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# Merit functions and error bounds for generalized variational inequalities

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## Abstract

We consider the generalized variational inequality and construct certain merit functions associated with this problem. In particular, those merit functions are everywhere nonnegative and their zero-sets are precisely solutions of the variational inequality. We further use those functions to obtain error bounds, i.e., upper estimates for the distance to solutions of the problem.

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## 1. Introduction

Let  $\mathcal{H}$  be a real Hilbert space. Given two mappings  $F, g: \mathcal{H} \rightarrow \mathcal{H}$  and a function  $f: \mathcal{H} \rightarrow (-\infty, +\infty]$ , we consider the generalized variational inequality problem, GVIP( $F, g, f$ ) for short, which is to find a point  $x \in \mathcal{H}$  such that

$$\langle F(x), y - g(x) \rangle + f(y) - f(g(x)) \geq 0 \quad \text{for all } y \in \mathcal{H}, \quad (1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathcal{H}$ . We assume that  $f$  is proper convex, with its effective domain  $\text{dom}(f) = \{x \in \mathcal{H} \mid f(x) < +\infty\}$  being closed. When  $f$  is the indicator function of a closed convex set  $C \subset \mathcal{H}$  (i.e.,  $f(x) = 0$  if  $x \in C$ ,  $f(x) = +\infty$  otherwise) then GVIP( $F, g, f$ ) reduces to the problem considered, for example, in [22]. If  $g(x) = x \forall x \in \mathcal{H}$  then GVIP( $F, g, f$ ) is the same as studied, for example, in [23].

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Furthermore, it is not difficult to see that if both  $g(x) = x$  and  $f$  is the indicator function of a closed convex set  $C$ , then  $\text{GVIP}(F, g, f)$  reduces to the classical variational inequality  $\text{VIP}(F, C)$  [2,5]: Find  $x$  such that

$$x \in C, \quad \langle F(x), y - x \rangle \geq 0 \quad \text{for all } y \in C.$$

We note that there exist also some other formulations of generalized variational inequalities, for example [10,11,20]. These formulations will not be considered in the present paper.

Many fruitful approaches to both theoretical and numerical treatment of  $\text{VIP}(F, C)$  make use of constructing a function  $M: \mathcal{H} \rightarrow (-\infty, +\infty]$  such that

$$M(x) \geq 0 \quad \forall x \in D \supset C \text{ and } M(x) = 0, \quad x \in D \quad \Leftrightarrow \quad x \text{ solves } \text{VIP}(F, C).$$

The set  $D$  is usually either the whole space or the set  $C$  itself, and  $M(\cdot)$  with such properties is commonly referred to as a *merit function*. We refer the reader to [4,12,16] for surveys, further references, and some specific examples of merit functions for classical variational inequalities  $\text{VIP}(F, C)$ . One of the many useful applications of merit functions is in deriving the so-called *error bounds*, i.e., upper estimates on the distance to the solution set  $S$  of  $\text{VIP}(F, C)$ :

$$\text{dist}(x, S) \leq \gamma M(x)^\lambda \quad \forall x \in D' \subset D,$$

where  $\gamma, \lambda > 0$  are independent of  $x$  (but possibly dependent on  $D'$ ). The bound is global if  $D' = D = \mathcal{H}$ , and it is local if the set  $D'$  is some neighbourhood of the solution set  $S$ . We refer the reader to [12] for a recent survey.

To our knowledge, merit functions have not been studied beyond the classical formulation  $\text{VIP}(F, C)$ , except for norms of certain equation reformulations (more on this in Section 2). In this paper, we exhibit three merit functions for  $\text{GVIP}(F, g, f)$ , and show how they can be used to obtain error bounds.

Our notation is standard. By  $\arg \min_{z \in Z} t(z)$  we mean the set of minimizers of the function  $t: \mathcal{H} \rightarrow (-\infty, +\infty]$  over the set  $Z \subset \mathcal{H}$ . The subdifferential of a convex function  $f$  at  $x \in \mathcal{H}$  is denoted by  $\partial f(x) = \{u \in \mathcal{H} \mid f(y) \geq f(x) + \langle u, y - x \rangle \quad \forall y \in \mathcal{H}\}$ . For  $t: \mathcal{H} \rightarrow \mathcal{H}$  and  $Z \subset \mathcal{H}$ , we denote  $t^{-1}(Z) := \{x \in \mathcal{H} \mid t(x) \in Z\}$ .

## 2. Natural residual

For  $\text{VIP}(F, C)$ , it is well known that  $x \in \mathcal{H}$  is a solution if, and only if,

$$0 = x - P_C(x - \alpha F(x)),$$

where  $P_C(\cdot)$  is the orthogonal projector onto  $C$  and  $\alpha > 0$  is arbitrary. Hence, the norm of the right-hand side in this equation can serve as a merit function for  $\text{VIP}(F, C)$ , which is commonly called *natural residual*.

We next derive a similar characterization for  $\text{GVIP}(F, g, f)$ . We note that analogous results are quite well-known also for other extensions of  $\text{VIP}(F, C)$ , e.g., [11,21]. So this characterization is, in a sense, as expected. However, it provides some building blocks for

deriving certain gap functions in the subsequent sections. And we note that our error bound for GVIP( $F, g, f$ ) based on the natural residual is new.

Recall that the proximal map [9,15],  $p_f^\alpha : \mathcal{H} \rightarrow \text{dom}(f)$ , is given by

$$p_f^\alpha(z) := \arg \min_{y \in \mathcal{H}} \left\{ f(y) + \frac{1}{2\alpha} \|y - z\|^2 \right\}, \quad z \in \mathcal{H}, \alpha > 0.$$

Note that the objective function above is proper strongly convex. Since  $\text{dom}(f)$  is closed,  $p_f^\alpha(\cdot)$  is well-defined and single-valued. Define

$$R_\alpha(x) := g(x) - p_f^\alpha(g(x) - \alpha F(x)), \quad x \in \mathcal{H}, \alpha > 0.$$

We next show that  $R_\alpha(\cdot)$  plays the role of natural residual for GVIP( $F, g, f$ ).

**Theorem 1.** *Let  $\alpha > 0$  be arbitrary. An element  $x \in \mathcal{H}$  solves GVIP( $F, g, f$ ) if, and only if,  $R_\alpha(x) = 0$ .*

**Proof.** We have that  $R_\alpha(x) = 0$  is equivalent to

$$g(x) = \arg \min_{y \in \mathcal{H}} \left\{ f(y) + \frac{1}{2\alpha} \|y - (g(x) - \alpha F(x))\|^2 \right\}.$$

By optimality conditions (which are necessary and sufficient, by convexity), the latter is equivalent to

$$0 \in \partial f(g(x)) + \frac{1}{\alpha} (g(x) - (g(x) - \alpha F(x))) = \partial f(g(x)) + F(x)$$

or

$$-F(x) \in \partial f(g(x)),$$

which in turn is equivalent, by the definition of the subgradient, to

$$f(y) \geq f(g(x)) - \langle F(x), y - g(x) \rangle \quad \forall y \in \mathcal{H},$$

which means that  $x$  solves GVIP( $F, g, f$ ).  $\square$

We next exhibit an error bound in terms of  $R_\alpha(\cdot)$ . Let  $\bar{x}$  be a solution of GVIP( $F, g, f$ ). We say that  $F$  is  $g$ -strongly monotone with respect to  $\bar{x}$  with modulus  $\mu > 0$  if

$$\langle F(\bar{x}) - F(y), g(\bar{x}) - g(y) \rangle \geq \mu \|\bar{x} - y\|^2 \quad \forall y \in \mathcal{H}. \tag{2}$$

First note that in this case the solution is in fact unique. Indeed, if  $\hat{x}$  is another solution of GVIP( $F, g, f$ ) then taking  $x = \bar{x}$ ,  $y = g(\hat{x})$  and  $x = \hat{x}$ ,  $y = g(\bar{x})$  in (1), we have that

$$\begin{aligned} \langle F(\bar{x}), g(\hat{x}) - g(\bar{x}) \rangle + f(g(\hat{x})) - f(g(\bar{x})) &\geq 0, \\ \langle F(\hat{x}), g(\bar{x}) - g(\hat{x}) \rangle + f(g(\bar{x})) - f(g(\hat{x})) &\geq 0. \end{aligned}$$

Adding the two inequalities and using (2), we obtain

$$0 \leq \langle F(\bar{x}) - F(\hat{x}), g(\hat{x}) - g(\bar{x}) \rangle \leq -\mu \|\bar{x} - \hat{x}\|^2,$$

which implies that necessarily  $\bar{x} = \hat{x}$ .

Finally, we say that  $F$  is Lipschitz-continuous with respect to  $\bar{x}$  with modulus  $L > 0$  if

$$\|F(\bar{x}) - F(x)\| \leq L\|\bar{x} - x\| \quad \forall x \in \mathcal{H}.$$

An analogous definition will be used for  $g$ .

**Theorem 2.** *Suppose that  $F$  is  $g$ -strongly monotone with modulus  $\mu > 0$  with respect to solution  $\bar{x}$  of  $\text{GVIP}(F, g, f)$ , and that  $F, g$  are Lipschitz-continuous with modulus  $L > 0$  with respect to  $\bar{x}$ . Then*

$$\|x - \bar{x}\| \leq \frac{L}{\mu} \left(1 + \frac{1}{\alpha}\right) \|R_\alpha(x)\| \quad \forall x \in \mathcal{H}, \alpha > 0.$$

**Proof.** Fix any  $x \in \mathcal{H}$  and  $\alpha > 0$ . By the definition of  $p_f^\alpha$ , we have that  $p_f^\alpha(g(x) - \alpha F(x))$  satisfies

$$0 \in \partial f(p_f^\alpha(g(x) - \alpha F(x))) + \frac{1}{\alpha}(p_f^\alpha(g(x) - \alpha F(x)) - (g(x) - \alpha F(x))).$$

Hence,

$$-F(x) + \frac{1}{\alpha}(g(x) - p_f^\alpha(g(x) - \alpha F(x))) \in \partial f(p_f^\alpha(g(x) - \alpha F(x))).$$

It follows that for all  $y \in \mathcal{H}$ ,

$$\begin{aligned} & f(y) - f(p_f^\alpha(g(x) - \alpha F(x))) \\ & + \left\langle F(x) - \frac{1}{\alpha}(g(x) - p_f^\alpha(g(x) - \alpha F(x))), y - p_f^\alpha(g(x) - \alpha F(x)) \right\rangle \geq 0. \end{aligned} \quad (3)$$

Taking  $y = g(\bar{x})$  in the inequality above,  $y = p_f^\alpha(g(x) - \alpha F(x))$  in (1), and adding the two inequalities, we obtain that

$$\begin{aligned} & \left\langle F(\bar{x}) - F(x) + \frac{1}{\alpha}(g(x) - p_f^\alpha(g(x) - \alpha F(x))), p_f^\alpha(g(x) - \alpha F(x)) - g(\bar{x}) \right\rangle \\ & \geq 0. \end{aligned} \quad (4)$$

By (2), we have that

$$\begin{aligned} \mu\|x - \bar{x}\|^2 & \leq \langle F(\bar{x}) - F(x), g(\bar{x}) - g(x) \rangle \\ & = \langle F(\bar{x}) - F(x), g(\bar{x}) - p_f^\alpha(g(x) - \alpha F(x)) \rangle \\ & \quad + \langle F(\bar{x}) - F(x), p_f^\alpha(g(x) - \alpha F(x)) - g(x) \rangle \\ & \leq \frac{1}{\alpha} \langle g(x) - p_f^\alpha(g(x) - \alpha F(x)), p_f^\alpha(g(x) - \alpha F(x)) - g(\bar{x}) \rangle \\ & \quad + \langle F(\bar{x}) - F(x), p_f^\alpha(g(x) - \alpha F(x)) - g(x) \rangle \\ & \leq -\frac{1}{\alpha} \|R_\alpha(x)\|^2 + \frac{1}{\alpha} \|R_\alpha(x)\| \|g(\bar{x}) - g(x)\| + \|F(\bar{x}) - F(x)\| \|R_\alpha(x)\| \\ & \leq L \left(1 + \frac{1}{\alpha}\right) \|x - \bar{x}\| \|R_\alpha(x)\|, \end{aligned}$$

where the second inequality is by (4), the third is by the Cauchy–Schwarz inequality, and the last is by the Lipschitz-continuity assumptions. The assertion of the theorem now follows.  $\square$

### 3. Regularized-gap function

The following *regularized-gap function* was introduced (in the finite-dimensional setting) for  $VIP(F, C)$  in [1,3]:

$$\langle F(x), x - P_C(x - \alpha F(x)) \rangle - \frac{1}{2\alpha} \|x - P_C(x - \alpha F(x))\|^2, \quad x \in \mathcal{H}, \alpha > 0.$$

In the special case when  $C$  is the nonnegative orthant, it is equivalent to the restricted *implicit Lagrangian* [7,16]. This function has a number of interesting properties. For example, it is nonnegative on the set  $C$  and its set of zeroes in  $C$  coincides with solutions of  $VIP(F, C)$ . Furthermore, it has better smoothness properties when compared to the natural residual.

We next construct the regularized-gap function for  $GVIP(F, g, f)$ . To this end, define  $G_\alpha : \mathcal{H} \rightarrow [0, +\infty]$  by

$$G_\alpha(x) := \max_{y \in \mathcal{H}} \left\{ \langle F(x), g(x) - y \rangle + f(g(x)) - f(y) - \frac{1}{2\alpha} \|g(x) - y\|^2 \right\}, \quad x \in \mathcal{H}, \alpha > 0. \tag{5}$$

We next derive an explicit formula for  $G_\alpha$ , and in particular we show that this function is nonnegative everywhere on  $\mathcal{H}$ .

**Lemma 3.** *It holds that*

$$G_\alpha(x) = \langle F(x), g(x) - p_f^\alpha(g(x) - \alpha F(x)) \rangle + f(g(x)) - f(p_f^\alpha(g(x) - \alpha F(x))) - \frac{1}{2\alpha} \|g(x) - p_f^\alpha(g(x) - \alpha F(x))\|^2, \quad x \in \mathcal{H}. \tag{6}$$

*In particular,  $G_\alpha(x) \geq 0$  for all  $x \in \mathcal{H}$ .*

**Proof.** Denote by  $t(y)$  the function being maximized in (5), with  $x \in \mathcal{H}$  fixed. If  $x \in g^{-1}(\text{dom}(f))$  then  $t(g(x)) = 0$ , and hence  $0 = t(g(x)) \leq \max_{y \in \mathcal{H}} t(y) = G_\alpha(x)$ . If  $x \notin g^{-1}(\text{dom}(f))$  then for any  $y \in \text{dom}(f)$  it holds that  $t(y) = +\infty$ , and hence  $+\infty = \max_{y \in \mathcal{H}} t(y) = G_\alpha(x)$ .

First note that if  $x \notin g^{-1}(\text{dom}(f))$  then formula (6) is correct, because  $f(g(x)) = +\infty$  while the other terms are all finite (recall that  $p_f^\alpha(z) \in \text{dom}(f)$  for any  $z \in \mathcal{H}$ ).

Consider now any  $x \in g^{-1}(\text{dom}(f))$ . Let  $z$  be the (unique, by strong concavity of  $t(y)$ ) element at which the maximum is realized in (5). Then  $z$  is uniquely characterized by the optimality condition

$$0 \in F(x) + \partial f(z) + \frac{1}{\alpha}(z - g(x)).$$

But this inclusion also uniquely characterizes the solution of the problem

$$\min_{z \in \mathcal{H}} \left\{ f(z) + \frac{1}{2\alpha} \|z - (g(x) - \alpha F(x))\|^2 \right\},$$

which by definition is  $p_f^\alpha(g(x) - \alpha F(x))$ . We conclude that  $z = p_f^\alpha(g(x) - \alpha F(x))$ , from which (6) follows by evaluating the maximum in (5).  $\square$

The next result shows that the set of zeroes of  $G_\alpha$  is precisely the solution set of  $\text{GVIP}(F, g, f)$ . Furthermore, it establishes the order of growth of  $G_\alpha$  when compared to the natural residual  $R_\alpha$ .

**Theorem 4.** *It holds that*

$$G_\alpha(x) \geq \frac{1}{2\alpha} \|R_\alpha(x)\|^2 \quad \forall x \in \mathcal{H}, \alpha > 0.$$

*In particular,  $G_\alpha(x) = 0$  if, and only if,  $x$  solves  $\text{GVIP}(F, g, f)$ .*

**Proof.** Fix any  $x \in \mathcal{H}$  and  $\alpha > 0$ . Taking  $y = g(x)$  in (3), we have that

$$f(g(x)) - f(p_f^\alpha(g(x) - \alpha F(x))) + \langle F(x), R_\alpha(x) \rangle \geq \frac{1}{\alpha} \|R_\alpha(x)\|^2.$$

Using this relation together with (6), we obtain

$$\begin{aligned} G_\alpha(x) &= \langle F(x), R_\alpha(x) \rangle + f(g(x)) - f(p_f^\alpha(g(x) - \alpha F(x))) - \frac{1}{2\alpha} \|R_\alpha(x)\|^2 \\ &\geq \frac{1}{\alpha} \|R_\alpha(x)\|^2 - \frac{1}{2\alpha} \|R_\alpha(x)\|^2 = \frac{1}{2\alpha} \|R_\alpha(x)\|^2. \end{aligned}$$

We now prove the last assertion of the theorem. If  $G_\alpha(x) = 0$  then  $R_\alpha(x) = 0$ , by the first assertion. Hence, by Theorem 1,  $x$  solves  $\text{GVIP}(F, g, f)$ . Conversely, suppose  $x$  is a solution. Again by Theorem 1,  $g(x) = p_f^\alpha(g(x) - \alpha F(x))$ . In particular, it holds that  $g(x) \in \text{dom}(f)$ . Now (6) implies that  $G_\alpha(x) = 0$ .  $\square$

As was already mentioned, one advantage of the regularized-gap function over the natural residual is the more attractive differentiability properties in certain situations. For example, for  $\text{VIP}(F, C)$  it holds that  $G_\alpha(\cdot)$  is differentiable whenever  $F(\cdot)$  is differentiable [3], while the same is not true for  $R_\alpha(\cdot)$ . Another advantage is exhibited by the following error bound result, where, unlike in Theorem 2, we do not need Lipschitz-continuity of  $F$ .

**Theorem 5.** *Suppose that  $F$  is  $g$ -strongly monotone with modulus  $\mu > 0$  with respect to solution  $\bar{x}$  of  $\text{GVIP}(F, g, f)$ , and that  $g$  is Lipschitz-continuous with modulus  $L > 0$  with respect to  $\bar{x}$ . Then*

$$\|x - \bar{x}\| \leq \sqrt{\frac{2\alpha}{2\alpha\mu - L^2}} \sqrt{G_\alpha(x)} \quad \forall x \in \mathcal{H}, \alpha > \frac{L^2}{2\mu}.$$

**Proof.** Again, let  $t(y)$  denote the function being maximized in (5), with  $x \in \mathcal{H}$  fixed. It holds that

$$\begin{aligned} G_\alpha(x) &\geq t(g(\bar{x})) \\ &= \langle F(x), g(x) - g(\bar{x}) \rangle + f(g(x)) - f(g(\bar{x})) - \frac{1}{2\alpha} \|g(x) - g(\bar{x})\|^2 \\ &\geq \langle F(\bar{x}), g(x) - g(\bar{x}) \rangle + \mu \|x - \bar{x}\|^2 \\ &\quad + f(g(x)) - f(g(\bar{x})) - \frac{1}{2\alpha} \|g(x) - g(\bar{x})\|^2, \end{aligned} \tag{7}$$

where the second inequality follows from (2). Taking  $y = g(x)$  in (1), we have

$$\langle F(\bar{x}), g(x) - g(\bar{x}) \rangle + f(g(x)) - f(g(\bar{x})) \geq 0.$$

Combining the latter relation with (7) and using the Lipschitz-continuity assumption, we obtain

$$G_\alpha(x) \geq \mu \|x - \bar{x}\|^2 - \frac{1}{2\alpha} \|g(x) - g(\bar{x})\|^2 \geq \left( \mu - \frac{L^2}{2\alpha} \right) \|x - \bar{x}\|^2,$$

which concludes the proof.  $\square$

#### 4. D-gap function

The *D-gap function* can be thought of as a difference of two regularized-gap functions with distinct parameters. It was introduced, via a rather different construction, in [7] for  $\text{VIP}(F, C)$  with  $C$  being the nonnegative orthant (i.e., for the nonlinear complementarity problem, where it is known as the *implicit Lagrangian* [16]). It was subsequently extended for a general closed convex set  $C$  (in finite-dimensional space) [13,19]. For properties and applications, see also [6,8,14,17,18].

In the setting of  $\text{GVIP}(F, g, f)$ , the expression

$$G_\alpha(x) - G_\beta(x), \quad x \in \mathcal{H}, \quad \alpha > \beta > 0,$$

will not be well-defined for  $x \notin g^{-1}(\text{dom}(f))$ , as both quantities are not finite. Nevertheless, we shall define the D-gap function by taking a formal difference of the expressions (6) for two parameters  $\alpha$  and  $\beta$ . As we shall show, this would result in an appropriate merit function for  $\text{GVIP}(F, g, f)$ . Furthermore, there is an advantage that, unlike the regularized-gap function, the resulting function will be not only nonnegative but also everywhere finite. To this end, taking a formal difference of expressions given by (6) with parameters  $\alpha$  and  $\beta$ , we define  $H_{\alpha,\beta} : \mathcal{H} \rightarrow [0, +\infty)$  by

$$\begin{aligned} H_{\alpha,\beta}(x) &= \langle F(x), p_f^\beta(g(x) - \beta F(x)) - p_f^\alpha(g(x) - \alpha F(x)) \rangle \\ &\quad + f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\ &\quad + \frac{1}{2\beta} \|g(x) - p_f^\beta(g(x) - \beta F(x))\|^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\alpha} \|g(x) - p_f^\alpha(g(x) - \alpha F(x))\|^2 \\
& = \langle F(x), R_\alpha(x) - R_\beta(x) \rangle + f(p_f^\beta(g(x) - \beta F(x))) \\
& \quad - f(p_f^\alpha(g(x) - \alpha F(x))) + \frac{1}{2\beta} \|R_\beta(x)\|^2 - \frac{1}{2\alpha} \|R_\alpha(x)\|^2, \\
& \quad x \in \mathcal{H}, \alpha > \beta > 0.
\end{aligned} \tag{8}$$

Note that since  $p_f^\alpha(z) \in \text{dom}(f)$  for all  $z \in \mathcal{H}$  and all  $\alpha > 0$ , it follows that  $H_{\alpha,\beta}$  given by (8) is indeed everywhere finite. We next show that the D-gap function is of the order of the natural residual. Hence, it is nonnegative and its zeroes are solutions of  $\text{GVIP}(F, g, f)$ .

**Theorem 6.** *It holds that*

$$\begin{aligned}
\frac{1}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \|R_\alpha(x)\|^2 & \geq H_{\alpha,\beta}(x) \geq \frac{1}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \|R_\beta(x)\|^2 \\
\forall x \in \mathcal{H}, \alpha > \beta > 0.
\end{aligned}$$

*In particular,  $x \in \mathcal{H}$  solves  $\text{GVIP}(F, g, f)$  if, and only if,  $H_{\alpha,\beta}(x) = 0$  with  $\alpha > \beta > 0$ .*

**Proof.** Fix any  $x \in \mathcal{H}$  and  $\alpha > \beta > 0$ . Using (3) with  $y = p_f^\beta(g(x) - \beta F(x))$ , we have that

$$\begin{aligned}
\frac{1}{\alpha} \langle R_\alpha(x), R_\alpha(x) - R_\beta(x) \rangle & \leq f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\
& \quad + \langle F(x), R_\alpha(x) - R_\beta(x) \rangle.
\end{aligned}$$

Combining this relation with (8), we obtain

$$\begin{aligned}
H_{\alpha,\beta}(x) & \geq \frac{1}{\alpha} \langle R_\alpha(x), R_\alpha(x) - R_\beta(x) \rangle + \frac{1}{2\beta} \|R_\beta(x)\|^2 - \frac{1}{2\alpha} \|R_\alpha(x)\|^2 \\
& = \frac{1}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \|R_\beta(x)\|^2 + \frac{1}{\alpha} \langle R_\alpha(x), R_\alpha(x) - R_\beta(x) \rangle \\
& \quad - \frac{1}{2\alpha} \|R_\alpha(x) - R_\beta(x)\|^2 - \frac{1}{\alpha} \langle R_\beta(x), R_\alpha(x) - R_\beta(x) \rangle \\
& = \frac{1}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \|R_\beta(x)\|^2 + \frac{1}{2\alpha} \|R_\alpha(x) - R_\beta(x)\|^2,
\end{aligned}$$

which implies the right-most inequality in the assertion.

On the other hand,

$$-F(x) + \frac{1}{\beta} R_\beta(x) \in \partial f(p_f^\beta(g(x) - \beta F(x)))$$

implies that

$$\begin{aligned}
\frac{1}{\beta} \langle R_\beta(x), R_\alpha(x) - R_\beta(x) \rangle & \geq f(p_f^\beta(g(x) - \beta F(x))) - f(p_f^\alpha(g(x) - \alpha F(x))) \\
& \quad + \langle F(x), R_\alpha(x) - R_\beta(x) \rangle.
\end{aligned}$$

Similarly to the analysis above, we then obtain

$$\begin{aligned} H_{\alpha,\beta}(x) &\leq \frac{1}{\beta} \langle R_\beta(x), R_\alpha(x) - R_\beta(x) \rangle + \frac{1}{2\beta} \|R_\beta(x)\|^2 - \frac{1}{2\alpha} \|R_\alpha(x)\|^2 \\ &= \frac{1}{2} \left( \frac{1}{\beta} - \frac{1}{\alpha} \right) \|R_\alpha(x)\|^2 - \frac{1}{2\beta} \|R_\alpha(x) - R_\beta(x)\|^2, \end{aligned}$$

which implies the left-most inequality in the assertion.

The last assertion now follows from Theorem 1.  $\square$

As a consequence of Theorems 6 and 2, we obtain the following error bound. Note that compared to Theorem 5 for the regularized-gap function, in Theorem 7 regularization parameters  $\alpha$  and  $\beta$  do not depend on the problem data.

**Theorem 7.** *Suppose that  $F$  is  $g$ -strongly monotone with modulus  $\mu > 0$  with respect to solution  $\bar{x}$  of GVIP( $F, g, f$ ), and that  $F, g$  are Lipschitz-continuous with modulus  $L > 0$  with respect to  $\bar{x}$ . Then*

$$\|x - \bar{x}\| \leq \frac{L}{\mu} \left( 1 + \frac{1}{\beta} \right) \sqrt{\frac{2\alpha\beta}{\alpha - \beta}} \sqrt{H_{\alpha,\beta}(x)} \quad \forall x \in \mathcal{H}, \quad \alpha > \beta > 0.$$

Finally, we note the following advantage of the D-gap function compared to the natural residual: as a difference of two regularized-gap functions, it inherits smoothness properties of the latter mentioned in Section 3.

## References

- [1] G. Auchmuty, Variational principles for variational inequalities, *Numer. Funct. Anal. Optim.* 10 (1989) 863–874.
- [2] R.W. Cottle, F. Giannessi, J.-L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, Wiley, New York, 1980.
- [3] M. Fukushima, Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, *Math. Programming* 53 (1992) 99–110.
- [4] M. Fukushima, Merit functions for variational inequality and complementarity problems, in: G. Di Pillo, F. Giannessi (Eds.), *Nonlinear Optimization and Applications*, Plenum Publishing Corporation, New York, 1996, pp. 155–170.
- [5] P.T. Harker, J.-S. Pang, Finite-dimensional variational inequality problems: A survey of theory, algorithms and applications, *Math. Programming* 48 (1990) 161–220.
- [6] C. Kanzow, M. Fukushima, Solving box constrained variational inequality problems by using the natural residual with D-gap function globalization, *Oper. Res. Lett.* 23 (1998) 45–51.
- [7] O.L. Mangasarian, M.V. Solodov, Nonlinear complementarity as unconstrained and constrained minimization, *Math. Programming* 62 (1993) 277–297.
- [8] O.L. Mangasarian, M.V. Solodov, A linearly convergent derivative-free descent method for strongly monotone complementarity problems, *Comput. Optim. Appl.* 14 (1999) 5–16.
- [9] J.-J. Moreau, Proximité et dualité dans un espace Hilbertien, *Bull. Soc. Math. France* 93 (1965) 273–299.
- [10] M.A. Noor, General nonlinear variational inequalities, *J. Math. Anal. Appl.* 126 (1987) 78–84.
- [11] M.A. Noor, An iterative algorithm for variational inequalities, *J. Math. Anal. Appl.* 158 (1991) 448–455.
- [12] J.-S. Pang, Error bounds in mathematical programming, *Math. Programming* 79 (1997) 299–332.

- [13] J.-M. Peng, Equivalence of variational inequality problems to unconstrained optimization, *Math. Programming* 78 (1997) 347–356.
- [14] J.-M. Peng, M. Fukushima, A hybrid Newton method for solving the variational inequality problem via the D-gap function, *Math. Programming* 86 (1999) 367–386.
- [15] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* 14 (1976) 877–898.
- [16] M.V. Solodov, Implicit Lagrangian, in: C. Floudas, P. Pardalos (Eds.), *Encyclopedia of Optimization*, Kluwer, Dordrecht, 2001.
- [17] M.V. Solodov, P. Tseng, Some methods based on the D-gap function for solving monotone variational inequalities, *Comput. Optim. Appl.* 17 (2000) 255–277.
- [18] N. Yamashita, M. Fukushima, On stationary points of the implicit Lagrangian for nonlinear complementarity problems, *J. Optim. Theory Appl.* 84 (1995) 653–663.
- [19] N. Yamashita, K. Taji, M. Fukushima, Unconstrained optimization reformulations of variational inequality problems, *J. Optim. Theory Appl.* 92 (1997) 439–456.
- [20] L.C. Zeng, Iterative algorithms for finding approximate solutions to completely generalized strongly nonlinear quasi-variational inequalities, *J. Math. Anal. Appl.* 201 (1996) 180–194.
- [21] L.C. Zeng, On a general projection algorithm for variational inequalities, *J. Optim. Theory Appl.* 97 (1998) 229–235.
- [22] Y.-B. Zhao, The iterative methods for monotone generalized variational inequalities, *Optimization* 42 (1997) 285–307.
- [23] D.L. Zhu, P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, *SIAM J. Optim.* 6 (1996) 714–726.