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Stationary Points of Bound Constrained Minimization Reformulations of Complementarity Problems[1,2]

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Abstract. We consider two merit functions which can be used for solving the nonlinear complementarity problem via nonnegatively constrained minimization. One of the functions is the restricted implicit Lagrangian (Refs. 1-3), and the other appears to be new. We study the conditions under which a stationary point of the minimization problem is guaranteed to be a solution of the underlying complementarity problem. It appears that, for both formulations, the same regularity condition is needed. This condition is closely related to the one used in Ref. 4 for unrestricted implicit Lagrangian. Some new sufficient conditions are also given.

Key Words. Complementarity problems, bound constrained reformulations, stationary points, regularity conditions.

1. Introduction

The classical nonlinear complementarity problem NCP(F) (Refs. 5-6) is to find a point $x \in \Re^n$ such that

$$x \geq 0, \qquad F(x) \geq 0, \qquad \langle x, F(x) \rangle = 0, \tag{1}$$

where $F: \Re^n \to \Re^n$ is a continuously differentiable function and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. We assume that the solution set S of (1) is nonempty. In the case where $F(\cdot)$ is affine, NCP(F) reduces to the linear complementarity problem (Ref. 7).

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A popular approach to solving NCP(F) is to construct a merit function $f(\cdot)$ such that solutions of (1) are related in a certain way to the optimal set of the problem of minimizing $f(\cdot)$ over some set C. Typically, C is either $\Re^n$ itself or the nonnegative orthant $\Re^n_+$. In the first case, an unconstrained reformulation is obtained; papers developing this approach include Refs. 2 and 8–13. In the second case, nonnegatively constrained problems have to be solved; work in this direction can be found in Refs. 1 and 2. Yet other approaches are nonsmooth and semismooth methods (Refs. 14–19). A recent survey of merit functions and related issues can be found in Ref. 20. While reformulating the problem, it is often desirable to preserve the smoothness of $F(\cdot)$ and the dimensionality n of the variables space. In this paper, we shall consider only some nonnegatively constrained smooth reformulations.

It is known that NCP(F) can be cast as a problem of minimizing the following restricted implicit Lagrangian function $f_1(\cdot)$ (Refs. 1–3) over the nonnegative orthant $\Re^n_+$:

$$f_1(x) := \sum_{i=1}^{n} \psi_1(x_i, F_i(x)),$$

where $\psi_1: \Re^2 \to \Re$ is given by

$$\psi_1(a, b) := ab + (1/2\alpha)[(a - \alpha b)_+^2 - a^2], \tag{2}$$

with $\alpha > 0$ being a parameter and $(\cdot)_+$ denoting the orthogonal projection map onto the nonnegative orthant $\Re^n_+$. In this form, the merit function was introduced in Ref. 2. In the case where $\alpha = 1$, it is equivalent to the merit function introduced in Ref. 1 for general variational inequality problems. It is known that $f_1(\cdot)$ is nonnegative on $\Re^n_+$ and assumes the value of zero precisely at the solutions of NCP(F). Moreover, $f_1(\cdot)$ is continuously differentiable whenever $F(\cdot)$ is; see Refs. 1–2.

In this paper, we will also consider the following merit function:

$$f_2(x) := \sum_{i=1}^{n} \psi_2(x_i, F_i(x)),$$

where $\psi_2: \Re^2 \to \Re$ is given by

$$\psi_2(a, b) := a(b)_+^2 + (-b)_+^2. \tag{3}$$

This function is nonnegative on $\Re^n_+$, and as we shall see, its minimal values of zero also identify solutions of NCP(F). This second reformulation appears to be new, so further study of its properties may be needed, including the study of numerical performance.

NCP(F) can be solved via smooth bound constrained minimization,

$$\min_{x \geq 0} f_j(x), \tag{4}$$

for $j=1$ or $j=2$. In particular, the set of global solutions of (4) coincides with the set S of solutions of (1). Because, in general, most optimization techniques are only guaranteed to find a stationary [Karush-Kuhn-Tucker (Ref. 21), or KKT for short] point of (4), the question arises as to when such a point is a solution of the underlying complementarity problem. In Refs. 1 and 3, it was shown for $f_1(\cdot)$ that, if x is a KKT point of the minimization problem and $\nabla F(x)$ is positive definite, then x is indeed a solution of NCP(F). Here, we give a more general regularity condition which is both necessary and sufficient and applies to either of the two reformulations; see Section 3. Our analysis is motivated by Ref. 4 where stationary points of the (unrestricted) implicit Lagrangian $f_3(\cdot)$, introduced in Ref. 2, and further studied and extended in Refs. 8, 13 and 22–26, are considered:

$$f_3(x) := \sum_{i=1}^{n} \psi_3(x_i, F_i(x)),$$

with

$$\psi_3(a, b) := ab + (1/2\alpha)[(a - \alpha b)_+^2 - a^2 + (b - \alpha a)_+^2 - b^2].$$

In Ref. 4, regularity conditions for both the unconstrained and nonnegatively constrained cases are given. However, the restricted implicit Lagrangian (2) is a more natural choice for the nonnegatively constrained optimization; we will therefore consider this function. Curiously, it turns out that $f_1(\cdot)$ and $f_2(\cdot)$ require the same regularity condition (see Definition 3.1) for a KKT point of (4) to solve NCP(F). Furthermore, this condition appears to be closely related to the one used in Ref. 4 for $f_3(\cdot)$. However, the regularity condition employed here is, in a sense, independent of the merit function [in Ref. 4, it is tied to $\psi_3(\cdot,\cdot)$].

A few words about our notation. For a real-valued matrix A of any dimension, A^{T} denotes its transpose. For a differentiable function $f: \Re^n \rightarrow \Re$, ∇f will denote the n-dimensional vector of partial derivatives with respect to x. For a differentiable vector function $F: \Re^n \rightarrow \Re^n$, ∇F will denote the $n \times n$ Jacobian matrix whose rows are the gradients of the components of F. For a vector $z \in \Re^n$ and an index set $I \subset \{1, \ldots, n\}$, z_I will stand for the vector obtained by removing the components of z whose indices are not in the set I. We will also adopt the notation $\langle x, y \rangle_I := \langle x_I, y_I \rangle$.

We finally define some classes of matrices which will be used in the paper.

Definition 1.1. Let M be an $n \times n$ real-valued matrix. Then:

(i) M is a P-matrix if, $\forall z \in \Re^n$, $z \neq 0$, $\exists i$ such that $z_i[Mz]_i > 0$;
(ii) M is a strictly semimonotone matrix if, $\forall z \in \Re_+^n$, $z \neq 0$, $\exists i$ such that $z_i[Mz]_i > 0$;

(iii) M is a P_0-matrix if, $\forall z \in \Re^n$, $z \neq 0$, $\exists i$ such that $z_i \neq 0$ and $z_i[Mz]_i \geq 0$;
(iv) M is a P_0^+-matrix if, $\forall z \in \Re^n_+$, $z \neq 0$, $\exists i$ such that $z_i \neq 0$ and $z_i[Mz]_i \geq 0$;
(v) M is a strictly copositive matrix if, $\forall z \in \Re^n_+$, $z \neq 0$, $\langle z, Mz \rangle > 0$.

Clearly, the class of P-matrices is contained in the class of P_0-matrices and the class of strictly semimonotone matrices. Also, P_0 and strictly semimonotone matrices are subclasses of P_0^+-matrices. We refer the reader to Ref. 7 for a detailed discussion.

2. Preliminary Results

We start with some preliminary results.

Lemma 2.1. Let $j \in \{1, 2\}$. Then, $\psi_j(a, b) \geq 0$ for all $(a, b) \in \Re_+ \times \Re$, and the following statements are equivalent:

(i) $\psi_j(a, b) = 0$, $a \geq 0$;
(ii) $a \geq 0$, $b \geq 0$, $ab = 0$;
(iii) $\dfrac{\partial \psi_j}{\partial b}(a, b) = 0$, $a \geq 0$.

Proof. The fact that $\psi_1(a, b) \geq 0$ for all $(a, b) \in \Re_+ \times \Re$ and the equivalence of the first and second statements have been established in Refs. 1 and 2. For the third statement, observe that

$$\frac{\partial \psi_1}{\partial b}(a, b) = a - (a - \alpha b)_+. \tag{5}$$

It is well known that the condition $a - (a - \alpha b)_+ = 0$ is equivalent to the second statement above [note that, for $\psi_1(a, b)$, condition $a \geq 0$ in the last statement is extraneous].

Now, we consider the second function. Since both terms in (3) are nonnegative on $\Re_+ \times \Re$, it follows that $\psi_2(a, b) \geq 0$ on that set. Next, we will show that the first condition implies the second, the second implies the third, and the third implies the first.

Suppose that $a \geq 0$ and $\psi_2(a, b) = 0$. Then, $a(b)_+^2 = 0$ and $(-b)_+^2 = 0$. Hence $b \geq 0$, and we have $0 = a(b)_+^2 = ab^2$, which further implies that $ab = 0$.

Now, let $a \geq 0$, $b \geq 0$, $ab = 0$. Note that

$$\frac{\partial \psi_2}{\partial b}(a, b) = 2a(b)_+ - 2(-b)_+. \tag{6}$$

Since $b \geq 0$, it follows that

$$\frac{\partial \psi_2}{\partial b}(a, b) = 2ab = 0.$$

Finally, suppose that

$$\frac{\partial \psi_2}{\partial b}(a, b) = 0, \qquad a \geq 0.$$

If $b < 0$, then

$$0 = \frac{\partial \psi_2}{\partial b}(a, b) = -2(-b)_+ = 2b,$$

which is a contradiction. Hence $b \geq 0$, and we have

$$0 = \frac{\partial \psi_2}{\partial b}(a, b) = 2a(b)_+.$$

Therefore, $(-b)_+^2 = 0$ and $a(b)_+^2 = 0$, which imply that $\psi_2(a, b) = 0$. The proof is complete. □

We will adopt the following notation:

$$\frac{\partial \psi_j}{\partial a}(x, F(x)) := \begin{pmatrix} \frac{\partial \psi_j}{\partial a}(x_1, F_1(x)) \\ \vdots \\ \frac{\partial \psi_j}{\partial a}(x_n F_n(x)) \end{pmatrix},$$

$$\frac{\partial \psi_j}{\partial b}(x, F(x)) := \begin{pmatrix} \frac{\partial \psi_j}{\partial b}(x_1, F_1(x)) \\ \vdots \\ \frac{\partial \psi_j}{\partial b}(x_n F_n(x)) \end{pmatrix}.$$

Lemma 2.2. Let $j \in \{1, 2\}$. The following statements are equivalent:

(i) a point $x \in \Re^n$ solves NCP(F);

(ii) $x \in \Re^n_+$ and $f_j(x) = 0$;

(iii) $x \in \Re^n_+$ and $\frac{\partial \psi_j}{\partial b}(x, F(x)) = 0.$

Proof. Suppose $x \in \Re^n_+$ and $f_j(x) = 0$. Since

$$f_j(x) = \sum_{i=1}^{n} \psi_j(x_i, F_i(x)),$$

if follows from Lemma 2.1 that

$$\psi_j(x_i, F_i(x)) = 0, \qquad \text{for all } i = 1, \ldots, n.$$

Again applying Lemma 2.1, we obtain that the latter is equivalent to

$$x_i \geq 0, \quad F_i(x) \geq 0, \quad x_i F_i(x) = 0, \qquad \text{for all } i = 1, \ldots, n,$$

which means that x solves the NCP(F).

We now prove that the first and third statements are equivalent. By Lemma 2.1,

$$x \in \Re^n_+ \quad \text{and} \quad \frac{\partial \psi_j}{\partial b}(x, F(x)) = 0,$$

if and only if

$$x_i \geq 0, \quad F_i(x) \geq 0, \quad x_i F_i(x) = 0, \qquad \text{for all } i = 1, \ldots, n,$$

which means that x solves the NCP(F). □

We will need another intermediate result.

Lemma 2.3. Let $j \in \{1, 2\}$ and $a \geq 0$. Then, the following properties hold:

(i) $\frac{\partial \psi_j}{\partial b}(a, b) > 0 \Leftrightarrow a > 0, b > 0;$

(ii) $\frac{\partial \psi_j}{\partial b}(a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0;$

(iii) $\frac{\partial \psi_j}{\partial b}(a, b) < 0 \Leftrightarrow a \geq 0, b < 0,$

in that case also $\frac{\partial \psi_j}{\partial a}(a, b) = 0;$

(iv) $\frac{\partial \psi_j}{\partial a}(a, b) \geq 0$ for all $(a, b) \in \Re^2$.

Proof. First note that Part (ii) has already been proven in Lemma 2.1. We start with the first function. By (5), we obtain that

$$\frac{\partial \psi_1}{\partial b}(a, b) = \begin{cases} \alpha b, & \text{if } a - \alpha b \geq 0, \\ a, & \text{if } a - \alpha b < 0. \end{cases}$$

Thus, $a > 0$, $b > 0$ imply that

$$\frac{\partial \psi_1}{\partial b}(a, b) > 0.$$

On the other hand, if

$$\frac{\partial \psi_1}{\partial b}(a, b) > 0,$$

in the first case we have

$$a \geq \alpha b = \frac{\partial \psi_1}{\partial b}(a, b) > 0,$$

and in the second case we have

$$\alpha b > a = \frac{\partial \psi_1}{\partial b}(a, b) > 0.$$

It follows that $a > 0$, $b > 0$.

Now, suppose that

$$\frac{\partial \psi_1}{\partial b}(a, b) < 0.$$

Given that $a \geq 0$, this can only happen when $b < 0$. On the other hand, when $b < 0$, we have that

$$a - \alpha b > 0 \quad \text{and} \quad \frac{\partial \psi_1}{\partial b}(a, b) = \alpha b < 0.$$

For the last assertion, note that

$$\frac{\partial \psi_1}{\partial a}(a, b) = b + (1/\alpha)[(a - \alpha b)_+ - a]. \tag{7}$$

In the case where $a - \alpha b \geq 0$, we have that

$$\frac{\partial \psi_1}{\partial a}(a, b) = 0.$$

In the case where $a - \alpha b < 0$,

$$\frac{\partial \psi_1}{\partial a}(a, b) = b - a/\alpha > 0.$$

It follows that

$$\frac{\partial \psi_1}{\partial a}(a, b) \geq 0, \qquad \text{for all } (a, b) \in \Re^2.$$

Finally, note that when $b < 0$, we have that $a - \alpha b \geq 0$, and therefore

$$\frac{\partial \psi_1}{\partial a}(a, b) = 0.$$

Now, consider the second function. By (6),

$$0 < \frac{\partial \psi_2}{\partial b}(a, b) \Leftrightarrow a(b)_+ > (-b)_+ \Leftrightarrow a > 0, b > 0,$$

and given that $a \geq 0$,

$$0 > \frac{\partial \psi_2}{\partial b}(a, b) \Leftrightarrow a(b)_+ < (-b)_+ \Leftrightarrow b < 0.$$

Finally,

$$\frac{\partial \psi_2}{\partial a}(a, b) = (b)_+^2, \tag{8}$$

which is always nonnegative. Note that

$$\frac{\partial \psi_2}{\partial a}(a, b) = 0, \qquad \text{if } b < 0.$$

This completes the proof. □

It is possible to establish some additional properties for our merit functions. For example, using the fact that $\psi_2(a, b) \to \infty$ if $ab \to \infty$ and $a > 0$, it can be shown that the level sets of $f_2(\cdot)$ are bounded if $F(\cdot)$ is monotone and NCP(F) is strictly feasible. The argument is similar to that in Ref. 27, Theorem 4.1. However, in this paper, we focus on the study of stationary points.

3. Regularity Conditions

We now give a definition of a regular point which does not explicitly depend on the merit function used. Conditions of this type for various merit functions have been employed in Refs. 4, 10, 12, 14, 16, 18.

Definition 3.1. A point $x \in \Re^n_+$ is said to be regular if $\nabla F(x)^T$ reverses the sign of no nonzero vector $z \in \Re^n$ satisfying

$$z_P > 0, \qquad z_C = 0, \qquad z_N < 0, \tag{9}$$

where

$$C := \{i \mid x_i \geq 0,\ F_i(x) \geq 0,\ x_i F_i(x) = 0\},$$

$$P := \{i \mid x_i > 0,\ F_i(x) > 0\},$$

$$N := \{i \mid x_i \geq 0,\ F_i(x) < 0\}.$$

We recall that the matrix $\nabla F(x)^T$ is said to reverse the sign of a vector $z \in \Re^n$ (Ref. 7) if

$$z_i [\nabla F(x)^T z]_i \leq 0, \qquad \forall i \in \{1, \ldots, n\}. \tag{10}$$

Definition 3.1 states that a point $x \in \Re^n_+$ is regular if the only vector $z \in \Re^n$ satisfying both (9) and (10) is the zero vector.

We note that, in Ref. 4, the following definition of a regular point was used in connection with the implicit Lagrangian, i.e., for $j = 3$.

Definition 3.2. Given $x \in \Re^n$, let

$$C := \left\{ i \,\middle|\, \frac{\partial \psi_j}{\partial b}(x_i, F_i(x)) = 0 \right\},$$

$$P := \left\{ i \,\middle|\, \frac{\partial \psi_j}{\partial b}(x_i, F_i(x)) > 0 \right\},$$

$$N := \left\{ i \,\middle|\, \frac{\partial \psi_j}{\partial b}(x_i, F_i(x)) < 0 \right\}.$$

The point $x \in \Re^n$ is said to be regular if, for any nonzero vector $z \in \Re^n$ satisfying

$$z_P > 0, \qquad z_C = 0, \qquad z_N < 0, \tag{11}$$

there exists a vector $y \in \Re^n$ satisfying

$$y_P \geq 0, \qquad y_C = 0, \qquad y_N \leq 0, \tag{12}$$

and such that

$$\langle y, \nabla F(x)^{\top} z\rangle > 0. \tag{13}$$

As we shall see, in the case of our merit functions $f_1(\cdot)$ and $f_2(\cdot)$, the two definitions are equivalent. However, for the implicit Lagrangian $f_3(\cdot)$, they are not. Definition 3.1 appears to be more elegant as it does not depend on the merit function used and does not involve an extra auxiliary vector y. Also, it has a clear connection to one of the characterizations of P-matrices [recall that a P-matrix reverses the sign of no nonzero vector (Ref. 7)]. Thus, if $\nabla F(x)$ is a P-matrix, then the regularity condition is satisfied automatically.

Lemma 3.1. For $j \in \{1, 2\}$ and $x \in \mathfrak{R}^n_+$, Definitions 3.1 and 3.2 are equivalent.

Proof. First note that, by Lemma 2.3, the index sets P, C, N in the two definitions coincide. Thus, every $z \in \mathfrak{R}^n$ which satisfies (9) also satisfies (11), and vice versa.

First, suppose that x is regular according to Definition 3.1. Then, for any nonzero z satisfying (9), there exists an index $i \in P$ such that $[\nabla F(x)^{\top} z]_i > 0$, or an index $i \in N$ such that $[\nabla F(x)^{\top} z]_i < 0$. Take y such that $y_i > 0$ in the first case and $y_i < 0$ in the second case; in both cases, set $y_k := 0$ for all $k \neq i$. Clearly, such y satisfies (12) and also (13) holds.

Suppose now that x is regular according to Definition 3.2. If x were not regular in the sense of Definition 3.1, then there would exist a $z \neq 0$ satisfying (9) and (10). This implies that

$$[\nabla F(x)^{\top} z]_P \leq 0 \quad \text{and} \quad [\nabla F(x)^{\top} z]_N \geq 0.$$

Because by (12),

$$y_P \geq 0, \quad y_C = 0, \qquad y_N \leq 0,$$

we obtain that

$$\langle y, \nabla F(x)^{\top} z\rangle = \langle y, \nabla F(x)^{\top} z\rangle_P + \langle y, \nabla F(x)^{\top} z\rangle_C + \langle y, \nabla F(x)^{\top} z\rangle_N \leq 0,$$

which contradicts (13). Hence, x is regular in the sense of Definition 3.1. □

It is worth pointing out that for the implicit Lagrangian $f_3(\cdot)$, Definitions 3.1 and 3.2 are not equivalent; in fact, for the unrestricted implicit Lagrangian, the first definition is not even well posed because a stationary point need not belong to the nonnegative orthant.

We now state our main result. The proof is similar to the one in Ref. 4.

Theorem 3.1. Let $j \in \{1, 2\}$, and let $x \in \Re^n$ be a KKT point of the problem

$$\min_{x \geq 0} f_j(x).$$

Then, x solves NCP(F) if and only if x is regular.

Proof. If x solves NCP(F), it immediately follows that $x \in \Re^n_+$, $C = \{1, \ldots, n\}$, and $P = \emptyset$, $N = \emptyset$. Thus, there exists no nonzero vector satisfying (9). Hence, x is regular.

Now, suppose that x is a regular stationary point. By the first-order optimality conditions (Ref. 21), it follows that

$$x \geq 0, \qquad \nabla f_j(x) \geq 0, \qquad \langle x, \nabla f_j(x) \rangle = 0. \tag{14}$$

Let

$$I := \{i \mid [\nabla f_j(x)]_i > 0\}.$$

By (14), clearly $x_I = 0$. Therefore $I \subset (C \cup N)$. Hence, by Lemma 2.3,

$$\frac{\partial \psi_j}{\partial b}(x_i, F_i(x)) \leq 0, \qquad \text{for all } i \in I.$$

Suppose that x is not a solution of NCP(F). Then, by Lemma 2.2,

$$z = \frac{\partial \psi_j}{\partial b}(x, F(x)) \neq 0.$$

By regularity, for this z there exists a $y \in \Re^n$ satisfying (12) and (13). Note that, since $I \subset (C \cap N)$, it follows that $y_I \leq 0$. Hence,

$$\langle y, \nabla f_j(x) \rangle = \langle y, \nabla f_j(x) \rangle_I \leq 0. \tag{15}$$

Note that

$$y_C = 0, \qquad \text{by (12)},$$

$$\left[\frac{\partial \psi_j}{\partial a}(x, F(x))\right]_N = 0, \qquad \text{by Lemma 2.3},$$

$$y_P \geq 0, \qquad \text{by (12)},$$

$$\left[\frac{\partial \psi_j}{\partial a}(x, F(x))\right]_P \geq 0, \qquad \text{by Lemma 2.3}.$$

Therefore,

$$\left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle$$

$$= \left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle_P + \left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle_C + \left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle_N$$

$$= \left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle_P$$

$$\geq 0.$$

Hence, by (13),

$$\langle y, \nabla f_j(x) \rangle$$

$$= \left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) + \nabla F(x)^{\mathrm{T}} \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle$$

$$= \left\langle y, \frac{\partial \psi_j}{\partial a}(x, F(x)) \right\rangle + \langle y, \nabla F(x)^{\mathrm{T}} z \rangle$$

$$> 0,$$

which contradicts (15). Hence, x must be a solution of NCP(F). □

Remark 3.1. We note that, in Ref. 28, a different constrained optimization reformulation of an NCP is considered. This reformulation is based on the squared Fischer–Burmeister function and requires a somewhat weaker regularity condition.

Another curious observation is that strict complementarity in the KKT conditions (14) of the minimization problem (4) holds whenever a solution of NCP(F) is nondegenerate. This can be seen as follows. If $x \in S$ then, by Lemma 2.2,

$$\frac{\partial \psi_j}{\partial b}(x, F(x)) = 0.$$

Hence,

$$\nabla f_j(x) = \frac{\partial \psi_j}{\partial a}(x, F(x)).$$

Suppose that x is a nondegenerate solution of NCP(F), that is,

$$x_i + F_i(x) > 0, \qquad \text{for all } i.$$

Then, if $x_i = 0$ for some i, we have

$$[\nabla f_j(x)]_i = \left[\frac{\partial \psi_j}{\partial a}(x, F(x))\right]_i = F_i(x) > 0.$$

Hence, the KKT conditions (14) are satisfied with strict complementarity.

We now give some sufficient conditions for regularity.

Corollary 3.1. Let $j \in \{1, 2\}$, and let x be a KKT point of (4). Then, the following statements hold:

(i) if $\nabla F(x)$ is a P-matrix, then x solves NCP(F);
(ii) if x is feasible [i.e., $F(x) \geq 0$] and if $\nabla F(x)$ is strictly semimonotone, then x solves NCP(F).

Proof. The first assertion follows immediately from Definition 3.1 and the characterization of P-matrices (Ref. 7). The second assertion can be proven the same way as in Ref. 4. □

We now give a new sufficient condition for the function $f_2(\cdot)$.

Corollary 3.2. Let x be a KKT point of (4) for $j = 2$. Then, the following properties hold:

(i) if x is feasible [i.e., $F(x) \geq 0$] and if $\nabla F(x)$ is a P_0^+-matrix, then x solves NCP(F);
(ii) if $\nabla F(x)$ is positive semidefinite and strictly copositive, then x solves NCP(F).

Proof. First, note that $F(x) \geq 0$ implies that $N = \varnothing$. In particular, we have

$$\frac{\partial \psi_2}{\partial b}(x, F(x)) \geq 0.$$

Suppose that x does not solve NCP(F). By Lemma 2.2,

$$\frac{\partial \psi_2}{\partial b}(x, F(x)) \neq 0,$$

so $P \neq \emptyset$. Since $\nabla F(x)$ *is a* P_0^+-matrix, it follows that, for some index j,

$$\left[\frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j > 0$$

and

$$\left[\frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j \left[\nabla F(x)^\top \frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j \geq 0.$$

Note that $j \in P$. Now, choose $y \in \mathfrak{R}^n$ so that

$$y_j = \left[\frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j > 0$$

and

$$y_i = 0, \qquad \text{for all } i \neq j.$$

Then, we have that

$$\begin{aligned}
&\left\langle y, \nabla F(x)^\top \frac{\partial \psi_2}{\partial b}(x, F(x)) \right\rangle \\
&= y_j \left[\nabla F(x)^\top \frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j \\
&= \left[\frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j \left[\nabla F(x)^\top \frac{\partial \psi_2}{\partial b}(x, F(x))\right]_j \\
&\geq 0. \qquad (16)
\end{aligned}$$

Furthermore, observe that

$$\left[\frac{\partial \psi_2}{\partial a}(x, F(x))\right]_P = \begin{pmatrix} (F_1(x))_+^2 \\ \vdots \\ (F_n(x))_+^2 \end{pmatrix}_P > 0.$$

Hence, because $j \in P$,

$$\left\langle y, \frac{\partial \psi_2}{\partial a}(x, F(x)) \right\rangle = y_j \left[\frac{\partial \psi_2}{\partial a}(x, F(x))\right]_j > 0.$$

Combining the latter relation with (16), we obtain

$$\langle y, \nabla f_2(x)\rangle = \left\langle y, \frac{\partial \psi_2}{\partial a}(x, F(x))\right\rangle + \left\langle y, \nabla F(x)^{\mathrm{T}} \frac{\partial \psi_2}{\partial b}(x, F(x))\right\rangle > 0. \tag{17}$$

On the other hand, $j \in P$ implies that $x_j > 0$, and hence by (14), $[f_2(x)]_j = 0$. Therefore,

$$\langle y, \nabla f_2(x)\rangle = y_j [f_2(x)]_j = 0,$$

which contradicts (17). Hence, x must be a solution of NCP(F).

Now, we prove the second assertion. Suppose that x does not solve NCP(F). Let

$$I := \{i \mid [\nabla f_2(x)]_i > 0\}.$$

By (14) and the fact that $I \subset (C \cup N)$, we have

$$\left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla f_2(x)\right\rangle = \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla f_2(x)\right\rangle_I \leq 0. \tag{18}$$

We first consider the case where $P \neq \varnothing$. As we have already established,

$$\left[\frac{\partial \psi_2}{\partial a}(x, F(x))\right]_P > 0.$$

By Lemma 2.3,

$$\left[\frac{\partial \psi_2}{\partial b}(x, F(x))\right]_C = 0, \qquad \left[\frac{\partial \psi_2}{\partial a}(x, F(x))\right]_N = 0, \qquad \left[\frac{\partial \psi_2}{\partial b}(x, F(x))\right]_P > 0.$$

Therefore,

$$\begin{aligned}
&\left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x))\right\rangle \\
&= \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x))\right\rangle_P + \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x))\right\rangle_C \\
&\quad + \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x))\right\rangle_N \\
&= \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x))\right\rangle_P \\
&> 0.
\end{aligned}$$

Taking into account that $\nabla F(x)$ is positive semidefinite, we obtain

$$\left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla f_2(x) \right\rangle$$

$$= \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla F(x)^{\mathrm{T}} \frac{\partial \psi_2}{\partial b}(x, F(x)) \right\rangle$$

$$+ \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x)) \right\rangle$$

$$> 0,$$

which is in contradiction with (18). Hence, in the case $P \neq \varnothing$, x must be a solution of NCP(F).

Now, suppose that $P = \varnothing$. Then,

$$\frac{\partial \psi_2}{\partial b}(x, F(x)) \leq 0,$$

and by Lemma 2.2,

$$\frac{\partial \psi_2}{\partial b}(x, F(x)) \neq 0.$$

It follows from strict copositiveness of $\nabla F(x)$ that

$$\left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla F(x)^{\mathrm{T}} \frac{\partial \psi_2}{\partial b}(x, F(x)) \right\rangle > 0.$$

Also by Lemma 2.3,

$$\left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x)) \right\rangle \geq 0.$$

Therefore,

$$\left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla f_2(x) \right\rangle$$

$$= \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \nabla F(x)^{\mathrm{T}} \frac{\partial \psi_2}{\partial b}(x, F(x)) \right\rangle$$

$$+ \left\langle \frac{\partial \psi_2}{\partial b}(x, F(x)), \frac{\partial \psi_2}{\partial a}(x, F(x)) \right\rangle$$

$$> 0.$$

But the latter relation again contradicts (18); therefore, it must be the case that x solves NCP(F). □

Remark 3.2. Note that the first assertion of Corollary 3.2 is stronger than the corresponding result in Corollary 3.1. It is an open question whether the same is true for the restricted implicit Lagrangian $f_1(\cdot)$.

4. Concluding Remarks

Two nonnegatively constrained minimization reformulations for nonlinear complementarity problems were considered. A regularity condition was established which is necessary and sufficient for a stationary point of the minimization problem to be a solution of the underlying complementarity problem.

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