Complementarity and game-theoretical models for equilibria in energy markets: deterministic and risk-averse formulations

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Abstract Electricity and natural gas transmission and distribution networks are subject to regulation in price, service quality, emission limits. The interaction of competing agents in an energy market subject to various regulatory interventions is usually modeled through equilibrium problems that ensure profit maximization for all the agents. These type of models can be written in different manners, for example by means of mixed complementarity problems, variational inequalities, and game-theoretical formulations. More generally, we consider energy markets both in deterministic and stochastic settings and explore theoretical relations between the various formulations found in the literature and in practice. Our analysis shows that the profit-maximization complementarity model is equivalent to a game with agents minimizing costs if the setting is deterministic or risk neutral. On the other hand, when the agents exhibit risk aversion which is natural in this type of markets, the equivalence no longer holds. This gives rise to an interesting economical interpretation. As a complement to our theoretical study, and for the European natural gas market with deterministic data, we present some numerical results showing the impact of market power on equilibrium prices.

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1 Introduction

In spite of an undeniable worldwide trend of liberalization, industries dealing with energy networks (and to a lesser extent with water supply) continue to be subject to regulation in price, entry, and service quality of the network. Regarding electricity and natural gas transmission and distribution, the specific mechanism chosen for regulation impacts significantly competition and affects the network prices, investment and reliability.

In general, good performance of the regulatory framework results in lower operation and transmission costs, better service quality, and investment to expand the network and face future changes in demand and supply. Regulation plays an important role also with respect to environmental concerns, for example encouraging carbon trading to reduce CO2 emissions.

In a market of energy that is subject to various regulatory interventions it is very important to fully understand the interaction of competing agents. Due to the presence of relatively few companies generating power in a given region, electricity markets are naturally set in an oligopolistic competition framework. A similar situation arises in the natural gas industry.

In a centralized environment the paradigm of cost minimization defines energy prices based on marginal costs or shadow prices obtained by optimization. In a liberalized setting, by contrast, prices are computed through equilibrium models aimed at ensuring profit maximization for all the agents. These type of models can be formulated in different ways; for example, by means of mixed complementarity problems, bi-level programming, mathematical programs with equilibrium constraints. We mention [5], [17], [18], [2], [1], [26], [19], [8], [4], [34], without the claim of being exhaustive.

In this work we explore the relations between mixed complementarity, variational inequality, and game-theoretical formulations of energy markets both in deterministic and stochastic settings. Our analysis shows that the profit-maximization complementarity formulation is equivalent to a game with agents minimizing costs if the setting is deterministic or risk neutral. On the other hand, when the agents in the market exhibit risk aversion, which is natural in this type of markets, the equivalence no longer holds. More precisely, the risk-averse game becomes equivalent to a complementarity model where agents maximize the expected remuneration and hedge risk only in the cost.

In the development that follows, we consider a stylized energy market that is general enough to cover the generation capacity expansion model [9] as well as the European natural gas market model in [14]. For the latter market and in a deterministic setting, we also present some numerical results showing the impact of market power on equilibrium prices.

Some comments about our notation and terminology are in order. For x, y in any given space, we denote by $\langle x, y \rangle$ the usual (Euclidean) inner product, and we write $x \perp y$ to say that $\langle x, y \rangle = 0$. By $\mathscr{N}_D(x)$ we denote the normal cone to the convex set D at x, that is $\mathscr{N}_D(x) = \{w : \langle w, y - x \rangle \le 0, \text{ for all } y \in D\}$ if $x \in D$ and $\mathscr{N}_D(x) = \emptyset$ otherwise.

The variational inequality (VI) [11] associated to a mapping F and a convex set D consists in determining a point $\bar{x} \in D$ such that the following inequality holds, for every $y \in D$: $\langle F(\bar{x}), y - \bar{x} \rangle \geq 0$. In terms of the normal cone, this means that $0 \in F(\bar{x}) + \mathcal{N}_D(\bar{x})$. The latter inclusion is called a generalized equation (GE). The Mixed complementarity problem (MCP) is a VI (equivalently, GE) with the set D defined by box constraints (where some bounds can be infinite).

2 A Simple Network of Agents

Our initial market is composed of producers, traders, and one end-consumption sector like in Figure 1. Producers generate some kind of good (electricity, natural gas) that is sold to traders in an amount $S_P^i q_p^i$ for the i-th producer. The j-th trader buys from the producers an amount $B_T^j q_T^j$ and sells to consumers the product, after transporting and possibly modifying it, in an amount $S_T^j q_T^j$.

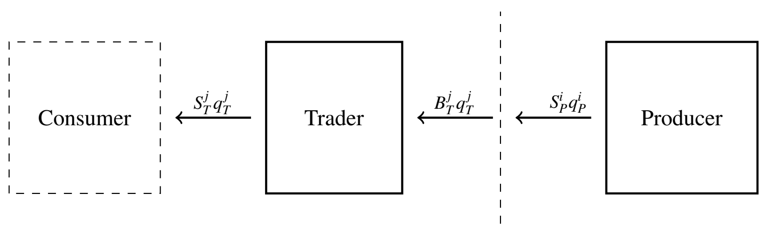

Fig. 1 A Simple Market.

We shall see in Section 5.1 that the model can easily incorporate pipeline and storage operators, marketers, and other outsourcing agents like in [14]. For simplicity, and without loss of generality, in our presentation we analyze a network with only producers and traders that captures the main properties of the market model. Differently from [14], we consider a setup suitable for [9], in which decision variables are separated in two stages. For producers, for instance, some investment to increase capacity has to be decided at stage 0, in order to decide how much produce at stage 1. Another example is, in the presence of uncertainty, when the second stage variables are a recourse to correct first stage decisions, taken before knowing the realization of uncertainty; [6].

In what follows, at equilibrium, all variables are denoted with a bar; for instance, $\bar{\pi}$ stands for an equilibrium price.

2.1 Producers, Traders, and Market Clearing

There are N_P producers, each one with decision variable (z_P^i, q_P^i). As mentioned, the variable z_P^i could refer to decisions concerning capacity or technological investments with a smooth convex cost $I_P^i(z_P^i)$. The variable q_P^i is related to operational activities involving a smooth convex cost $c_{P_i}(q_P^i)$. All the producer decision variables are taken in some set X_P^i which represents technological and resource constraints. After transformation of the raw materials, expressed by a matrix S_P^i of suitable dimensions, the producer has the quantity $S_P^i q_P^i$ for sale. In our model, we assume that producers are of the *price taker* type: there exists a market price that they can not influence directly. So, for a given price π_P (exogenous to the players) each producer tries to maximize profit by solving the following problem:

$$\begin{cases} \max \ \left\langle S_P^i q_P^i, \pi_P \right\rangle - c_P^i(q_P^i) - I_P^i(z_P^i) \\ \text{s.t.} \quad (z_P^i, q_P^i) \in X_P^i \,. \end{cases} \tag{1}$$

The trader's model is similar; for $j = 1, \ldots, N_T$, the j-th trader has decision variable (z_P^j, q_T^j). Given a transformation matrix B_T^j of suitable size, the trader buys $B_T^j q_T^j$ from the producers at price π_P. After modifying and/or transporting the product via a matrix S_T^j of suitable dimensions, the quantity $S_T^j q_T^j$ is sold to consumers at price π_T. The trader may have some additional (smooth convex) operational expenses $c_T^j(q_T^j)$ along the process, and maximizes revenue by solving the following problem:

$$\begin{cases} \max \ \left\langle S_T^j q_T^j, \pi_T \right\rangle - \left\langle B_T^j q_T^j, \pi_P \right\rangle - c_T^j(q_T^j) - I_T^j(z_T^j) \\ \text{s.t.} \quad (z_T^j, q_T^j) \in X_T^j \,. \end{cases} \tag{2}$$

We shall see below that, as in [14], traders have a special role in the market, and can exert market power by withholding supply from end costumers.

When the market is at equilibrium, there is no excess of generation and the producers' supply meets the traders' demand:

$$\sum_{i=1}^{N_P} S_P^i \bar{q}_P^i - \sum_{j=1}^{N_T} B_T^j \bar{q}_T^j = 0 \qquad\qquad (\text{mult. } \bar{\pi}_P). \tag{3}$$

The rightmost notation means that the producers are remunerated at a price that clears the market: $\bar{\pi}_P$ is the multiplier corresponding to (3) at an equilibrium.

An environmentally responsible regulator can also impose a CO2 clearing condition, similar to (3), but involving different emission factors, depending on the technology employed to generate energy, see for instance [22], [31]. The essential feature of such constraints is that they *couple* the actions of different agents, and in this sense (3) suffices for our development.

2.2 Consumer Modeling

The representation of the end-consumption sector can be done in different ways, depending on the manner price-taking producers operate in an imperfectly competitive market. Market imperfections can originate in regulatory measures such as price caps and emission limits, and/or in traders exerting market power. We now review some alternatives that fit our general modeling.

2.2.1 Consumer via Inverse-Demand Function

When a price-sensitive demand curve is available, the consumers needs are represented implicitly by their inverse-demand function. Following [14], we model the demand-curve by an affine function $P \cdot + d_0$, depending on given intercept d_0 and matrix P. The dimension of d_0 is the same as of the traders' selling price (π_T in (2)); the matrix P is of order $|\pi_T| \times |S_T^j q_T^j|$. At equilibrium the constraint

$$\sum_{j=1}^{N_T} P S_T^j \bar{q}_T^j + d_0 - \bar{\pi}_T = 0 \tag{4}$$

must be satisfied.

The inverse-demand function is useful to model the influence that the traders may exert on the market, a typical phenomenon in oligopolies. Instead of selling all the goods at price π_T (exogenous, hence not controllable), the trader sells a portion δ^j at price $\sum_{k=1}^{N_T} P S_T^k q_T^k + d_0$ (that depends on the amount of product the trader offers to the market). The factor $\delta^j \in [0,1]$ determines the strength of the influence the trader can have on the market. Accordingly, now the trader's problem (2) is

$$\begin{cases} \max \left\langle S_T^j q_T^j, \pi_T \right\rangle - \left\langle B_T^j q_T^j, \pi_P \right\rangle - c_T^j(q_T^j) - I_T^j(z_T^j) \\ \qquad\qquad + \delta^j \left\langle S_T^j q_T^j, \sum_{k=1}^{N_T} P S_T^k q_T^k + d_0 - \pi_T \right\rangle \\ \text{s.t.} \quad (z_T^j, q_T^j) \in X_T^j . \end{cases} \tag*{$(2)_{\delta^j}$}$$

For future use, note that the initial problem (2) amounts to setting $\delta^j = 0$ for all the traders. Like for (2), both prices π_P and π_T are exogenous for the traders.

2.2.2 Consumer via Explicit Demand Constraint

Sometimes instead of inverse-demand function there is a load duration curve segmented into blocks defining a vector D, which represents the consumers' demand. Accordingly, letting q^0 denote a nonnegative variable, at the equilibrium the constraint

$$\sum_{j=1}^{N_T} S_T^j \bar{q}_T^j + \bar{q}^0 - D = 0 \qquad \text{(mult. } \bar{\pi}_T) \tag{5}$$

should be satisfied. To prevent traders from exerting market power, and following [9], the deficit variable is related in a dual manner to a price cap imposed by the regulating agency:

$$\bar{\pi}_T \leq PC \qquad \text{(mult. } \bar{q}^0)$$

with PC being the maximal allowed price. Note that, in view of their definitions, the variables q^0 and π_T have the same dimension.

In what follows, we refer to (1),$(2)_{\delta^j}$,(3),(4) as *implicit* model; while (1),(2),(3),(5) and the price-cap condition define the *explicit* model.

3 Equilibrium: Mixed Complementarity Formulation

For both consumers models, the equilibrium problem consists in computing prices $\bar{\pi}$ and decision variables $(\bar{z}, \bar{q})$ such that:

- for the i-th producer, problem (1) written with price $\pi_P := \bar{\pi}_P$ is solved by $(\bar{z}_P^i, \bar{q}_P^i)$; and
- for the j-th trader, problem $(2)_{\delta^j}$ written with prices $(\pi_P, \pi_T) := (\bar{\pi}_P, \bar{\pi}_T)$ is solved by $(\bar{z}_T^i, \bar{q}_T^i)$, keeping in mind that if the explicit model is used then $\delta^j = 0$ for all the traders.
- The market is cleared and (3) holds.
- Regarding the price at which the traders sell the final product,
  - if the implicit model is used, the relation (4) holds;
  - if the explicit model is used, both (5) and the price cap conditions (cf. (8) below) hold.

For the sake of clarity we derive first the mixed complementarity problem (MCP) when the consumers model is explicit, i.e., the trader's problem is (2) and both (5) and the price cap condition hold.

3.1 MCP in the Presence of Explicit Demand Constraint

We start by writing down the Karush-Kuhn-Tucker (KKT) optimality conditions for the profit maximization problems of the producers and traders. Typically, in (1) and (2) the feasible sets X_P^i and X_T^j are polyhedra, say, of the form

$$Z_P^i z_P^i + Q_P^i q_P^i \geq b_P^i \quad \text{and} \quad Z_T^j z_T^j + Q_T^j q_T^j \geq b_T^j,$$

respectively. Let μ_P^i and μ_T^j denote the corresponding Lagrange multipliers. The KKT conditions for the producers' problems (1), dropping the super-indices i to alleviate notation, are

$$\begin{aligned} 0 &= I_P'(z_P) - Z_P^\top \mu_P \\ 0 &= c_P'(q_P) - Q_P^\top \mu_P - S_P^\top \pi_P \\ 0 &\le Z_P z_P + Q_P q_P - b_P \perp \mu_P \ge 0\,. \end{aligned} \tag{6}$$

Similarly for the traders, dropping the super-indices j, we write

$$\begin{aligned} 0 &= I_T'(z_T) - Z_T^\top \mu_T \\ 0 &= c_T'(q_T) - Q_T^\top \mu_T + B_T^\top \pi_P - S_T^\top \pi_T \\ 0 &\le Z_T z_T + Q_T q_T - b_T \perp \mu_T \ge 0\,. \end{aligned} \tag{7}$$

The system is completed with (3), (5), and the price cap, written in the form:

$$0 \le PC - \pi_T \perp q^0 \ge 0\,. \tag{8}$$

To write the associated GE in a compact form, we use the primal and dual variables defined by

$$\mathfrak{p} := \left((z_P^i)_{i=1}^{N_P}, (q_P^i)_{i=1}^{N_P}, (z_T^j)_{j=1}^{N_T}, (q_T^j)_{i=1}^{N_T}, q^0 \right) \text{ and } \mathfrak{d} := \left((\mu_P^i)_{i=1}^{N_P}, (\mu_T^j)_{j=1}^{N_T}, \pi_P, \pi_T \right)$$

$$\text{over the sets} \quad \mathfrak{P} := \mathbb{R}^{\sum_{i=1}^{N_P}(|z_P^i|+|q_P^i|)+\sum_{j=1}^{N_T}(|z_T^j|+|q_T^j|)} \times \mathbb{R}^{m^0}_{\ge 0} \tag{9}$$

$$\text{and} \quad \mathfrak{D} := \mathbb{R}_{\ge 0}^{\sum_{i=1}^{N_P}|\mu_P^i|+\sum_{j=1}^{N_T}|\mu_T^j|} \times \mathbb{R}^{|\pi_P|+|\pi_T|} \tag{10}$$

where $|q^0| = |\pi_T|$, by construction. For convenience, we introduce the operations $\texttt{diag}(\cdot)$, $\texttt{col}(\cdot)$, and $\texttt{row}(\cdot)$ for matrices $M^k, k = 1, \ldots, K$:

$$\texttt{diag}(M^k) := \begin{pmatrix} M^1 & & \\ & \ddots & \\ & & M^K \end{pmatrix}, \texttt{col}(M^k) := \begin{pmatrix} M^1 \\ \vdots \\ M^K \end{pmatrix}, \texttt{row}(M^k) := \left[M^1 \ldots M^K \right].$$

With this notation, the matrix below has $|\mathfrak{D}|$ rows and $|\mathfrak{P}|$ columns:

$$B := \begin{pmatrix} \texttt{diag}(Z_P^i) & \texttt{diag}(Q_P^i) & 0 & 0 & 0 \\ 0 & 0 & \texttt{diag}(Z_T^j) & \texttt{diag}(Q_T^j) & 0 \\ 0 & \texttt{row}(S_P^i) & 0 & -\texttt{row}(B_T^j) & 0 \\ 0 & 0 & 0 & \texttt{row}(S_T^j) & I \end{pmatrix}, \tag{11}$$

where I is an identity matrix of order $|\pi_T| = |q^0|$. Finally, we define the following operator acting on primal variables only, and the following dual vector:

$$F(\mathfrak{p}) := \begin{pmatrix} (I_P^{i\,\prime}(z_P^i))_{i=1}^{N_P} \\ (c_P^{i\,\prime}(q_P^i))_{i=1}^{N_P} \\ (I_T^{j\,\prime}(z_T^j))_{j=1}^{N_T} \\ (c_T^{j\,\prime}(q_T^j))_{j=1}^{N_T} \\ PC \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} (b_P^i)_{i=1}^{N_P} \\ (b_T^j)_{j=1}^{N_T} \\ 0 \\ D \end{pmatrix}. \tag{12}$$

The GE that results from putting together the relations in (6), (7), (3), (5), and (8) is

$$0 \in \begin{bmatrix} 0 & -B^\top \\ B & 0 \end{bmatrix} \begin{pmatrix} \mathfrak{p} \\ \mathfrak{d} \end{pmatrix} + \begin{pmatrix} F(\mathfrak{p}) \\ -b \end{pmatrix} + \mathscr{N}_{\mathfrak{P}\times\mathfrak{D}}(\mathfrak{p},\mathfrak{d}). \tag{13}$$

3.2 MCP in the Presence of Inverse-Demand Function

As the traders' conditions are more involved when there is market power, we shall keep the super-indices j (as otherwise there might be some confusion); the optimality system for the traders then reads as follows:

$$\begin{aligned} 0 &= I_T^{j\,\prime}(z_T^j) - Z_T^{j\,\top}\mu_T^j \\ 0 &= c_T^{j\,\prime}(q_T^j) - Q_T^{j\,\top}\mu_T^j + B_T^{j\,\top}\pi_P - (1-\delta^j)S_T^{j\,\top}\pi_T \\ &\qquad -\delta^j S_T^{j\,\top}\Big(\sum_{k=1}^{N_T} PS_T^k q_T^k + d_0\Big) - \delta^j S_T^{j\,\top}P^\top S_T^j q_T^j \\ 0 &\le Z_T^j z_T^j + Q_T^j q_T^j - b_T^j \perp \mu_T^j \ge 0\,. \end{aligned} \tag{14}$$

As before, the KKT conditions (6) and (14), together with the market clearing condition (3) and the implicit representation of consumers via (4), give a GE on both primal and dual variables. There are a few differences with (13), though:

- There is no deficit q^0, so the primal variables and primal feasible set are now

  $$\tilde{\mathfrak{p}} := \Big((z_P^i)_{i=1}^{N_P}, (q_P^i)_{i=1}^{N_P}, (z_T^j)_{j=1}^{N_T}, (q_T^j)_{j=1}^{N_T}\Big) \text{ and } \tilde{\mathfrak{P}} := \mathbb{R}^{\sum_{i=1}^{N_P}(|z_P^i|+|q_P^i|)+\sum_{j=1}^{N_T}(|z_T^j|+|q_T^j|)}.$$

  Accordingly, instead of the matrix B from (11), we consider the sub-matrix $\tilde{B}$ obtained by eliminating from B the last row and column. Dual variables remain unchanged, so the GE uses $\tilde{B}$ and an additional row to represent (4).
- The market power terms in the third line in (14) enter the primal operator:

$$\tilde{F}(\tilde{\mathfrak{p}}) := \begin{pmatrix} I_P^{i\,\prime}(z_P^i) \\ c_P^{i\,\prime}(q_P^i) \\ I_T^{j\,\prime}(z_T^j) \\ c_T^{j\,\prime}(q_T^j) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta^j S_T^{j\,\top}\Big(\sum_{k=1}^{N_T} PS_T^k q_T^k + d_0\Big) + \delta^j S_T^{j\,\top}P^\top S_T^j q_T^j \end{pmatrix}.$$

To alleviate the writing we omitted the super-indices ranges: $i = 1,\dots,N_P$ and $j = 1,\dots,N_T$, which are clear from the context; see (12).

- Replacing (5) by (4) modifies the dual vector as follows: $\tilde{b} := \left(b_P^i, b_T^j, 0, -d_0\right)^\top$, where, once again, i and j run in their respective ranges, as in (12).

Finally, the GE with the implicit model is

$$0 \in \tilde{A}\begin{pmatrix}\tilde{\mathfrak{p}}\\ \mathfrak{d}\end{pmatrix} + \begin{pmatrix}\tilde{F}(\tilde{\mathfrak{p}})\\ -\tilde{b}\end{pmatrix} + \mathscr{N}_{\tilde{\mathfrak{P}}\times\mathfrak{D}}(\mathfrak{p},\mathfrak{d}), \tag{15}$$

for a matrix $\tilde{A}$ that, unlike the one in (13), is not skewed symmetric (and, moreover, has a last line relating primal and dual elements):

$$\tilde{A} := \begin{bmatrix} 0 & -\tilde{B}^\top \begin{pmatrix} 0\\ 0\\ 0\\ -\mathtt{col}((1-\delta^j)S_T^{j\top}) \end{pmatrix} \\ \tilde{B} & 0 \\ \begin{bmatrix} 0 & 0 & \mathtt{row}(PS_T^j) & 0 \end{bmatrix} & 0 \quad -I \end{bmatrix}.$$

We shall see in Section 4 that GEs of the form (13) can be reduced to VI in smaller dimensions, which can in turn be interpreted in terms of a Nash game with shared constraints. By contrast, the GE (15) cannot be reformulated the same way directly. We next rewrite (15) in an equivalent form that does have the desired properties.

3.3 Inverse-Demand Function and an Extra Variable

Taking inspiration from the explicit model, we introduce a new primal variable p^0, gathering the portion of supply that the traders cannot influence by exerting market power. Thus, we require the relation

$$\sum_{j=1}^{N_T}(1-\delta^j)S_T^j\bar{q}_T^j - \bar{p}^0 = 0 \tag{16}$$

to be satisfied when the market is at an equilibrium point. In view of its definition, this new variable has the same dimension as the deficit variable q^0 from (5) in the explicit model (and, hence, $|p^0| = |\pi_T|$).

The GE gathering (6), (14), (3), (4), and (16) now employs the primal objects

$$\hat{\mathfrak{p}} := \left(\tilde{\mathfrak{p}}, p^0\right) \text{ and } \widehat{\mathfrak{P}} := \mathbb{R}^{\sum_{i=1}^{N_P}(|z_P^i|+|q_P^i|)+\sum_{j=1}^{N_T}(|z_T^j|+|q_T^j|)+|p^0|}, \tag{17}$$

noting that the dual variables remain the same from the explicit model, given in (10). The primal sets in the implicit and explicit models, from (17) and (9) respectively, only differ in their last component (q^0 and p^0, respectively). While in the explicit model the deficit is nonnegative (as a multiplier of the price cap (8)), in the implicit model the new primal variable is unconstrained. So the normal cone to q^0 will be the null vector and we can require satisfaction of the inverse-demand relation (4) in the corresponding new component of the GE. This eliminates the primal-dual coupling in the last line of matrix $\tilde{A}$ in (15). Similarly for ensuring (16), recalling that the π_T-component of the dual set is the whole space.

The resulting GE is

$$0 \in \begin{bmatrix} 0 & -\hat{B}^\top \\ \hat{B} & 0 \end{bmatrix} \begin{pmatrix} \hat{\mathfrak{p}} \\ \mathfrak{d} \end{pmatrix} + \begin{pmatrix} \hat{F}(\hat{\mathfrak{p}}) \\ -\hat{b} \end{pmatrix} + \mathcal{N}_{\widehat{\mathfrak{P}} \times \mathfrak{D}}(\mathfrak{p}, \mathfrak{d}), \tag{18}$$

where we defined a matrix $\hat{B}$ of order $|\widehat{\mathfrak{P}}|$ and $|\mathfrak{D}|$:

$$\hat{B} := \begin{pmatrix} \tilde{B} & 0 \\ \begin{bmatrix} 0 & 0 & 0 & \mathtt{row}((1-\delta^j)S_T^j) \end{bmatrix} & -\hat{I} \end{pmatrix}, \tag{19}$$

using an identity $\hat{I}$ of order $|\pi_T| = |p^0|$, and the primal operator and dual vector:

$$\widehat{F}(\hat{\mathfrak{p}}) := \begin{pmatrix} I_P^{i\,\prime}(z_P^i) \\ c_P^{i\,\prime}(q_P^i) \\ I_T^{j\,\prime}(z_T^j) \\ c_T^{j\,\prime}(q_T^j) - \delta^j S_T^{j\,\top}(\sum_{k=1}^{N_T} PS_T^k q_T^k + d_0) - \delta^j S_T^{j\,\top} P^\top S_T^j q_T^j \\ -\sum_{k=1}^{N_T} PS_T^k q_T^k - d_0 \end{pmatrix} \text{ and } \\ \hat{b} := \begin{pmatrix} b_P^i \\ b_T^j \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

Like for (12), in both vectors $i = 1, \ldots, N_P$ and $j = 1, \ldots, N_T$.

4 Equivalent Mixed Complementarity Formulations

Both GEs (13) and (18) are defined using very simple normal cones, and have a very specific primal-dual structure. The size of both GEs is the same: the respective primal and dual sets only differ in their last primal component: $q^0 \geq 0$ in the explicit model, and unconstrained p^0 in the implicit one.

To establish the relation of the MCP models with a game-theoretical formulation, we state a result from [27]; see also [15]. Our GEs are a particular case of the setting covered by the reduction method in [27], as the primal sets $\mathfrak{P}$ are cones in both models. Here, the relation with a game could actually be also shown directly, by comparing the KKT conditions of the MCP model with those for a game. We prefer to state the more general result, because it includes a nice characterization of dual variables as solutions to a certain linear programming problem, defined *a posteriori*, once the primal solution is available. As dual variables have economical meaning as prices, this is an interesting feature; see Remark 4.

Theorem 1. *The following statements are equivalent:*

Primal-Dual GE: *the primal-dual pair* $(\bar{\mathfrak{p}}, \bar{\mathfrak{d}})$ *satisfies* (13).
Primal GE + Dual LP: *the primal variable* $\bar{\mathfrak{p}}$ *solves the generalized equation*

$$0 \in F(\mathfrak{p}) + \mathscr{N}_{\mathfrak{P}^0}(\mathfrak{p}) \tag{21}$$

where $\mathfrak{P}^0 := \mathfrak{P} \cap \mathscr{S}$ *and* $\mathscr{S} := \left\{ \mathfrak{p} : b - B\mathfrak{p} \in \mathbb{R}_{\leq 0}^{\sum_{i=1}^{N_P} |\mu_P^i| + \sum_{j=1}^{N_T} |\mu_T^j|} \times \{0 \in \mathbb{R}^{|\pi_P| + |\pi_T|}\} \right\}$.
As for the dual variable, $\bar{\mathfrak{d}}$ *solves the linear programming problem*

$$\begin{cases} \min \ \langle B\bar{\mathfrak{p}} - b, \mathfrak{d} \rangle \\ \text{s.t.} \ \ B^\top \mathfrak{d} - F(\bar{\mathfrak{p}}) \in \mathscr{N}_{\mathfrak{P}}(\bar{\mathfrak{p}}) \\ \qquad \mathfrak{d} \in \mathfrak{D}\,. \end{cases} \tag{22}$$

Proof. The statement is just a rewriting of Propositions 1 and 2 in [27] in our notation. Specifically, the respective correspondence for primal elements is $(p, d(p), P) = (\mathfrak{p}, -F(\mathfrak{p}), \mathfrak{P})$, for the dual ones $(y, Y) = (\mathfrak{d}, \mathfrak{D})$, and for the matrix and vector $(A, f) = -(B^\top, b)$. Our set $\mathscr{S}$ corresponds to Z in Proposition 2, using the fact that in our setting the polar cone therein, $Y^0 = \mathfrak{D}^0$, has a very simple expression. □

Existence of solutions to the GE (21) can be guaranteed under mild assumptions, such as continuity of F and convexity and compactness of $\mathfrak{P}^0$, [11, Corollary 2.2.5]. These conditions are natural in our context: components of F consist of derivatives of smooth convex functions and the feasible set $\mathfrak{P}^0$ represents limited resources. Furthermore, the existence of solutions of (21) implies the existence of solutions of the (bounded) linear program (22), whose optimal value is zero.

The interest of Theorem 1 is twofold. First, the GE (21) is in primal variables only, stated over a set that (for both of our models) is a simple polyhedron. It is therefore a VI with linear constraints. We shall see that in some cases the multipliers corresponding to the constraints provide the equilibrium prices. Once a primal solution is at hand, the dual component of the MCP solution can be found by solving an easy linear program. This feature is attractive to identify (undesirable) situations in which equilibrium prices are not unique, even if the primal part of the equilibrium points is unique (the linear program solution will not be unique in this case; see Remark 4 below). A second advantage of the equivalent formulation is that, in addition to providing a mechanism for ensuring existence of solutions of the game, the reformulation reveals the particular structure of the set $\mathfrak{P}^0$, amenable to decomposition. More precisely, without the coupling constraints (some components in $b - B\mathfrak{p}$, hence in $\mathscr{S}$), the feasible set is decomposable (like $\mathfrak{P}$ from (9), (17)).

This decomposable structure can be exploited by decomposition methods, like the Dantzig-Wolfe algorithms developed in [20]; see also [13] and [3].

4.1 Game for the Explicit Model

For the market in Section 2, instead of viewing the agents as maximizing revenue like in the complementarity model, we consider a Generalized Nash Equilibrium Problem (GNEP) [10] with players minimizing costs. The coupling constraints in the game are (3) and (5). In addition to the traders and producers, there is an additional player, indexed by number "0", in charge of capping prices. Specifically, the purpose of the game is to find $\tilde{\mathfrak{p}} = \left((\tilde{z}_P^i)_{i=1}^{N_P}, (\tilde{q}_P^i)_{i=1}^{N_P}, (\tilde{z}_T^j)_{j=1}^{N_T}, (\tilde{q}_T^j)_{i=1}^{N_T}, \tilde{q}^0\right)$ such that the following minimization problems are solved by $\tilde{\mathfrak{p}}$:

$$\textbf{Producers}\quad \begin{cases} \min_{(z_P^i,q_P^i)\in X_P^i} & I_P^i(z_P^i)+c_P^i(q_P^i) \\ \text{s.t.} & S_P^i q_P^i + \displaystyle\sum_{i\neq k=1}^{N_P} S_P^k \tilde{q}_P^k - \sum_{j=1}^{N_T} B_T^j \tilde{q}_T^j = 0 \end{cases} \tag{23}$$

$$\textbf{Traders}\quad \begin{cases} \min_{(z_T^j,q_T^j)\in X_T^j} & I_T^j(z_T^j)+c_T^j(q_T^j) \\ \text{s.t.} & -B_T^j \tilde{q}_T^j + \displaystyle\sum_{i=1}^{N_P} S_P^i \tilde{q}_P^i - \sum_{j\neq k=1}^{N_T} B_T^k \tilde{q}_T^k = 0 \\ & S_T^j q_T^j + \displaystyle\sum_{j\neq k=1}^{N_T} S_T^k \tilde{q}_T^k + \tilde{q}^0 - D = 0 \end{cases} \tag{24}$$

$$\begin{array}{l}\textbf{Consumer}\\ \textbf{representative}\end{array}\quad \begin{cases} \min_{q^0\geq 0} & \langle PC, q^0 - D\rangle \\ \text{s.t.} & \displaystyle\sum_{j=1}^{N_T} S_T^j \tilde{q}_T^j + q^0 - D = 0 \end{cases} \tag{25}$$

In the GNEP (23)-(25), the market between producers and traders is cleared, and demand is satisfied up to certain deficit, q^0. The deficit is minimized by the action of the additional player, who tries to reduce the impact of imposing a price cap. In Corollary 2 below it is shown that (the negative of) the multiplier of the coupling constraint (5) is precisely the traders' remuneration in (2). We shall also see that in the game formulation, the price cap is maintained in an indirect manner, via (25).

In the game, the solution of each individual problem depends on the decisions of the other agents in the market: for instance (24) is an optimization problem on the j-th trader variables (say, $\mathfrak{p}_j$), that depends on actions of other traders (say, on $\mathfrak{p}_{-j}$). A primal point $\bar{\mathfrak{p}}$ is a Nash equilibrium for the game (23)–(25) when each player's optimal decision (say, $\bar{\mathfrak{p}}_j$) is obtained by solving the individual problem (say, (24)) after fixing the other players' decisions to the corresponding entries on $\bar{\mathfrak{p}}$ (say $\bar{\mathfrak{p}}_{-j}$).

As this notion is so general that it includes points contradicting the natural intuition of what an equilibrium must be, it is further specialized to the notion of *variational equilibrium*, as follows. Note that the value function for the producers

$$v_P^i(x) := \begin{cases} \min_{(z_P^i, q_P^i) \in X_P^i} \; I_P^i(z_P^i) + c_P^i(q_P^i) \\ \text{s.t.} \qquad S_P^i q_P^i + \sum_{i \neq k=1}^{N_P} S_P^k \tilde{q}_P^k - \sum_{j=1}^{N_T} B_T^j \tilde{q}_T^j = x\,, \end{cases}$$

is convex. Furthermore, because in (23) all constraints are linear and the objective function is differentiable, there exists a Lagrange multiplier $\check{\pi}_P^i$ associated to the equality constraint. This multiplier represents a marginal cost, since it satisfies the inclusion $-\check{\pi}_P^i \in \partial v_P^i(0)$, [16, Theorem VII.3.3.2]. The issue with a generic Nash equilibrium like $\bar{\mathfrak{p}}$ above is that it may have multipliers associated to coupling constraints of the players' problems that are different for different players. In economical terms, this means that the equilibrium is "unfair", because it benefits some players more than others. To avoid this undesirable feature, we shall solve a VI derived from the game and find a variational equilibrium (VE) [10] of the GNEP (23)–(25), ensuring that the multipliers associated with the coupling constraints are the same.

By Theorem 1, the GE (13) is equivalent to solving the GE (21), written with the data from Subsection 2.2.2. Putting together (11), (9) and (12) yields for (21) the following:

$$0 \in F(\mathfrak{p}) + \mathcal{N}_{\mathfrak{P}^0}(\mathfrak{p}),$$

where the feasible set $\mathfrak{P}^0 := \prod_{i=1}^{N_P} X_P^i \times \prod_{j=1}^{N_T} X_T^j \times \mathbb{R}_{\geq 0}^{m^0} \cap \mathscr{S}$ depends on the coupling set $\mathscr{S} := \{\mathfrak{p} = (z_P^i, q_P^i, z_T^j, q_P^j, q^0) : (3) \text{ and } (5) \text{ hold}\}$.

The equivalence between the MCP formulation and the generalized Nash game results from Theorem 1.

Corollary 2 (Game Formulation for the Explicit Model). *The MCP in Subsection 2.2.2 and the game* (23)-(25) *are equivalent, in the following sense. Suppose the game has a variational equilibrium*

$$\bar{\mathfrak{p}} := \left((\check{z}_P^i)_{i=1}^{N_P}, (\check{q}_P^i)_{i=1}^{N_P}, (\check{z}_T^j)_{j=1}^{N_T}, (\check{q}_T^j)_{i=1}^{N_T}, \check{q}^0 \right),$$

with $(\check{\mu}_P^i)_{i=1}^{N_P}, (\check{\mu}_T^j)_{j=1}^{N_T}$ *being the corresponding multipliers for the constraints in* (23) *and* (24)*, and let* $\check{\pi}_P$ *and* $\check{\pi}_T$ *be the multipliers associated to the coupling constraints* (3) *and* (5)*.*

Then the primal-dual pair $(\bar{\mathfrak{p}}, \bar{\mathfrak{d}})$ *with* $\bar{\mathfrak{d}} := (\check{\mu}_P, \check{\mu}_T, -\check{\pi}_P, -\check{\pi}_T)$ *solves the MCP given by* (1)-(3)*,* (5)*, and* (8)*.*

Proof. By Theorem 1, for the result to hold, $\bar{\mathfrak{d}}$ needs to solve the linear program therein. For the objects in (13), and for the normal cone to the primal set $\mathfrak{P}$ from (9), this linear program is

$$\left\{\begin{array}{ll}\min\limits_{\substack{\mu_P,\mu_T\geq 0\\ \text{any }\pi_P,\pi_T}} & \sum_{i=1}^{N_P}\left\langle Z_P^i\check{z}_P^i+Q_P^i\check{q}_P^i-b_P^i,\mu_P^i\right\rangle+\sum_{j=1}^{N_T}\left\langle Z_T^j\check{z}_T^j+Q_T^j\check{q}_T^j-b_T^j,\mu_T^j\right\rangle\\ \text{s.t.} & Z_P^{i\top}\mu_P^i=I_P^{i\prime}(\check{z}_P^i)\,,\,Z_T^{j\top}\mu_T^j=I_T^{j\prime}(\check{z}_T^j)\\ & Q_P^{i\top}\mu_P^i+S_P^{i\top}\pi_P=c_P^{i\prime}(\check{q}_P^i)\\ & Q_T^{j\top}\mu_T^j-B_T^{j\top}\pi_P+S_T^{j\top}\pi_T=c_T^{j\prime}(\check{q}_T^j)\\ & \pi_T\leq PC\text{ and }\pi_T^k=PC^k\text{ whenever }\check{q}^{0k}>0\,.\end{array}\right. \tag{26}$$

The optimality conditions for problems (23) and (24) amount to $\check{\mu}_P\,,\check{\mu}_T\,,-\check{\pi}_P$ and $-\check{\pi}_T$ to satisfy the first four equalities in the feasible set of (26). Note also that, by complementarity, the (nonnegative) objective function attains its minimum value at $\check{\mu}_P\,,\check{\mu}_T$. The last line in (26), written with $-\check{\pi}_T$, is $\check{q}^0\perp PC+\check{\pi}_T\geq 0$. As these relations result from the optimality condition of (25), the desired result follows. $\square$

4.2 Game for the Implicit Model

We now apply Theorem 1 to the GE (18). Writing (21) with the data from Subsection 2.2.1, that is, using (19), (17) and (20), we have:

$$0\in\widehat{F}(\hat{\mathfrak{p}})+\mathscr{N}_{\mathfrak{P}^0}(\hat{\mathfrak{p}})\quad\text{ where }\mathfrak{P}^0:=\prod_{i=1}^{N_P}X_P^i\times\prod_{j=1}^{N_T}X_T^j\times\mathbb{R}^{|p^0|}\cap\mathscr{S}\,,$$

for $\mathscr{S}:=\{(z_P^i,q_P^i,z_T^j,q_P^j,p^0):\text{(3) and (16) hold}\}$.

The MCP formulation of (1), $(2)_{\delta^j}$, (3) and (4) is now equivalent to finding a variational equilibrium of the following GNEP:
the point $\tilde{\mathfrak{p}}=\left((\tilde{z}_P^i)_{i=1}^{N_P},(\tilde{q}_P^i)_{i=1}^{N_P},(\tilde{z}_T^j)_{j=1}^{N_T},\,(\tilde{q}_T^j)_{i=1}^{N_T},\tilde{q}^0\right)$ solves the problems

Producers same as (23),

$$\textbf{Traders}\;\left\{\begin{array}{ll}\min\limits_{(z_T^j,q_T^j)\in X_T^j} & I_T^j(z_T^j)+c_T^j(q_T^j)\\ & -\delta^j\left\langle\sum_{k=1}^{N_T}PS_T^kq_T^k+d_0,S_T^jq_T^j\right\rangle\\ \text{s.t.} & \sum_{i=1}^{N_P}S_P^iq_P^i-\sum_{k=1}^{N_T}B_T^kq_T^k=0\\ & \sum_{k=1}^{N_T}(1-\delta^k)S_T^kq_T^k-p^0=0\,,\end{array}\right. \tag{27}$$

$$\textbf{Consumer representative} \quad \begin{cases} \max\limits_{p^0} \left\langle \sum\limits_{k=1}^{N_T} PS_T^k q_T^k + d_0, p^0 \right\rangle \\ \text{s.t.} \quad \sum\limits_{j=1}^{N_T} (1-\delta^j) S_T^j q_T^j - p^0 = 0\,. \end{cases} \tag{28}$$

The game (23),(27)-(28) can be interpreted as follows. The additional player tries to maximize the traders' revenue that is market-power free (given in terms of the inverse-demand function). The traders see their influence on the market as a way of reducing costs, or of increasing their income (the negative δ^j term in the objective function from (27)). Transactions between producers and traders are cleared, as before. Regarding the traders remuneration π_T (that is, the multiplier of constraint (16)), we now show that the additional player controls it in a manner ensuring satisfaction of (4).

Corollary 3 (Game Formulation for the Implicit Model). *The MCPs in Subsections 2.2.1 and 3.3 and the game* (23),(27)-(28) *are equivalent in the following sense. Suppose the game has a variational equilibrium*

$$\bar{\mathfrak{p}} := \left((\check{z}_P^i)_{i=1}^{N_P}, (\check{q}_P^i)_{i=1}^{N_P}, (\check{z}_T^j)_{j=1}^{N_T}, (\check{q}_T^j)_{i=1}^{N_T}, \check{p}^0 \right),$$

with $(\check{\mu}_P^i)_{i=1}^{N_P}, (\check{\mu}_T^j)_{j=1}^{N_T}$ *being the corresponding multipliers for the constraints in* (23) *and* (27)*, and let* $\check{\pi}_P$ *and* $\check{\pi}_T$ *be the multipliers associated to the coupling constraints* (3) *and* (16)*.*

Then the primal-dual pair $(\bar{\mathfrak{p}}, \bar{\mathfrak{d}})$ *with* $\bar{\mathfrak{d}} := (\check{\mu}_P, \check{\mu}_T, -\check{\pi}_P, -\check{\pi}_T)$ *solves the MCP* (18)*, which is equivalent to* (15)*.*

Proof. Like for Corollary 2, we only need to show that $\bar{\mathfrak{d}}$ solves the linear program in Theorem 1. In this case, the normal cone to the primal set $\widehat{\mathfrak{P}}$ from (17) is just the null vector and, hence, the linear program is

$$\begin{cases} \min\limits_{\substack{\mu_P,\mu_T \ge 0 \\ \text{any } \pi_P,\pi_T}} \sum\limits_{i=1}^{N_P} \left\langle Z_P^i \check{z}_P^i + Q_P^i \check{q}_P^i - b_P^i, \mu_P^i \right\rangle + \sum\limits_{j=1}^{N_T} \left\langle Z_T^j \check{z}_T^j + Q_T^j \check{q}_T^j - b_T^j, \mu_T^j \right\rangle \\ \text{s.t.} \quad Z_P^{i\top} \mu_P^i = I_P^{i\,\prime}(\check{z}_P^i), \quad Z_T^{j\top} \mu_T^j = I_T^{j\,\prime}(\check{z}_T^j) \\ \qquad Q_P^{i\top} \mu_P^i + S_P^{i\top} \pi_P = c_P^{i\,\prime}(\check{q}_P^i) \\ \qquad Q_T^{j\top} \mu_T^j - B_T^{j\top} \pi_P + (1-\delta^j) S_T^{j\top} \pi_T = c_T^{j\,\prime}(\check{q}_T^j) \\ \qquad\qquad -\delta^j S_T^{j\top} (\sum\limits_{k=1}^{N_T} PS_T^k \check{q}_T^k + d_0) - \delta^j S_T^{j\top} P^\top S_T^j \check{q}_T^j \\ \qquad \pi_T = \sum\limits_{k=1}^{N_T} PS_T^k \check{q}_T^k + d_0\,. \end{cases} \tag{29}$$

It is easy to see that all the relations in KKT conditions of this problem are verified by $\bar{\mathfrak{d}}$, except for the last equality, corresponding to (4). For the latter, observe that since $\check{\pi}_T$ is the multiplier of the coupling constraint (16), and the variable p^0

is unconstrained in the problem of the extra player (28), the p^0-component of the optimality conditions for the game gives that $0 = -\sum_{k=1}^{N_T} PS_T^k \check{q}_T^k - d_0 - \check{\pi}_T$. $\square$

Remark 4 (Uniqueness of Prices). With the explicit model, equilibrium prices will be unique if the linear program (26) has the unique solution. Likewise for the implicit model, which depends on the linear program (29). This problem can be further simplified, by eliminating the variable π_T, as follows:

$$\begin{cases} \min\limits_{\substack{\mu_P,\mu_T \geq 0 \\ \text{any } \pi_P}} & \sum_{i=1}^{N_P} \left\langle Z_P^i \check{z}_P^i + Q_P^i \check{q}_P^i - b_P^i, \mu_P^i \right\rangle + \sum_{j=1}^{N_T} \left\langle Z_T^j \check{z}_T^j + Q_T^j \check{q}_T^j - b_T^j, \mu_T^j \right\rangle \\ s.t. & Z_P^{i\top} \mu_P^i = I_P^{i\,\prime}(\check{z}_P^i), \quad Z_T^{j\top} \mu_T^j = I_T^{j\,\prime}(\check{z}_T^j) \\ & Q_P^{i\top} \mu_P^i + S_P^{i\top} \pi_P = c_P^{i\,\prime}(\check{q}_P^i) \\ & Q_T^{j\top} \mu_T^j - B_T^{j\top} \pi_P + S_T^{j\top} (\sum_{k=1}^{N_T} PS_T^k \check{q}_T^k + d_0) = c_T^{j\,\prime}(\check{q}_T^j) - \delta^j S_T^{j\top} P^\top S_T^j \check{q}_T^j \,. \end{cases}$$

5 The European Network of Natural Gas

We now consider a network with a third kind of player, called *outsourcer*, in charge of modifying or transporting the product before the traders supply it to the end consumers.

Like before, producers only deal with traders and, therefore, solve problems (1). By contrast, traders now deal also with the outsourcer players, who charge a unitary price π_O for their activity. The exchange between the trader and the oursourcer player involves transformation of the product, represented by matrices $S_{T\to O}^j$, $B_{T\leftarrow O}^j$, S_O^k, B_O^k as schematically represented in Figure 2, with the product flow.

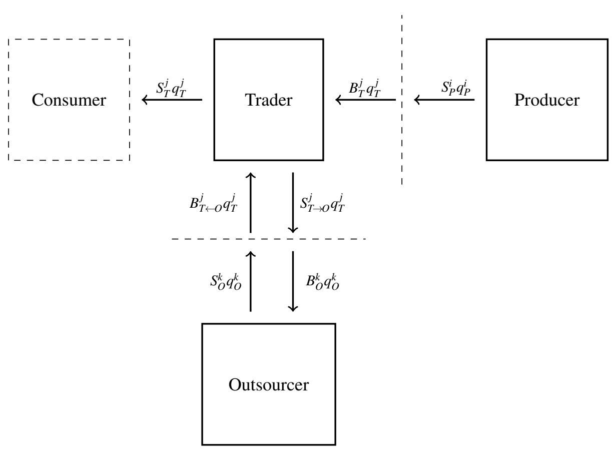

Fig. 2 Market flow.

The j-th trader problem (2) is modified accordingly:

$$\begin{cases} \max \left\langle S_T^j q_T^j, \pi_T \right\rangle - \left\langle B_T^j q_T^j, \pi_P \right\rangle - \left\langle S_{T\to O}^j q_T^j, \pi_O \right\rangle - c_T^j(q_T^j) - I_T^j(z_T^j) \\ \text{s.t.} \quad (z_T^j, q_T^j) \in X_T^j \,. \end{cases}$$

As for the outsourcing players, denoting once more the investment-operational decision variables of the k-th agent by (z_O^k, q_O^k) and similarly for the costs and feasible set, the corresponding maximization problem is

$$\begin{cases} \max \left\langle B_O^k q_O^k, \pi_O \right\rangle - c_O^k(q_O^k) - I_O^k(z_O^k) \\ \text{s.t.} \quad (z_O^k, q_O^k) \in X_O^k \,. \end{cases} \tag{30}$$

To clear the market, in addition to (3) and (4), the exchange between traders and outsourcing players should be balanced and, hence,

$$\sum_{j=1}^{N_T} S_{T\to O}^j \bar{q}_T^j - \sum_{k=1}^{N_O} B_O^k \bar{q}_O^k = 0 \,.$$

The additional balance $\sum_{k=1}^{N_O} S_O^k \bar{q}_O^k - \sum_{j=1}^{N_T} B_{T\leftarrow O}^j \bar{q}_T^j = 0$, is omitted, because it is often automatic from the condition above.

Our previous framework, starting in Section 2, covers the new network. This network, considered in [14] to analyze the market of natural gas in Europe (using a MCP formulation corresponding to the model in Section 2.2.1) is now considered as a test-case in the numerical experience that follows.

5.1 Numerical Assessment

The full European gas network described in [14] covers 54 countries and 36 markets; the market has 7 types of players representing producers, traders, and 5 different outsourcing activities. Specifically, there are 28 producers, 22 traders, 10 liquefiers, 15 re-gasifiers, 22 storage operators, 74 pipeline operators, and 36 marketers.

To illustrate the analysis that can be derived from the models presented above, we coded the MCP and game implicit models in Matlab (R2012a), using PATH [7, 12] to solve the variational problems. The runs were performed on a PC operating under Ubuntu 12.04-64 bit with a processor Intel Atom 1.80GHz $\times$ 4 and 2GB of memory.

The data in [14] gives a game problem with 4620 variables and 488 constraints. We solved the equilibrium problem of the implicit model, with and without market power. In the first instance, the trader's problem $(2)_{\delta^j}$ has $\delta^j \equiv 0$. In the second, $\delta^j = 0.75$ for Russia, Norway, the Netherlands, and Algeria; and $\delta^j = 0.25$ for the Caspian Sea, Denmark, and the UK.

To ensure that the implementation is error-free, we first ran both formulations, that is the game (23),(27),(28) and the MCP (15), and checked whether the corresponding output was alike. Table 1 summarizes the results.

Table 1 Output for the implicit model

FORMULATION	MARKET POWER?	PATH RESIDUAL	CPU (seconds)
Game	no	2E-08	36.7
MCP	no	7E-08	47.3
Game	yes	7E-11	77.1
MCP	yes	2.5E-11	201.5

The ∞-norms of the differences of the primal solutions obtained with both approaches were very small in all the cases. We observed larger differences in the dual components, in percentages ranging up to 6% (for the competitive case, without market power). However, this is still an insignificant difference in this context, which allows us to conclude that the output of both formulations is indeed "the same" and the implementations are correct.

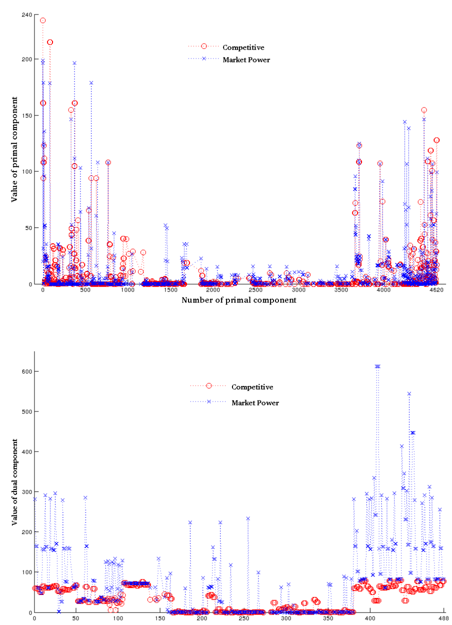

Fig. 3 Comparison of the primal and dual output

An interesting information in Table 1 is the CPU times. In general PATH was very fast, but solving time increased significantly for the MCP formulation when there is market power. At this point, one could ask why this increase is of importance, given that the solution times were still within some minutes. The answer is that this increase, still significant in percentage terms, would blow up once stochasticity is introduced to the model. According to PATH final convergence report, when there is market power, the solver needed much more inner iterations to converge. We observed that when decreasing the solver precision from 10^{-8} to 10^{-6}, both formulations were again solved in about 70 seconds.

A comparison of the results obtained with and without market power can be found in Figure 3, whose top and bottom graphs correspond to the primal and dual output, respectively. The impact of market power is especially noticeable in the dual variables, corresponding to prices: in the bottom graph in Figure 3 the red circles (competitive prices) are systematically lower than the blue crosses (market power). The graphs also show that the largest price increase is in the last components of the dual output, corresponding to the variable $\bar{\pi}_T$, i.e., to the remuneration of the traders.

6 Equilibrium for Stochastic Models

Realistic models for the energy industry often include *uncertainty*: for instance in (5), the actual electrical load may deviate from the predicted one due to random variations of temperature, switch off/on of local consumers, or daylight. Similarly in (1), for the generation costs $c^i_P(\cdot)$ or the available resources defining the feasible sets X^i_P. To reflect such variations, a stochastic model of uncertainty must be built and the risk-averse decision process must be put in a suitable setting.

In what follows we no longer distinguish between producers, traders, and outsourcer players. Instead, we analyze a market with agents trying to maximize profit on a market regulated by coupling constraints or by a price cap. Accordingly, we unify the notation for problems (1)-(2), and consider that the agents solve $\max \left\langle \pi, S^i q^i \right\rangle - I^i(z^i) - c^i(q^i)$, which is equivalent to $\min I^i(z^i) + c^i(q^i) - \left\langle \pi, S^i q^i \right\rangle$.

6.1 Hedging Risk: The Setting

Consider the probability space defined by a measure $\mathbb{P}$ on a sample space Ω equipped with a sigma-algebra $\mathscr{F}$. Decision variables are now random functions in the space $L_p(\Omega,\mathscr{F},\mathbb{P})$ for $p \in [1,+\infty)$, with dual $L_{p^*}(\Omega,\mathscr{F},\mathbb{P})$ for $p^* \in (1,+\infty]$ such that $1/p + 1/p^* = 1$. We sometimes use the shorter notation L_p and L_{p^*} for these spaces, which are paired by the duality product

$$\langle x^*, x\rangle_{\mathbb{P}} = \int_{\omega} \langle x^*(\omega), x(\omega)\rangle \, d\mathbb{P}(\omega)\,.$$

In the presence of uncertainty, a natural reaction of agents in the market is to hedge against undesirable events. For the i-th agent, aversion to volatility is expressed by a coherent (convex) risk measure $\rho^i(\cdot)$, assumed to be a proper function, as in [6, Chapter 6]. One possibility in the space $L_1(\Omega,\mathscr{F},\mathbb{P})$ is to take the *Average Value-at-Risk* of level $1-\varepsilon_i$, a recent renaming of the *Conditional Value-at-Risk* [28]. Namely, given a confidence level $0<1-\varepsilon<1$, if the random outcome $X\in L_1$ represents a loss (lower values are preferred), the measure is given by the expression

$$AV@R_{\varepsilon}(X) := \min_u \left\{ u + \frac{1}{1-\varepsilon}\mathbb{E}[X(\omega)-u]^+ \right\},$$

where $[\cdot]^+ := \max\{0,\cdot\}$ is the positive-part function and $\mathbb{E}(\cdot)$ denotes the expected-value function taken with respect to $d\mathbb{P}$. We consider the more general functions

$$\rho^i(X) := (1-\kappa_i)\mathbb{E}(X) + \kappa_i AV@R_{\varepsilon_i}(X), \tag{31}$$

depending on a given risk-aversion parameter $\kappa_i \in [0,1]$.

It is shown in [6, Theorem 6.4] that any proper coherent risk measure is in fact the support function of the domain of its conjugate; see also [24]. In particular (see [6, Theorem 6.4, and (6.69) in Ex. 6.16]), (31) has the dual representation

$$\rho^i(X) = \sup_{x^*\in X^*} \langle x^*, X\rangle_{\mathbb{P}}, \text{ where} \tag{32}$$

$$X^* := \left\{ x^* \in L_\infty(\Omega,\mathscr{F},\mathbb{P}) : \begin{array}{l} 1-\kappa_i \le x^*(\omega) \le 1-\kappa_i+\kappa_i/\varepsilon_i \ a.e.\ \omega\in\Omega \\ \mathbb{E}(x^*) = 1 \end{array} \right\}.$$

6.2 Stochastic Mixed Complementarity Formulation

For convenience, from now on we make two simplifying assumptions:

- The concept of stochastic equilibrium and its connections with a game formulation is examined for a market with agents maximizing profit as in (1), dropping sub-indices P throughout, using an explicit model (the analysis below remains valid for the implicit model too). Accordingly, the market clearing relation (3) disappears; only a stochastic variant of (4) is in order. Incidentally, this is the framework considered in [9].
- The stochastic counterparts of the agents' problems are set in a two-stage framework. For example, in (23) the "investment" variables z_P^i are of the "here-and-now" type, to be decided before the uncertainty realizes. By contrast the "generation" variables q_P^i are of the type "wait-and-see": they are decided at a second stage, once ω becomes known, so q_P^i depends on ω. So, dropping the sub-index, the random vectors q^i belong to the space $L_p(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^{m^i})$, that is $q(\omega)\in\mathbb{R}^{m^i}$ for all $\omega\in\Omega$, while the prices are in the dual space $L_{p^*}(\Omega,\mathscr{F},\mathbb{P})$.

Given a price cap $PC \in L_{p^*}(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^{m^0})$, the complementarity formulation of stochastic equilibrium with risk aversion (considered also in [9]) is:
Find $\left((\bar z^i\in\mathbb{R}^{n^i})_{i=1}^N, (\bar q^i\in L_p)_{i=1}^N, \bar q^0\in L_p, \bar\pi\in L_{p^*}\right)$ such that

$$\begin{array}{l}\textbf{Risk-averse}\\ \textbf{agents}\end{array}\begin{cases}\min\ I^i(z^i)+\rho^i\Big(c^i(q^i(\omega),\omega)-\big\langle\pi(\omega),S^iq^i(\omega)\big\rangle\Big)\\ \text{s.t.}\quad (z^i,q^i(\omega))\in X^i(\omega)\quad \text{a.e. }\omega\in\Omega\end{cases}\tag{33}$$

$$\begin{array}{ll}\begin{array}{l}\textbf{Coupling}\\ \textbf{constraints}\end{array} & \sum_{i=1}^{N}S^iq^i(\omega)+q^0(\omega)=D(\omega)\quad \text{a.e. }\omega\in\Omega\ \ (\text{mult.}\pi(\omega))\\ \textbf{Price cap} & 0\le q^0(\omega)\perp PC(\omega)-\pi(\omega)\ge 0\quad \text{a.e. }\omega\in\Omega.\end{array}\tag{34}$$

When compared to (1), the agent's problem is now set as a minimization, because the risk averse measure controls losses and not incomes. The objective function in (33) is in fact equivalent to the one considered in [9], taking into account that the investment functions I^i and the first-stage variables z^i are deterministic, recalling that risk measures are equivariant to translations.

6.3 Stochastic Variational Equilibria: Definition

Consider the following stochastic game.
Find $\tilde{\mathfrak{p}}=\Big((\tilde{z}^i\in\mathbb{R}^{n^i})_{i=1}^N,(\tilde{q}^i\in L_p)_{i=1}^N,\tilde{q}^0\in L_{p^*}\Big)$ solving the problems:

$$\begin{array}{l}\textbf{Risk-averse}\\ \textbf{agents}\end{array}\begin{cases}\min\ I^i(z^i)+\rho^i\Big(c^i(q^i(\omega),\omega)\Big)\\ \text{s.t.}\quad (z^i,q^i(\omega))\in X^i(\omega)\quad \text{a.e. }\omega\in\Omega\\ \qquad S^iq^i(\omega)+\sum_{i\neq k=1}^{N}S^k\tilde{q}^k(\omega)+\tilde{q}^0(\omega)=D(\omega)\quad \text{a.e. }\omega\in\Omega\,.\end{cases}\tag{35}$$

$$\begin{array}{l}\textbf{Risk-averse player}\\ \textbf{representing consumers}\end{array}\begin{cases}\min\ \rho^0\Big(\big\langle PC(\omega),q^0(\omega)-D(\omega)\big\rangle\Big)\\ \text{s.t.}\quad q^0(\omega)\ge 0\quad \text{a.e. }\omega\in\Omega\\ \qquad \sum_{i=1}^{N}S^i\tilde{q}^i(\omega)+q^0(\omega)=D(\omega)\quad \text{a.e. }\omega\in\Omega.\end{cases}\tag{36}$$

We define next the concept of variational equilibrium for this stochastic game. Recall that one is generally not interested in arbitrary Nash equilibria, but rather in VE defined as solutions to VIs derived from the game. In the general stochastic context like the one under consideration, instead of going via an explicit VI, we characterize VE using the Lagrange multipliers of the game coupling constraints.

Definition 5 (Stochastic VE). For a stochastic GNEP (35)-(36), the point $\bar{\mathfrak{p}}=\Big((\bar{z}^i)_{i=1}^N,(\bar{q}^i)_{i=1}^N,\bar{q}^0\Big)$ is a *variational equilibrium* if there exists a Lagrange multiplier $\bar{\pi}\in L_{p^*}$ associated to the coupling constraint,

$$\sum_{i=1}^{N} S^i q^i(\omega) + q^0(\omega) = D(\omega) \quad \text{a.e. } \omega \in \Omega\,, \tag{37}$$

the same for all the players, such that $\bar{\mathfrak{p}}$ still solves the agents' problems after relaxing the coupling constraints (as in Proposition 6 below).

We now show that, under mild conditions. the concept is well defined.

Proposition 6 (Existence of Stochastic Multipliers). *For the game* (35)-(36)*, the following holds.*

1. *There exists $\bar{\pi}^0 \in L_{p^*}(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^{m^0})$ such that whenever $\bar{q}^0$ solves* (36)*, it also solves the relaxed problem*

$$\min_{q^0(\omega)\geq 0 \text{ a.e. } \omega} \rho^0\Big(\big\langle PC(\omega), q^0(\omega) - D(\omega)\big\rangle\Big) - \big\langle q^0(\omega), \bar{\pi}^0(\omega)\big\rangle_{\mathbb{P}}\,.$$

2. *Suppose for each problem* (35) *the functions $I^i : \mathbb{R}^{n^i} \to \mathbb{R}$ are smooth and convex, while $c^i : \mathbb{R}^{m^i} \times \Omega \to \mathbb{R}$ are random finite-valued, lower-semicontinuous and convex for almost every $\omega \in \Omega$. Assume, in addition, that the sets $X^i(\omega) \subset \mathbb{R}^{n^i+m^i}$ are nonempty, closed, convex, and some constraint qualification condition holds. If the function $C^i : \mathbb{R}^{n^i} \times L_p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^{m^i}) \to L_p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R})$ given by*

$$[C^i(z^i, q^i)](\omega) := c^i(z^i, q^i(\omega), \omega) \quad \textit{is continuous and well-defined}, \tag{38}$$

then there exists $\bar{\pi}^i \in L_{p^*}(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^{m^0})$ such that whenever $(\bar{z}^i, \bar{q}^i)$ solves (35)*, it also solves the relaxed problem*

$$\begin{cases} \min\ I^i(z^i) + \rho^i(C(z^i, q^i(\omega), \omega)) - \big\langle S^i q^i(\omega), \bar{\pi}^i(\omega)\big\rangle_{\mathbb{P}} \\ \text{s.t.}\ \ z^i \in \mathbb{R}^{n^i}, q^i \in L_p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^{m^i}), \\ \qquad (z^i, q^i(\omega)) \in X^i(\omega) \quad \text{a.e. } \omega \in \Omega\,. \end{cases} \tag{39}$$

Proof. Since the objective function in (36) satisfies (38) and a constraint qualification condition holds automatically for the feasible set, the first item is just a particular case of the second one. Accordingly, we prove the assertion for the problem

$$\begin{cases} \min\ I(z) + \rho(c(z, q(\omega), \omega)) \\ s.t.\ \ z \in \mathbb{R}^n, q \in L_p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^m), \\ \qquad (z, q(\omega)) \in X(\omega) \quad \text{a.e. } \omega \in \Omega\,, \\ \qquad Sq(\omega) = \tilde{D}_0(\omega) \quad \text{a.e. } \omega \in \Omega\,, \end{cases} \tag{40}$$

corresponding to (35) without super-indices i (setting $\tilde{D}_0 := D_0 - \sum_{i\neq k} S^k \tilde{q}^k$). This problem is equivalent to

$$\min I(z) + \rho(C(z,q)) + \iota_{\mathscr{C}}(z,q) + \iota_{\mathbb{R}^n \times \mathscr{S}}(z,q)\,,$$

where $\iota_X(\cdot)$ denotes the indicator function of a set X (i.e., it returns zero for points in X and $+\infty$ otherwise), and where we defined the closed convex sets

$\mathscr{C} := \{(z,q) \in \mathbb{R}^n \times L_p(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^m) : (z,q(\omega)) \in X(\omega)\text{a.e. } \omega \in \Omega\}$ and $\mathscr{S} = \{q \in L_p(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^m) : Sq(\omega) = \tilde{D}_0(\omega)\text{a.e. } \omega \in \Omega\}$. By (38) the objective function is well defined and, with our assumptions, it is convex. Therefore, $(\bar{z},\bar{q})$ solves the GE

$$0 \in \partial\left(I + \rho \circ C + \iota_{\mathscr{C}} + \iota_{\mathbb{R}^n \times \mathscr{S}}\right)(\bar{z},\bar{q}),$$

and the constraint qualification assumption yields that

$$\partial\left(I + \rho \circ C + \iota_{\mathscr{C}} + \iota_{\mathbb{R}^n \times \mathscr{S}}\right)(\bar{z},\bar{q}) = \partial\left(I + \rho \circ C + \iota_{\mathscr{C}}\right)(\bar{z},\bar{q}) + \partial \iota_{\mathbb{R}^n \times \mathscr{S}}(\bar{z},\bar{q}).$$

Since $\partial \iota_{\mathbb{R}^n \times \mathscr{S}}(\bar{z},\bar{q}) = \{0\} \times \mathscr{N}_{\mathscr{S}}(\bar{q})$, we have that

$$0 \in \partial\left(I + \rho \circ C + \iota_{\mathscr{C}}\right)(\bar{z},\bar{q}) + \{0\} \times \mathscr{N}_{\mathscr{S}}(\bar{q}). \tag{41}$$

We claim that for any $\bar{q} \in \mathscr{S}$ the normal cone is given by

$$\mathscr{N}_{\mathscr{S}}(\bar{q}) = \{\nu : \nu(\omega) = S^\top \pi(\omega) \quad \text{a.e. } \omega \in \Omega, \pi \in L_{p^*}(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^{m^0})\}.$$

The $\supset$ inclusion is straightforward. To see the converse one, first note that for any $q \in \mathscr{S}$ the identity $S(q(\omega) - \bar{q}(\omega)) = 0$ holds for a.e. $\omega \in \Omega$ (for simplicity, we omit the symbol a.e. $\omega \in \Omega$ below, noting that relations hold almost everywhere when appropriate.) Thus, $\bar{q} + \theta(q - \bar{q}) \in \mathscr{S}$ for any $\theta \in L_\infty(\Omega,\mathscr{F},\mathbb{P};\mathbb{R})$. By the definition of normal cone,

$$\langle \nu(\omega), \theta(\omega)(q(\omega) - \bar{q}(\omega))\rangle_{\mathbb{P}} \leq 0,$$

and by [29, Corollary 1.9(e)], there exists $\theta \in L_\infty(\Omega,\mathscr{F},\mathbb{P};\mathbb{R})$ such that $|\theta(\omega)| = 1$ and $\langle \nu(\omega), \theta(\omega)(q(\omega) - \bar{q}(\omega))\rangle = |\langle \nu(\omega), q(\omega) - \bar{q}(\omega)\rangle|$. Therefore,

$$\int_\Omega |\langle \nu(\omega), q(\omega) - \bar{q}(\omega)\rangle| d\mathbb{P}(\omega) \leq 0 \quad \Longrightarrow \quad \langle \nu(\omega), q(\omega) - \bar{q}(\omega)\rangle = 0.$$

In particular, for any $u \in \text{Ker}(S)$ and $q(\omega) := \bar{q}(\omega) + u \in \mathscr{S}$, we have that $\langle \nu(\omega), u\rangle = 0$, which means that $\nu(\omega) \in [\text{Ker}(S)]^\perp = \text{Im}(S^\top)$.

As a result, there exists a function $\eta : \Omega \to \mathbb{R}^{m^0}$ such that $\nu(\omega) = S^\top \eta(\omega)$, and since $\nu \in L_{p^*}(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^m)$, the multiplier $\pi : \Omega \to \mathbb{R}^{m^0}$ in $L_{p^*}(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^{m^0})$ exists and is defined by $\pi(\omega) := [S^+]^\top \nu(\omega)$, where S^+ is the Moore–Penrose pseudo inverse. This establishes the claim, since

$$S^\top \pi(\omega) = S^\top [S^+]^\top \nu(\omega) = S^\top [S^+]^\top S^\top \eta(\omega) = S^\top \eta(\omega) = \nu(\omega).$$

In view of our claim, the inclusion (41) can be rewritten in the form

$$0 \in \partial\left(I + \rho \circ C + \iota_{\mathscr{C}}\right)(\bar{z},\bar{q}) - (0, S^\top \bar{\pi})$$

for some $\bar{\pi} \in L_{p^*}(\Omega,\mathscr{F},\mathbb{P};\mathbb{R}^{m^0})$, and the result follows. □

6.4 Relation between Risk-Averse Games and MCP

We now are in a position to give the equivalent mixed-complementarity counterpart of our risk-averse game.

Theorem 7 (MCP Formulation for the Risk-Averse Game). *In the setting of Proposition 6, suppose the risk-averse GNEP* (35)-(36) *has a variational equilibrium* $\bar{\mathfrak{p}} := \left((\bar{z}^i)_{i=1}^N, (\bar{q}^i)_{i=1}^N, \bar{q}^0\right)$*, and let* $(\bar{\mu}^i)_{i=1}^N$ *and* $\bar{\pi}$ *denote the respective* L_{p^*}*-multipliers for the endogenous constraints in* (35) *and the coupling constraints* (37)*.*

Then the primal-dual pair $(\bar{\mathfrak{p}}, \bar{\mathfrak{d}})$ *with* $\bar{\mathfrak{d}} := (\bar{\mu}, \bar{\pi})$ *solves the risk-averse MCP derived from the following problems*

$$\textbf{Risk-averse agents} \quad \begin{cases} \min\ I^i(z^i) + \rho^i\Big(c^i(q^i(\omega),\omega)\Big) - \langle \pi, S^i q^i \rangle_{\mathbb{P}} \\ \text{s.t.}\ \ (z^i, q^i(\omega)) \in X^i(\omega) \quad \text{a.e. } \omega \in \Omega \end{cases} \tag{42}$$

Coupling constraints *as in* (37)

Risk-averse Price cap $0 \le q^0(\omega) \perp x_0^*(\omega) PC(\omega) - \pi(\omega) \ge 0 \quad \text{a.e. } \omega \in \Omega$

for x_0^* *solving*

$$\begin{cases} \min\ \displaystyle\sum_{i=1}^N \Big\langle \big\langle x_0^*(\omega) PC(\omega), S^i \bar{q}^i(\omega) \big\rangle \Big\rangle_{\mathbb{P}} \\ \text{s.t.}\ \ \mathbb{E}(x_0^*) = 1 \\ \qquad 1 - \kappa_0 \le x_0^*(\omega) \le 1 - \kappa_0 + \dfrac{\kappa_0}{\varepsilon_0} \quad \text{a.e. } \omega \in \Omega. \end{cases} \tag{43}$$

Proof. To derive a complementarity formulation, we first consider (35). By Proposition 6, $(\bar{z}^i, \bar{q}^i)$ solves the relaxed problem (39) with $\bar{\pi} = \bar{\pi}^i$, by Definition 5. Since the optimality conditions of the relaxed problem coincide with those of problem (42), the stated result for the agents follows.

In the case of comsumers' representative, by Proposition 6 and Definition 5, $\bar{q}^0$ solves problem (36) as well as the relaxed problem

$$\min_{q^0 \ge 0} \rho^0\Big(\big\langle PC(\omega), q^0(\omega) - D(\omega)\big\rangle\Big) - \Big\langle \big\langle q^0(\omega), \bar{\pi}(\omega) \big\rangle \Big\rangle_{\mathbb{P}}.$$

We now show that the optimality conditions of the relaxed problem coincide with those of (43), together with the risk-averse price cap condition. Since $PC \in L_{p^*}$ and $D \in L_p$, the affine operator $A : L_p \to L_p$ defined by

$$[A(q^0)](\omega) := \big\langle PC(\omega), q^0(\omega) - D(\omega) \big\rangle$$

is continuous and, hence, the optimality condition for the relaxed problem is

$$0 \in \partial\big(\rho^0 \circ A + i_{\ge 0}\big)(\bar{q}^0) - \bar{\pi} = \partial(\rho^0 \circ A)(\bar{q}^0) + \mathcal{N}_{\ge 0}(\bar{q}^0) - \bar{\pi}.$$

By the normal cone definition, there exist $g \in \partial(\rho^0 \circ A)(\bar{q}^0)$ and $\nu \in L_{p^*}$ such that

$$0 \le \bar{q}^0(\omega) \perp -\bar{v}(\omega) \ge 0 \quad \text{and} \quad 0 = g(\omega) + \bar{v}(\omega) - \bar{\pi}(\omega),$$

almost everywhere in Ω. To get an explicit expression for g above, we apply [35, Thm.2.83] to compute the subdifferential $\partial(\rho^0 \circ A)(\bar{q}^0)$, recalling that the mapping A is affine and continuous, and the risk measure is increasing and finite-valued:

$$g \in \partial(\rho^0 \circ A)(\bar{q}^0) \iff g(\omega) = PC(\omega)s(\omega) \quad \text{for } s \in \partial\rho^0\Big(A(\bar{q}^0)\Big).$$

The definitions of the subdifferential and of the conjugate function give the equivalence $s \in \partial\rho^0(A(\bar{q}^0)) \iff A(\bar{q}^0) \in \partial\rho^{0^*}(s)$. By the dual representation (32), the conjugate of ρ^0 is the indicator function of the (convex and bounded) dual set X^*, that is $\rho^0 = \imath^*_{X^*}$. Then, $\rho^{0^*} = \imath^{**}_{X^*} = i_{X^*}$. Since the subdifferential of the indicator function of a closed convex set is the normal cone of the set, by the definition of the normal cone, the subgradient $g \in \partial(\rho^0 \circ A)(\bar{q}^0)$ has components $g(\omega) = PC(\omega)s(\omega)$ for $s \in X^*$ satisfying $\left\langle A(\bar{q}^0), x^* - s\right\rangle_{\mathbb{P}} \le 0$ for all $x^* \in X^*$. So s maximizes $\left\langle A(\bar{q}^0), x^*\right\rangle_{\mathbb{P}}$ over X^*, and in view of (37), $s = \bar{x}^*_0$ from (43). The risk-averse price cap condition follows from plugging $g(\omega) = PC(\omega)\bar{x}^*_0(\omega)$ in the optimality condition. □

Theorem 7 shows that, like in the deterministic framework, the stochastic game is equivalent to a complementarity model with risk aversion. Nevertheless, the stochastic MCP model is *not of the form* (33), where agents *hedge individually their profit*. Instead, a VE for the game (35)-(37) gives a stochastic equilibrium for a market that is cleared because (37) is satisfied, and where the risk-averse agents are *remunerated in mean at a price that is controlled by a risk-averse price cap*.

In the game, aversion to risk is peculiar in the sense that agents hedge volatility by controlling only variations in the generation costs. In the game problem (42), the remuneration is taken in mean without hedging risk, while in the MCP (33)-(34) each agent tries to control the risk in their individual revenue. In the game the control of volatile prices is "delegated" to some higher instance. This is the same instance that caps the remunerations, only that now the cap is chosen adaptively, in a manner that is optimal for the market, in the sense of (43). By contrast, in the risk-averse MCP, the instance limiting prices only takes into account stochasticity but does not perceive the fact of capping prices as a risky action, perturbing the market.

Our final result shows that the three models become equivalent in a risk-neutral market.

Corollary 8 (Equivalence for Risk-Neutral Agents). *Suppose that for all the agents $\rho^i = \mathbb{E}$, the expected-value function. Then finding a variational equilibrium for the GNEP* (35),(36)-(37) *is equivalent to solving the MCP* (33)-(34) *which is in turn equivalent to the MCP* (42)-(43).

Proof. Straightforward from Theorem 7, noting that the expected-value function is recovered by setting $\kappa_i = 0$ in (31), with the singleton dual set $X^* = \{x^* \equiv 1\}$ in (32). In particular, a risk-neutral representative of the consumers can only take $\bar{x}^*_0 \equiv 1$, which yields the stochastic price cap from (34). The equivalence with the last MCP results from the linearity of the expected-value function. □

Concluding Remarks

Like it has been done in the deterministic case in Section 5.1, it would be interesting to analyze and compare the performances of the risk-averse game versus the risk-averse MCP on a numerical example. However, due to the positive-part function in (31), risk measures are not differentiable and for both models the GE mapping has multi-valued components. In this context, a direct application of a solver like PATH is no longer possible (and there is currently no other established software that can do the job). In [9], the MCP (33)-(34) is "solved" ignoring nondifferentiability issues and treating the mapping as if it were single-valued. This heuristic seems to produce sound results for the considered example, but cannot be regarded as a reliable solution method, of course. In order to handle nonsmoothness, some special technique should be used, for example the approximation procedure in [21].

Finally, instead of handling uncertainty in two stages, a multistage setting can also be of interest. This, keeping in mind that multistage risk averse models remain a delicate subject, involving intricate issues such as time consistency and information monotonicity; see [23]. Last but not least, and as discussed in [32, Section 5], risk-averse variants of sampling approaches like [25] and [33] lack implementable stopping criteria. Multistage risk-averse models present numerous challenges already in an optimization framework, we refer to [30] and references therein for more details.

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Index