the problem of interest. In particular, we show empirically that over thousands of starting points in the experiment, PATH (with our regularization) finds genuine equilibria in all but three cases. By contrast, with the same points, but without regularization ($\beta = 0$) the output of PATH gave an equilibrium only once; see the right plots in Figure 3.

5.1 Squaring the system and the considered market

The argument of the min-operation in (22) is separated into two different lines, the agent variables g_i, p_i and the variables decided by the ISO, l, w, π, λ, τ. This separation is done to draw the attention to an important feature that arises when defining a complementarity system to solve the approximate EPEC model (22). Specifically, at least from the implementational point of view of PATH, the system obtained when putting together the optimality conditions for each agent for $i = 1, \ldots, N$ will be solvable only if it is square. This amounts to each agent seeing a *different ISO problem* in the lower level (like the three ISOs in Figure 1). In particular, for $j = 1, \ldots, N$ when appropriate,

$$\text{the ISO decision variables in (22) now have a super-index } i\text{: } l_j^i, s_j^i, \pi^i, \lambda_j^i, \tau^i\,. \qquad (23)$$

Likewise, the multipliers in (22) for constraints that involve ISO's variables are denoted with a super-index, $\gamma^{i,1}, \gamma^{i,2}, \mu^{i,4}, \mu_j^{i,5}, \mu_j^{i,6}, \mu_j^{i,7}$.

Since each individual ISO includes decision variables of all the agents, from the point of view of the ith agent, the generation in the ISO lower-level problem is split as follows:

$$g_i, \text{the decision variable of agent } i\,, \text{ and a vector } g_{-i} \text{ the generation of other agents}$$

(similarly for the price, dispatch and marginal rent, p_{-i}, l_{-i}^i and λ_{-i}^i). Vectors with subindex $-i$ represent the perception that agent i has of the action of the competitors, when considering their bids in the ISO lower-level problem. This distinction needs to be done because, when putting together the optimality conditions of the different problems to create the complementarity system, the Lagrangian derivatives are taken only with respect to the variables of the ith-player.

5.2 Data and values at equilibrium

For the numerical assessment we created a family of market instances, depending on two input parameters, the number N of players and the index of the marginal $j_{\mathtt{mg}} \leq N$. The remaining information is created as follows.

– The marginal cost and maximum bidding price increase with the index of the player:

$$\varphi_j = \frac{j}{2}\,, \quad p_j^{\max} = 2\varphi_j = j\,.$$

– The maximum capacity of each agent $j = 1, \ldots, N$ is

$$g_j^{\max} = \begin{cases} 0.1(N - j/N) & j \neq j_{\mathtt{mg}} \\ 20/3 + 0.1(N - j/N) & j = j_{\mathtt{mg}}\,. \end{cases}$$

– The demand is $d = 0.1\sum_{j=1}^{j_{\mathtt{mg}}}(N - j/N)$. To understand the consequences of this setting, recall from Proposition 1 that generators bidding a price cheaper than the marginal one are dispatched by the ISO at their maximal capacity. The value chosen for the demand absorbs all the generation capacity of agents j with $j < j_{\mathtt{mg}}$, but not that of the marginal agent. The term 20/3 is added for $j = j_{\mathtt{mg}}$ to ensure that $d < \sum_{j=1}^{j_{\mathtt{mg}}} g_j^{\max}$.

In view of Corollary 12, at least for sufficiently small β, for a market with the configuration above the dispatches at equilibrium should be

$$l_{j_k}^{j_k}(\beta) = \begin{cases} g_{j_k}^{\max} = 0.1N(1 - j_k/N) & \text{for } k = 1, \ldots, \mathtt{mg} - 1 \\ 0.1N(1 - j_{\mathtt{mg}}/N) < g_{j_{\mathtt{mg}}}^{\max} & \text{for } k = \mathtt{mg} \\ 0 & \text{for } k = \mathtt{mg} + 1, \ldots, N\,. \end{cases}$$

Since by definition $\varphi_{j_{\mathtt{mg}}} = j_{\mathtt{mg}}/2$ and $\min(p_{j_{\mathtt{mg}}}^{\max}, \varphi_{j_{\mathtt{mg}}+1}) = \min(j_{\mathtt{mg}}, j_{\mathtt{mg}+1}/2) = j_{\mathtt{mg}+1}/2$, we expect an equilibrium price in $[j_{\mathtt{mg}}/2, j_{\mathtt{mg}+1}/2]$, the interval of bidding prices of agent $j_{\mathtt{mg}}$. This is confirmed by the values of $\pi^* = 1.5$ plotted on the left top graph in Figure 1, corresponding to $j_{\mathtt{mg}} = 2$.

5.3 Benchmarking performance of the regularization strategy

After coding the model in GAMS, we use PATH to directly solve the bilinear EPEC (22), using the additional variables (23). All runs were performed on a notebook running under Ubuntu 18.04.4 LTS, with i7 CPU 1.90GHzx8 cores and 31.3GiB of memory.

The results reported below were obtained for a market with $N = 3$ and $j_{\mathtt{mg}} = 2$, making 2500 runs for a fixed value of the penalization parameter (related to different stating points)

$$\beta \in [0, 0.75]\%d = [0, 0.0375] .$$

A word of caution is in order. Even for this very simple market instance, to obtain an equilibrium when solving a complementarity system with PATH is possible, but delicate. The bilinear relations present in the complementarity system (22) make the output extremely dependent on the starting point. Accordingly, for $j \leq N$ we let

$$p_j^0(k) = \varphi_j + \tfrac{k-1}{50}(p_j^{\max} - \varphi_j) \text{ and } g_j^0(r) = g_j^{\max} - \tfrac{r-1}{50} g_j^{\max} . \tag{24}$$

By parsing k and r in $\{1, 2, \ldots, 50\}$, there are 2500 different starting bids $\left(p_j^0(k), g_j^0(r)\right)$ covering the whole range of prices and generation, that is, $[\varphi_j, p_j^{\max}]$ and $[0, g_j^{\max}]$, respectively.

The 2500 runs correspond to solving the same problem by calling PATH with each one of such points. In general, we observed that PATH runs were very fast, taking less than 6 minutes for completing the 2500 solves for a single β. Over the set of 2500 starting points, PATH solved successfully 784, 474, 776, and 644, runs for $\beta = 0, 0.25, 0.5, 0.75$, respectively.

For the considered market, as stated in Subsection 5.2, the ISO should dispatch agent 1 at maximum capacity, and complete the demand with generation from agent 2 and agent 3 if not dispatched. The bidding prices in our results were consistent with the output computed using the theory. Regarding the equilibrium price, for positive parameter β, the three ISO's multipliers had a similar median $\pi(\beta) \approx 1.5$, the expected value. By contrast, without regularization ($\beta = 0$), the three ISOs found different values for the final π, and some values being higher than 5, precluding the possibility of the output being an equilibrium.

In an effort to obtain a better output without regularization, we fixed the starting point to $k = r = 1$ in (24), and re-ran the model, this time for decreasing values of β, feeding the output of one run as initial point of the next solve of PATH. This strategy, that we refer to as a warm-start with decreasing β, appears to be useful. This is shown by the four graphs in Figure 1, with the output filtered for a batch of 41 complementarity problems that gave en equilibrium, having output $\pi^j(\beta) \approx 1.5$ for $j = 1, 2, 3$ and $\beta \in [0, 0.0375]$.

The top plot on the left shows the bidding prices p_j and the multiplier $\pi(\beta)$, the proxy of the equilibrium price in our model. For these runs, the bid prices $p_2 = p_3$ have a constant value equal to the bid at equilibrium ($\pi^* = 1.5$ in Subsection 5.2). There is some fluctuation in the values of p_1 because the bidding price of agent 1 is below the equilibrium price π^*; in fact, any $p_1 \in [0.5, 1]$ is a solution. The warm-start strategy with decreasing β reduced the variability, and all the displayed values represent equilibrium points, including when $\beta = 0$.

The remaining three graphs in Figure 1, labeled ISOi, report the dispatch decided by the ISO version seen by each agent problem in the lower level. In each graph, the dispatch is stacked from cheaper to more expensive, until completion of the demand $d = 0.5$ (the blue top thick line). We note that the three ISOs agreed on the dispatch of the cheapest generator. To visualize the effect of the "reserve" associated with the parameter β, the corresponding dispatch, l_1^i, is accompanied with a dotted line with the generation bid $g_1^i = g_1^{\max} = 1$. For the other agents, there is some disagreement in the dispatch of the ISOs, because the bidding prices p_2 and p_3 are equal, making the ISO indifferent to which one is dispatched to meet the demand (related to the indifference regions mentioned in Remark 2 and shown in the right graphs in Figure 1).

In the left plot in Figure 1, the equilibrium price is not visible in the graph because it always coincides with the bid of agent 2 at a value equal to the marginal cost of agent 3:

$$\pi(0) = \pi(\beta) = \varphi_{j_{\mathtt{mg}}+1} = \varphi_3 = 1.5 .$$

Notice that the marginal agent, $j_{\mathtt{mg}} = 2$, could have bid any price below its maximum,

$$p_{j_{\mathtt{mg}}}^{\max} = p_2^{\max} = 2 .$$

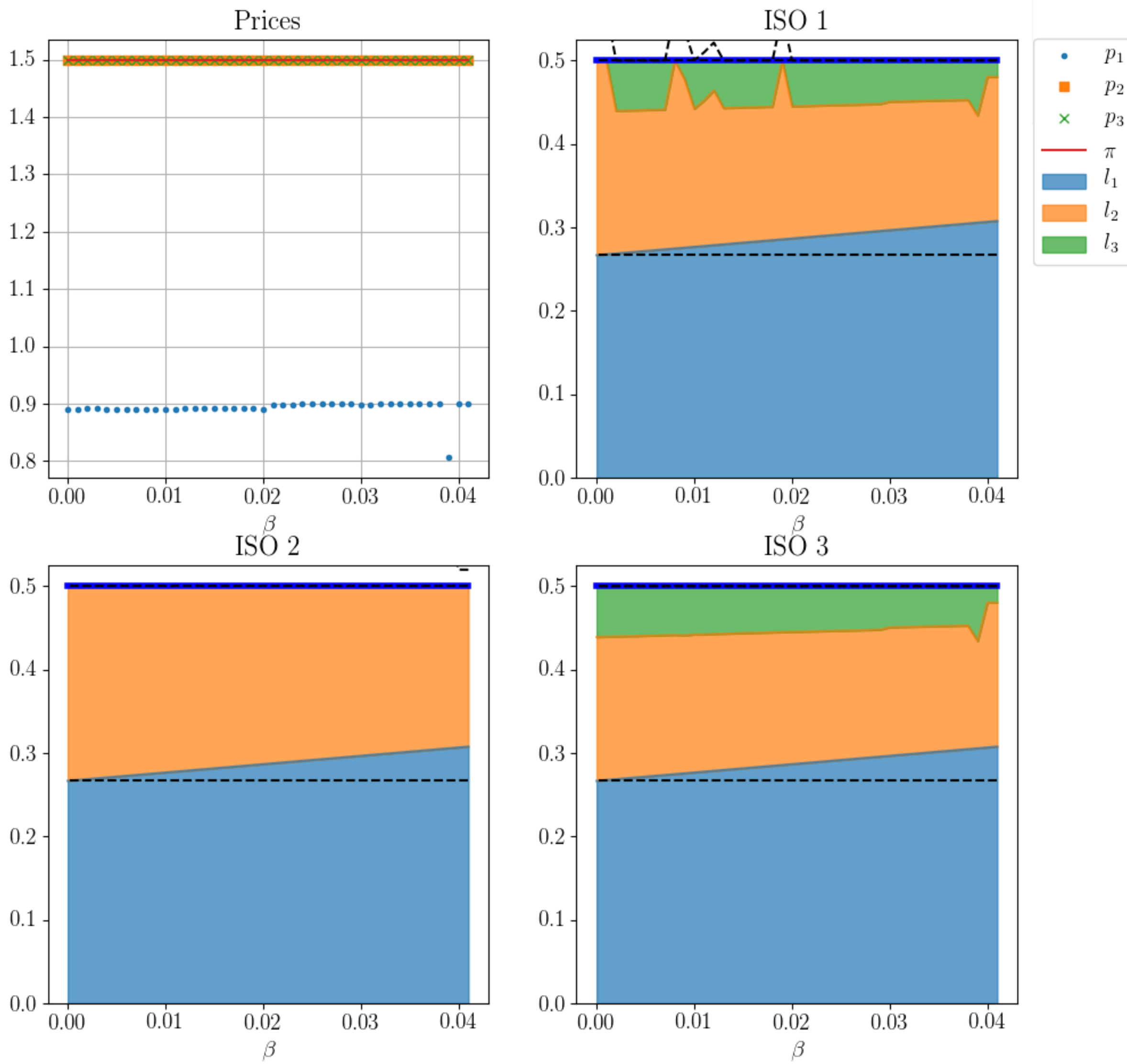

Figure 1: A selection of 41 runs, over the 2500 runs, that yielded an equilibrium when using the warm-start strategy with decreasing β. The top graph on the left shows an alignment of the equilibrium price with the bidding price of player 2, at the value 1.5 (also equal to p_3). The three other figures show the dispatch decided by the ISO in each of the lower-level problems of the three generators. The difference between the dotted line and the blue area represents the ISO's reserve, that increases with the value of β. The perception of the marginal agent, ISO2, gives the dispatch at equilibrium. The ISO seen by agent 1 is affected by the indifference region created by having $p_2 = p_3$. This is perceptible by ISO1 alternating between dispatching agent 2 and agent 3, creating the peaks in the top right graph. A similar behaviour is observed on ISO3, for larger values of β.

Thanks to the regularization, and the warm-starting procedure with decreasing values of β, even for $\beta = 0$ PATH yielded the smallest price in the allowed range.

In order to understand better this issue, we ran PATH only with $\beta = 0$, over several batches with different initial points, without success. We also replaced bilinear terms by new variables, and added new constraints using McCormick inequalities, but this new variant also had numerical issues when $\beta = 0$. These failures can be regarded as numerical evidence of the difficulties of solving bilinear complementarity formulations of equilibrium problems. Eventually, we found the best heuristic was the warm-start approach with decreasing values of the regularization parameter β.

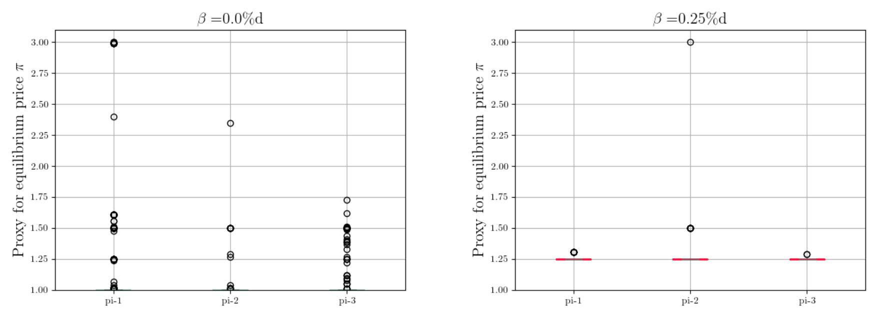

Figure 2: Boxplot for proxies of equilibrium prices. On the left ($\beta = 0$) the multiplier varies wildly, in high ranges that depend on the player, there is no agreement among the respective ISOs. On the right, the regularization yields the same median for the three players, at the optimal value.

Proceeding further with our analysis, we noticed that warm-starting appears to be necessary because of the ambiguities the ISO faces when more than one agent bids the same price (recall Remark 2 and the peaks with the dispatch alternating between agent 2 and 3, on right graphs in Figure 1). We conjectured that those regions of indifference were hindering the solution process of the ISO. This conjecture was confirmed when repeating the experiment, this time eliminating the possibility of ambiguities. To this aim, we re-defined the maximum bidding price of the marginal player so that it stays below the marginal cost of the next agent (first one that is not dispatched):

$$p^{\max}_{j_{\mathtt{mg}}} = \frac{\varphi_{j_{\mathtt{mg}}} + \varphi_{j_{\mathtt{mg}}+1}}{2} < \varphi_{j_{\mathtt{mg}}+1} \quad \Leftrightarrow \quad p^{\max}_2 = 1.25 < \varphi_3 = 1.5\,.$$

In this new instance, the most expensive generator cannot bid competitive prices and so the dispatched agents are more easily defined (the ISOs do not face ambiguous situations). The number of successful runs now were 1050, 1076, 70, and 106 for $\beta = 0, 0.25, 0.5, 0.75\%$d, respectively. When comparing the boxplots of the multiplier $\pi(\beta)$ for $\beta = 0$ and $\beta = 0.25$ reported in the right and left plots in Figure 2, we notice that the positive value of β consistently gave the optimal price of equilibrium, that is $\bar{\pi} = 1.25$ for most of the runs. The model with $\beta = 0$, by contrast, exhibits much more variability in the output, both in the value obtained by each ISO with different starting points, and in the values obtained by the three ISOs with one starting point.

In fact, examining closely the results, for $\beta = 0$ we found just *one* run with correct bidding prices $p_1 = 0.5$, $p_2 = 1.25$ and $p_3 = 1.5$. Moreover, in this case the multiplier $\pi(0)$ took the higher value $p_3 = 1.5$, instead of the maximum price of the marginal agent $p_2^{\max} = 1.25$. The top left graph on Figure 3 shows the bidding prices π_i informed by the three ISO with $\beta = 0$. We notice a clear disagreement on the values of the price of the marginal player (represented by orange squares). The top right graph on Figure 3 reports the output for the multiplier $\pi(0)$. In both graphs in the top, the vertical line marks the position of the only run that gave an equilibrium. The comparison with the bottom plots, reporting the results obtained with the model with regularization, is striking. The parameter $\beta = 0.25\%d$ represents a negligible amount of reserve, likely to be absorbed by the small corrections that are needed when actually operating the system. With the regularized model, the three ISOs agreed on the prices of the

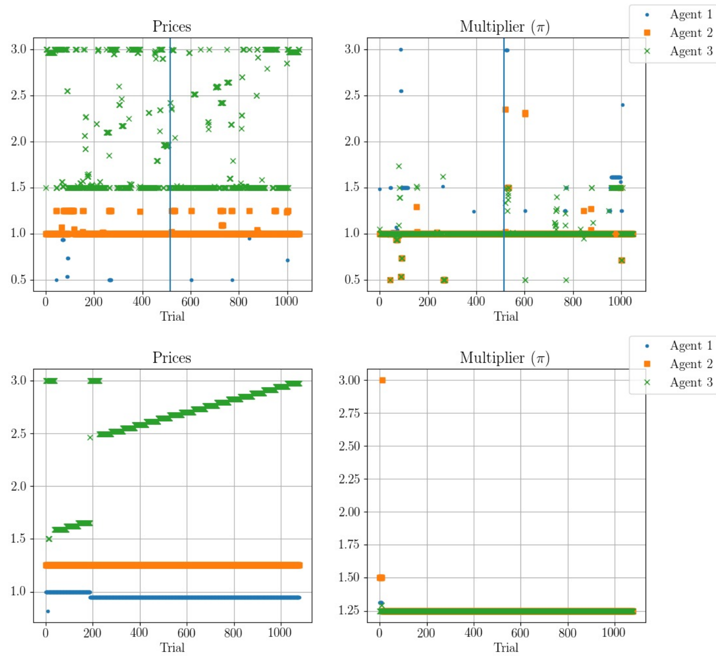

Figure 3: Bidding prices (top left) and multiplier (top right) for different starting points with $\beta = 0$. The only equilibrium, marked with the vertical line, has the value $\pi(0) = 1.5$. In the bottom graphs, for $\beta = 0.25\%d$, all runs except 3 gave the equilibrium price $\pi(\beta) = 1.25$, smaller than with $\beta = 0$, as expected.

marginal player (the orange squares on the left graph). As shown in the left graph, almost no variability is observed on the prices of the dispatched agents, with index 1 and 2. Agent 3 varying prices has no consequences, because the agent is not dispatched. Furthermore, *barring 3 runs only*, the model yielded the correct bidding prices and the multiplier $\pi(\beta) = 1.25$, which is an equilibrium price of minimal norm for the considered market.

Concluding Remarks

We presented theoretical analysis pointing out some downsides of EPEC models in a general bilevel setting. While the considered instance is simple, it serves well to provide a clear insight that the model can choose the highest possible value for the marginal price, in reasonable market configurations (a situation certainly undesirable).

Our proposal of replacing the lower-level ISO problem by a new regularization addresses this issue, as it ensures that in the limit the *minimal norm* price signal is produced. The regularized problem, which remains a linear programming problem if a polyhedral norm is employed, has interesting economic interpretations derived from analyzing both its dual and primal versions. Namely, the regularization can be seen as endowing the ISO with a small reserve that allows to control the marginal rent of the dispatched agents, and indirectly discourages the model from choosing larger values for the equilibrium price. The reserve can be thought of as being available out of the market, or simply being incorporated in the corrections that modify the generation when operating the system in real time.

Our theoretical analysis is complemented with a thorough numerical assessment. The experiments, designed to shed a light on the numerical difficulty inherent to solving EPECs, show the interest of the proposal as a stabilizing device that helps guiding the process towards an output that is usually an equilibrium, even if the mixed complementarity formulation resulting from EPEC is not monotone.

Acknowledgements The authors acknowledge the support of TACEMM, the Transatlantic Consortium of Energy Markets and Modeling. Research of the second author is partly funded by CEPID CeMEAI, CNPq Grant 306089/2019-0, and PRONEX–Optimization. The second author also would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme The mathematics of energy systems, supported by: EPSRC grant number EP/R014604/1. The fifth author is supported by CNPq Grant 303913/2019-3, by FAPERJ Grant E-26/202.540/2019, and by PRONEX–Optimization.

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