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Some Optimization Reformulations of the Extended Linear Complementarity Problem *

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Abstract. We consider the extended linear complementarity problem (XLCP) introduced by Mangasarian and Pang [22], of which the horizontal and vertical linear complementarity problems are two special cases. We give some new sufficient conditions for every stationary point of the natural bilinear program associated with XLCP to be a solution of XLCP. We further propose some unconstrained and bound constrained reformulations for XLCP, and study the properties of their stationary points under assumptions similar to those for the bilinear program.

Keywords: complementarity problems, bilinear program, optimization reformulations

Dedication: (Dedicated to Olvi Mangasarian on the occasion of his 65th birthday)
I feel very fortunate to have started my research endeavours under the guidance of Professor Olvi Mangasarian. Not only did I benefit greatly from his scientific vision and advice, but I was also deeply influenced by his creative and elegant approach to the research process itself. Moreover, working with Olvi is always a pleasure. I find his sparkling enthusiasm truly inspiring. In fact, Olvi's enthusiasm and genuine interest are the first things I remarked when I first walked into his office as an exchange student. He wanted to know everything – about me, my family, the country I come from, my research plans and interests. Olvi is one of those rare people who make you feel warm and welcome from the very first encounter. Needless to say, this first impression only strengthened over the years of my association with him. Olvi is very generous with his time, with sharing his extensive knowledge and deep elegant research ideas. He is always ready to extend his help and advice, whatever the occasion might be. I would like to take this opportunity to express my sincere admiration and gratitude to Professor Olvi Mangasarian, a very nice person and a pioneering researcher.

1. Introduction

The extended linear complementarity problem (XLCP) introduced by Mangasarian and Pang [22], is to find a pair of vectors x and y in $\Re^n$ such that

$$Mx - Ny \in \mathcal{P}, \quad x \geq 0, \quad y \geq 0, \quad \langle x, y \rangle = 0, \tag{1}$$

where M and N are two real matrices of order $m \times n$, $\mathcal{P}$ is a polyhedral set in $\Re^m$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Throughout the paper we assume that the feasible set of XLCP is nonempty :

$$\{(x, y) \mid Mx - Ny \in \mathcal{P}, x \geq 0, y \geq 0\} \neq \emptyset.$$

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In the special case when $m = n$ and $\mathcal{P}$ is a singleton, XLCP reduces to the horizontal linear complementarity problem which has been a subject of extensive research is recent years [34, 14, 1, 32, 12, 31]. If one further assumes that N is the identity matrix, then the classical linear complementarity problem [2] is obtained.

Associated with XLCP is the natural bilinear program (BLP)

$$\begin{array}{ll} \text{minimize} & \langle x, y \rangle \\ \text{subject to} & Mx - Ny \in \mathcal{P},\ x \geq 0,\ y \geq 0. \end{array} \tag{2}$$

Clearly, a pair of vectors (x, y) solves XLCP if and only if (x, y) is a global minimizer of BLP with zero optimal value. In [22], Mangasarian and Pang established a number of properties of XLCP and related BLP. Among other things, it was shown that if the matrix $MN^\top$ is copositive on the dual of the recession cone of the set $\mathcal{P}$, then every Karush-Kuhn-Tucker (KKT) point of (2) is a solution of XLCP. A further study of XLCP and associated BLP was undertaken by Gowda [13]. For example, copositiveness of $MN^\top$ was replaced by a more general X-row-sufficiency property. It was also pointed out in the latter reference that the XLCP stated here is essentially equivalent to the "general linear complementarity problem" considered by Ye [33] in the context of interior points algorithms.

In this paper, we give some new sufficient conditions for every KKT point of BLP (2) to be a solution of XLCP (1). Our primary goal, however, is to derive some optimization reformulations which are, just like BLP, equivalent to the original problem XLCP, yet their feasible sets have simpler structure. Consider, for a moment, the classical linear complementarity problem of finding a $z \in \Re^n$ such that

$$z \geq 0, \quad Qz + q \geq 0, \quad \langle z, Qz + q \rangle = 0,$$

where Q is an $n \times n$ matrix and $q \in \Re^n$. The natural quadratic problem that one associates with LCP is (see [2])

$$\begin{array}{ll} \text{minimize} & \langle z, Qz + q \rangle \\ \text{subject to} & z \geq 0,\ Qz + q \geq 0. \end{array}$$

It is well known that this quadratic program is, in general, not very useful for solving the LCP, in part because its feasible set typically has a fairly complicated structure. Clearly, situation with the BLP (2) associated with XLCP is very similar. In recent years, considerable amount of research on complementarity problems has focused on obtaining optimization reformulations with simple constraints. Of particular interest are unconstrained reformulations (for example, [24, 19, 15, 6, 16, 26, 4, 18, 20, 17, 23]) and reformulations where the feasible set contains only nonnegativity constraints (see [10, 24, 9, 3, 30, 7]). Research in this direction is vast and is by no means limited to the cited references. For example, bound constrained reformulations can be also constructed for the variational inequality problem [27].

Motivated by the above mentioned developments for standard complementarity problems, we propose a number of smooth optimization reformulations for XLCP of the form

$$\begin{array}{ll} \text{minimize} & f(x, y) := p(x, y) + \psi(x, y) \\ \text{subject to} & (x, y) \in \mathcal{B}, \end{array} \tag{3}$$

where the set $\mathcal{B}$ in $\Re^{2n}$ contains only some nonnegativity constraints if any at all, the function p is a smooth external penalization for the polyhedral set $\mathcal{P}$,

$$p(x,y) \begin{cases} = 0 & \text{if } Mx - Ny \in \mathcal{P} \\ > 0 & \text{otherwise ,} \end{cases}$$

and ψ is a nonnegative "complementarity function" satisfying the condition

$$(x,y) \in \mathcal{B}, \quad \psi(x,y) = 0 \quad \Longleftrightarrow \quad x \geq 0, \quad y \geq 0, \quad \langle x, y \rangle = 0.$$

Obviously, the appropriate choice of ψ depends on the choice of the set $\mathcal{B}$. And of course, some additional conditions will have to be imposed on the function ψ to obtain useful reformulations. We consider two types of problems. For the first one, $\mathcal{B}$ is the nonnegative orthant in $\Re^{2n}$, i.e.

$$\mathcal{B} = \{(x,y) \in \Re^{2n} \mid x \geq 0, y \geq 0\}.$$

We consider several complementarity functions. First of all, we can choose as ψ the square of the bilinear function itself :

$$\psi_1(x,y) := \frac{1}{2}(\langle x, y \rangle)^2.$$

This function ψ_1 has been used for other types of problems in [8, 9]. Another possible choice which we shall study is

$$\psi_2(x,y) := \frac{1}{2}\sum_{i=1}^{n} x_i^2 y_i^2.$$

A related least squares formulation has been employed for standard complementarity problems in [25]. The function ψ_2 has an advantage over ψ_1 in that it grows somewhat slower on the nonnegative orthant.

Given that all feasible x and y are nonnegative for this choice of $\mathcal{B}$, it is tempting to consider the bilinear function itself

$$\psi_3(x,y) := \langle x, y \rangle,$$

which is simpler than either ψ_1 or ψ_2. However, as we shall see, using this function results in reformulations with weaker properties (see also remarks in [9]).

Note that the functions ψ_1 and ψ_2 are essentially of growth order four. We therefore also consider the following functions which have quadratic growth (thus, in general, one would expect them to be more attractive computationally). These functions are well known in complementarity literature :

$$\psi_4(x,y) := \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{x_i^2 + y_i^2} - x_i - y_i\right)^2,$$

which is the (squared) Fischer-Burmeister function [5, 16, 4]; the implicit Lagrangian function [24, 29]

$$\psi_5(x,y) := \langle x, y \rangle + \frac{1}{2\alpha}\left(\|(x - \alpha y)_+\|^2 - \|x\|^2 + \|(y - \alpha x)_+\|^2 - \|y\|^2\right),$$

where $(\cdot)_+$ stands for the projection operator onto the nonnegative orthant, and $\alpha > 1$ is a parameter; and the restricted implicit Lagrangian [10, 24, 29]

$$\psi_6(x, y) := \langle x, y\rangle + \frac{1}{2\alpha}\left(\|(x - \alpha y)_+\|^2 - \|x\|^2\right).$$

For functions ψ_4 and ψ_5 we also consider the unconstrained reformulations with

$$\mathcal{B} = \Re^{2n}.$$

The paper is organized as follows. In Section 2, we study the BLP (2) and give some sufficient conditions which guarantee that every KKT point of BLP solves the XLCP. In Section 3 we consider the bound constrained reformulations, and in Section 4 the unconstrained reformulations. We establish the equivalence between their stationary points and solutions of XLCP under some of the same assumptions used for the bilinear program. Section 5 contains concluding remarks.

Throughout this paper, we assume that the polyhedral set $\mathcal{P}$ in $\Re^m$ appearing in the statement of XLCP (1) is represented as

$$\mathcal{P} = \{u \in \Re^m \mid Gu \geq g\},$$

where G is some $k \times m$ real matrix and $g \in \Re^k$. For this representation, the recession cone [28] of the set $\mathcal{P}$ is the set

$$0^+\mathcal{P} = \{u \in \Re^m \mid Gu \geq 0\},$$

and its dual is

$$\begin{aligned}(0^+\mathcal{P})^* &= \{v \in \Re^m \mid \langle v, u\rangle \geq 0 \text{ for all } u \in 0^+\mathcal{P}\}\\ &= \{v \in \Re^m \mid v = G^\top \mu \text{ for some } \mu \geq 0\},\end{aligned}$$

where $G^\top$ denotes the transpose of matrix G. Finally, we recall that a square matrix Q is said to be copositive on a cone $\mathcal{K}$ if $\langle Qv, v\rangle \geq 0$ for all $v \in \mathcal{K}$, and strictly copositive if the latter inequality is strict for all $0 \neq v \in \mathcal{K}$.

2. The Bilinear Program

We start with some sufficient conditions for every KKT point of the bilinear program

$$\begin{aligned}&\text{minimize} && \langle x, y\rangle\\ &\text{subject to} && x \geq 0,\ y \geq 0,\ G(Mx - Ny) \geq g\end{aligned} \tag{4}$$

to be a solution of the XLCP. Note that the first condition in Theorem 1 below has been established in [22]. We include its proof for completeness. The other two conditions appear to be new. It should be noted that the second condition in Theorem 1 cannot be satisfied for the horizontal LCP, because in that case $\mathcal{P}$ is a singleton and, hence, $(0^+\mathcal{P})^*$ is the whole space. However, it is not difficult to construct examples when this condition holds in the more general case of XLCP where $\mathcal{P}$ is not a singleton. With respect to the last condition in Theorem 1, observe that it is equivalent to saying that $(0,0)$ is a solution of XLCP. Of course, to obtain this trivial solution, no reformulations are needed. However, the XLCP may have other solutions, and the BLP may have other KKT points. Theorem 1 guarantees that if $(0,0) \in \mathcal{P}$ then all KKT points of BLP are solutions of XLCP.

THEOREM 1 *Suppose that one of the following three conditions is satisfied :*

(i) The matrix $MN^\top$ *is copositive on* $(0^+\mathcal{P})^*$.

(ii) It holds that $M^\top v \le 0$ *and* $N^\top v \ge 0$ *for all* $v \in (0^+\mathcal{P})^*$.

(iii) It holds that $(0,0) \in \mathcal{P}$.

Then every KKT point of the BLP solves the XLCP.

Proof: If (x,y) is a KKT point of (4) then there exist vectors $\mu \in \Re^k$ and $t, s \in \Re^n$ such that (see, for example, [21])

$$y - M^\top G^\top \mu - t = 0, \quad x + N^\top G^\top \mu - s = 0,$$

$$0 = \langle x, t\rangle = \langle y, s\rangle = \langle \mu, GMx - GNy - g\rangle,$$

$$GMx - GNy - g \ge 0, \quad \mu \ge 0, \quad x, y, t, s \ge 0.$$

Suppose that assumption (i) is satisfied. Using the KKT conditions, we obtain

$$\begin{aligned} 0 \le \langle x, y\rangle &= \langle x, y\rangle - \langle x, t\rangle - \langle y, s\rangle + \langle t, s\rangle - \langle t, s\rangle \\ &= \langle x - s, y - t\rangle - \langle t, s\rangle \\ &= -\langle N^\top G^\top \mu, M^\top G^\top \mu\rangle - \langle t, s\rangle \\ &= -\langle MN^\top (G^\top \mu), G^\top \mu\rangle - \langle t, s\rangle \le 0, \end{aligned}$$

where the last inequality follows from copositivity of $MN^\top$ on $(0^+\mathcal{P})^*$ and the fact that $G^\top \mu \in (0^+\mathcal{P})^*$, and nonnegativity of t and s. It immediately follows that $\langle x, y\rangle = 0$, and hence (x,y) is a solution of XLCP.

Now suppose that assumption (ii) holds. We have that

$$\begin{aligned} 0 &= \langle x, t\rangle + \langle y, s\rangle \\ &= \langle x, y - M^\top G^\top \mu\rangle + \langle y, x + N^\top G^\top \mu\rangle \\ &= 2\langle x, y\rangle - \langle x, M^\top G^\top \mu\rangle + \langle y, N^\top G^\top \mu\rangle. \end{aligned} \tag{5}$$

Under the assumption that $M^\top v \le 0,\ N^\top v \ge 0$ for all $v \in (0^+\mathcal{P})^*$, it follows from nonnegativity of x and y that each of the terms in the right-hand-side of (5) is nonnegative. Hence they are all zero. In particular, $\langle x, y\rangle = 0$ and (x,y) solves the XLCP.

Finally, let the last condition (iii) hold. Since $(0,0) \in \mathcal{P}$, it follows that $0 \ge g$. From (5) we further obtain

$$\begin{aligned} 0 &= 2\langle x, y\rangle - \langle Mx - Ny, G^\top \mu\rangle \\ &= 2\langle x, y\rangle - \langle GMx - GNy - g, \mu\rangle - \langle g, \mu\rangle \\ &= 2\langle x, y\rangle - \langle g, \mu\rangle, \end{aligned}$$

where the last equality follows from the KKT conditions. Because $g \leq 0$ while $\mu \geq 0$, both terms in the right-hand-side of the latter inequality are nonnegative. Hence $\langle x, y\rangle = 0$ and (x, y) is a solution of XLCP. ∎

In [13], an *X-row-sufficiency* property was introduced. In particular, a pair of matrices M and N is said to be X-row-sufficient with respect to a polyhedral set $\mathcal{P}$ if the following property holds :

$$\begin{array}{l} v \in (0^+\mathcal{P})^*, \quad (M^\top v)_i\,(N^\top v)_i \leq 0, \quad i = 1, \dots, n \quad \Longrightarrow \\ (M^\top v)_i\,(N^\top v)_i = 0, \quad i = 1, \dots, n. \end{array}$$

It is easy to see that (i), i.e. copositiveness of $MN^\top$ on $(0^+\mathcal{P})^*$, implies X-row-sufficiency with respect to $\mathcal{P}$. Note that (ii), on the other hand, implies that the matrix $-MN^\top$ is copositive on $(0^+\mathcal{P})^*$.

COROLLARY 1 *If the feasible region of BLP is nonempty and one of the assumptions of Theorem 1 is satisfied, then XLCP is solvable.*

Proof: The result follows from the fact that the quadratic objective function of the BLP is always bounded below on the feasible region. Thus BLP has solutions whenever it is feasible. Theorem 1 further guarantees that these solutions of BLP solve the XLCP under the given assumptions. ∎

3. Bound Constrained Reformulations

We now turn our attention to bound constrained reformulations. For simplicity, we shall consider only one exterior penalty function for the set $\mathcal{P}$. In particular, we shall use

$$p(x, y) := \frac{1}{2}\|(-GMx + GNy + g)_+\|^2,$$

where $\|\cdot\|$ is the 2-norm. The problem under consideration is therefore the following :

$$\begin{array}{ll} \text{minimize} & (1/2)\|(-GMx + GNy + g)_+\|^2 + \psi(x, y) \\ \text{subject to} & x \geq 0, \; y \geq 0. \end{array} \tag{6}$$

In principle, one could use some of the other smooth penalty functions as well.

To establish the equivalence of KKT points of (6) and solutions of XLCP (1), we require the complementarity function ψ to possess (some of) the following properties :

I. $\psi(x, y) \geq 0$ for all $x \geq 0, \; y \geq 0$; moreover $\psi(x, y) = 0 \iff \langle x, y\rangle = 0$.

II. $x \geq 0, \quad y \geq 0, \quad \langle \nabla_x \psi(x, y), \nabla_y \psi(x, y)\rangle \leq 0 \quad \Longrightarrow \quad \psi(x, y) = 0$.

III. $\psi(x, y) = 0 \quad \Longrightarrow \quad \nabla_{(x,y)} \psi(x, y) = 0$.

IV. $x_i = 0, \; y_i \geq 0 \implies [\nabla_y \psi(x, y)]_i = 0$ and
$y_i = 0, \; x_i \geq 0 \implies [\nabla_x \psi(x, y)]_i = 0$.

V. $\langle x, \nabla_x \psi(x, y)\rangle + \langle y, \nabla_y \psi(x, y)\rangle = c\psi(x, y)$, where $c > 0$.

VI. $\langle \nabla_x \psi(x,y), \nabla_y \psi(x,y)\rangle \geq 0$ for all $x \geq 0,\ y \geq 0$.

The properties listed above are quite natural. Before proceeding with the analysis, we show that some complementarity functions of interest satisfy all of them.

LEMMA 1 *Complementarity functions $\psi_1,\ \psi_2$ and ψ_4 satisfy all properties I-VI; ψ_3 satisfies all properties except III; ψ_5 satisfies all properties except II; and ψ_6 satisfies all properties except II and III.*

Proof: For functions $\psi_1,\ \psi_2$ and ψ_3, the assertions can be checked by direct observation.

For the Fischer-Burmeister function ψ_4, all properties except V are well known (see, for example, [11]). We proceed to prove V. Let

$$\varphi(a,b) := \frac{1}{2}\left(\sqrt{a^2+b^2} - a - b\right)^2.$$

With this notation,

$$\psi_4(x,y) = \sum_{i=1}^{n} \varphi(x_i, y_i).$$

Let φ_a and φ_b denote partial derivatives of φ with respect to a and b, respectively. If $a = b = 0$, clearly $a\varphi_a(a,b) + b\varphi_b(a,b) = 0 = 2\varphi(a,b)$. Assume now that $a \neq 0$ or $b \neq 0$. Then we can write

$$\varphi_a(a,b) = \left(\sqrt{a^2+b^2} - a - b\right)\left(\frac{a}{\sqrt{a^2+b^2}} - 1\right),$$

and similarly for $\varphi_b(a,b)$. Then we have

$$\begin{aligned} a\varphi_a(a,b) + b\varphi_b(a,b) &= \left(\sqrt{a^2+b^2} - a - b\right)\left(\frac{a^2}{\sqrt{a^2+b^2}} - a + \frac{b^2}{\sqrt{a^2+b^2}} - b\right) \\ &= \left(\sqrt{a^2+b^2} - a - b\right)^2 = 2\varphi(a,b). \end{aligned}$$

Summing up for all i, we obtain Property V.

For the implicit Lagrangian ψ_5, properties I,III and VI are also well known (see [24, 3, 23]). Let

$$\varphi(a,b) := ab + \frac{1}{2\alpha}\left((a-\alpha b)_+^2 - a^2 + (b-\alpha a)_+^2 - b^2\right).$$

We then have

$$\psi_5(x,y) = \sum_{i=1}^{n} \varphi(x_i, y_i).$$

Observe that

$$\varphi_a(a,b) = b + \frac{1}{\alpha}\left((a-\alpha b)_+ - a - \alpha(b-\alpha a)_+\right)$$

and

$$\varphi_b(a,b) = a + \frac{1}{\alpha}\left(-\alpha(a-\alpha b)_+ + (b-\alpha a)_+ - b\right).$$

Property IV can now be verified directly. Furthermore,

$$\begin{aligned} a\varphi_a(a,b) + b\varphi_b(a,b) &= ab - \frac{a^2}{\alpha} + \frac{a}{\alpha}(a-\alpha b)_+ - a(b-\alpha a)_+ \\ &\quad +ab - \frac{b^2}{\alpha} - b(a-\alpha b)_+ + \frac{b}{\alpha}(b-\alpha a)_+ \\ &= 2ab - \frac{1}{\alpha}(a^2+b^2) + \frac{1}{\alpha}(a-\alpha b)_+^2 + \frac{1}{\alpha}(b-\alpha a)_+^2 \\ &= 2\varphi(a,b), \end{aligned}$$

from which Property V follows.

The restricted implicit Lagrangian ψ_6 can be analyzed similarly; we omit the details. ■

It should be noted that for some of the complementarity functions, the properties considered above can be further strengthened. In I-VI we only list what seem to be minimal conditions necessary in the context of this paper.

We next establish that provided ψ satisfies appropriate assumptions, every KKT point of (6) is a solution of XLCP under precisely the same conditions that guarantee this property for the BLP (4). We point out that (6) has the advantage over the latter in that its feasible set has simple structure.

THEOREM 2 *Suppose that one of the following four sets of assumptions is satisfied :*

(i) The function ψ satisfies I,II,III,IV; and the matrix $MN^\top$ is copositive on $(0^+\mathcal{P})^*$.

(ii) The function ψ satisfies I,III,V; and it holds that $M^\top v \le 0$ and $N^\top v \ge 0$ for all $v \in (0^+\mathcal{P})^*$.

(iii) The function ψ satisfies I,V; and $(0,0) \in \mathcal{P}$.

(iv) The function ψ satisfies I,III,IV,V,VI; and the matrix $MN^\top$ is strictly copositive on $(0^+\mathcal{P})^*$.

Then every KKT point of (6) solves XLCP.

Proof: Let us denote $v := G^\top(-GMx+GNy+g)_+$. Observe that $v \in (0^+\mathcal{P})^*$ because $(-GMx+GNy+g)_+ \ge 0$. Let (x,y) be a KKT point of (6). Then there exist two vectors $t, s \in \Re^n$ such that

$$-M^\top v + \nabla_x\psi(x,y) - t = 0, \quad N^\top v + \nabla_y\psi(x,y) - s = 0, \tag{7}$$

$$0 = \langle x,t\rangle = \langle y,s\rangle, \quad x,y,t,s \ge 0. \tag{8}$$

We further obtain

$$\begin{aligned}
-\langle MN^\top v, v\rangle &= -\langle M^\top v, N^\top v\rangle \\
&= \langle \nabla_x\psi(x,y) - t, \nabla_y\psi(x,y) - s\rangle \\
&\geq \langle \nabla_x\psi(x,y), \nabla_y\psi(x,y)\rangle - \langle t, \nabla_y\psi(x,y)\rangle - \langle s, \nabla_x\psi(x,y)\rangle, \qquad (9)
\end{aligned}$$

where the inequality follows from the nonnegativity of t and s.

Suppose that the assumptions in (i) are satisfied. Note that if for some $i \in \{1, \dots, n\}$ it holds that $t_i > 0$ then (8) implies that $x_i = 0$. In that case, by Property IV, it follows that $[\nabla_y\psi(x,y)]_i = 0$. Hence,

$$\langle t, \nabla_y\psi(x,y)\rangle = 0.$$

By the same argument, also

$$\langle s, \nabla_x\psi(x,y)\rangle = 0.$$

It now follows from (9) and the copositiveness of $MN^\top$ on $(0^+\mathcal{P})^*$ that

$$0 \geq \langle \nabla_x\psi(x,y), \nabla_y\psi(x,y)\rangle.$$

By Property II, we conclude that $\psi(x,y) = 0$. We also have that $x \geq 0$ and $y \geq 0$. Therefore, by Property I, $\langle x, y\rangle = 0$.

It remains to establish that $Mx - Ny \in \mathcal{P}$. By Property III, $\psi(x,y) = 0$ implies that $\nabla_{(x,y)}\psi(x,y) = 0$. Recalling the definition of v, it is easy to see that KKT conditions (7),(8) for problem (6) reduce to the KKT conditions for the following convex program :

$$\begin{array}{ll} \text{minimize} & (1/2)\|(-GMx + GNy + g)_+\|^2 \\ \text{subject to} & x \geq 0,\ y \geq 0. \end{array} \qquad (10)$$

It follows that (x,y) is a global solution of this problem. Hence because of the feasibility of the given XLCP, $(-GMx + GNy + g)_+ = 0$, i.e. $Mx - Ny \in \mathcal{P}$. Therefore (x,y) is a solution of XLCP.

Now suppose that the assumptions in (ii) are satisfied. Using KKT conditions (7),(8) and Property V, we obtain

$$\begin{aligned}
0 &= \langle x, t\rangle + \langle y, s\rangle \\
&= \langle x, \nabla_x\psi(x,y)\rangle + \langle y, \nabla_y\psi(x,y)\rangle - \langle x, M^\top v\rangle + \langle y, N^\top v\rangle \\
&= c\psi(x,y) - \langle x, M^\top v\rangle + \langle y, N^\top v\rangle. \qquad (11)
\end{aligned}$$

Since it holds that $M^\top v \leq 0$ and $N^\top v \geq 0$ for $v \in (0^+\mathcal{P})^*$, it follows from nonnegativity of x and y that all terms in the right-hand-side of the above equality are zero. In particular, $\psi(x,y) = 0$ and, by Property I, $\langle x, y\rangle = 0$. Now using Property III, the proof that $Mx - Ny \in \mathcal{P}$ follows as before by considering (10).

Suppose now that the conditions in (iii) hold. By (11) we further obtain

$$\begin{aligned}
0 &= c\psi(x,y) + \langle -Mx + Ny, v\rangle \\
&= c\psi(x,y) + \langle G(-Mx + Ny) + g, (-GMx + GNy + g)_+\rangle \\
&\quad -\langle g, (-GMx + GNy + g)_+\rangle \\
&= c\psi(x,y) + \|(-GMx + GNy + g)_+\|^2 - \langle g, (-GMx + GNy + g)_+\rangle.
\end{aligned}$$

Since $(0,0) \in \mathcal{P}$, we have $g \le 0$; from the above we see that $\psi(x,y)=0$ and $(-GMx + GNy - g)_+ = 0$. Together with nonnegativity of x and y, and taking into account Property I, this means that (x,y) solves XLCP.

Finally, suppose that the assumptions in (iv) are satisfied. By (9) and Properties IV and VI, we obtain that

$$-\langle MN^\top v, v\rangle \ge 0.$$

By strict copositiveness of $MN^\top$ on $(0^+\mathcal{P})^*$, it follows that $v=0$. By KKT conditions (7),(8) and Property V, we further obtain

$$\begin{aligned} 0 &= \langle x, \nabla_x \psi(x,y) - t\rangle + \langle y, \nabla_y \psi(x,y) - s\rangle \\ &= \langle x, \nabla_x \psi(x,y)\rangle + \langle y, \nabla_y \psi(x,y)\rangle \\ &= c\psi(x,y). \end{aligned}$$

By Property I, it now follows that $\langle x,y\rangle = 0$. The proof that $Mx - Ny \in \mathcal{P}$ follows as before, using (10) and Property III. ■

Remark. It can be seen from the proof of Theorem 2 that every KKT point of (6) which is feasible with respect to the set $\mathcal{P}$ is necessarily a solution of XLCP, provided ψ satisfies Properties I and V (note that all functions considered in this paper possess these two properties).

According to Lemma 1 and Theorem 2, all KKT points of the bound constrained minimization problem (6) are solutions of the XLCP under precisely the same conditions needed for the BLP, provided the functions $\psi_1,\ \psi_2$ or ψ_4 are used in (6). If one is to use in (6) the bilinear function itself (i.e. ψ_3), then the equivalence holds only if $(0,0) \in \mathcal{P}$. So this reformulation has much weaker properties, which is consistent with observations made in [9]. For the implicit Lagrangian ψ_5, the equivalences hold under the same assumptions as for $\psi_1,\ \psi_2$ or ψ_4, except that copositiveness of $MN^\top$ on $(0^+\mathcal{P})^*$ has to be replaced by strict copositiveness. This is also consistent with some results [11] for standard complementarity problems.

We also tried to use the feasible set

$$\mathcal{B} = \{(x,y) \mid x \ge 0\}$$

thus keeping only nonnegativity constraints for x, as one would naturally try for the restricted implicit Lagrangian ψ_6 (or another function considered in [30]). While some results can still be obtained with those reformulations, they seem to be overall weaker than Theorem 2.

4. Unconstrained Reformulations

We now consider the unconstrained problem

$$\min_{(x,y)\in\Re^{2n}} \frac{1}{2}\|(-GMx + GNy + g)_+\|^2 + \psi(x,y). \tag{12}$$

Naturally, one might expect this problem to be useful for resolving the XLCP only if ψ is an *unconstrained* complementarity function. The two most interesting (smooth) functions of this class are the Fischer-Burmeister function ψ_4 [5, 16, 4] and the implicit Lagrangian ψ_5 [24, 29].

To establish the equivalence of stationary points of (12) and solutions of XLCP (1), we require the complementarity function ψ to possess (some of) the following properties :

I. $\psi(x,y) \geq 0$ for all $x,\ y$; moreover
$\psi(x,y) = 0 \iff x \geq 0,\ y \geq 0,\ \langle x,y\rangle = 0.$

II. $\langle \nabla_x \psi(x,y), \nabla_y \psi(x,y)\rangle \geq 0$ for all $x,\ y$.

III. $\langle \nabla_x \psi(x,y), \nabla_y \psi(x,y)\rangle = 0 \implies \psi(x,y) = 0.$

IV. $\psi(x,y) = 0 \iff \nabla_{(x,y)}\psi(x,y) = 0.$

V. $\langle x, \nabla_x \psi(x,y)\rangle + \langle y, \nabla_y \psi(x,y)\rangle = c\psi(x,y)$, where $c > 0$.

These properties are standard, except for V which has been established in Lemma 1. Thus we omit the proofs.

LEMMA 2 *The function ψ_4 satisfies all properties I-V; ψ_5 satisfies all properties except for III.*

We next describe conditions which guarantee that stationary points of the unconstrained problem (12) are solutions of the XLCP.

THEOREM 3 *Suppose that one of the following three sets of conditions is satisfied :*

(i) The function ψ satisfies I,II,III,IV; and matrix $MN^\top$ is copositive on $(0^+\mathcal{P})^*$.

(ii) The function ψ satisfies I,II,IV; and matrix $MN^\top$ is strictly copositive on $(0^+\mathcal{P})^*$.

(iii) The function ψ satisfies I,V; and $(0,0) \in \mathcal{P}$.

Then every stationary point of (12) is a solution of XLCP.

Proof: Let (x,y) be a stationary point of problem (12). Then we have

$$-M^\top v + \nabla_x \psi(x,y) = 0 \tag{13}$$

and

$$N^\top v + \nabla_y \psi(x,y) = 0, \tag{14}$$

where $v := G^\top(-GMx + GNy + g)_+ \in (0^+\mathcal{P})^*$. It follows that

$$\begin{aligned} 0 &= -\langle M^\top v, N^\top v\rangle - \langle M^\top v, \nabla_y \psi(x,y)\rangle \\ &= -\langle MN^\top v, v\rangle - \langle \nabla_x \psi(x,y), \nabla_y \psi(x,y)\rangle. \end{aligned} \tag{15}$$

Suppose that the assumptions in (i) are satisfied. Copositiveness of $MN^\top$ on $(0^+\mathcal{P})^*$ and Property II imply that

$$\langle \nabla_x \psi(x,y), \nabla_y \psi(x,y)\rangle = 0.$$

By Property III, $\psi(x,y) = 0$. Hence, by Property I, $x \geq 0,\ y \geq 0$ and $\langle x, y\rangle = 0$.

It remains to show that $Mx - Ny \in \mathcal{P}$. By Property IV, $\psi(x,y) = 0$ implies that

$$\nabla_x \psi(x,y) = \nabla_y \psi(x,y) = 0.$$

Therefore the stationarity conditions (13), (14) for problem (12) reduce to the condition that

$$0 = \nabla_{(x,y)} \|(-GMx + GNy + g)_+\|^2.$$

In view of convexity and the fact that the set $\mathcal{P}$ is nonempty, the latter equality implies that $(-GMx + GNy + g)_+ = 0$, i.e. $Mx - Ny \in \mathcal{P}$, and we have that (x,y) solves XLCP.

Now suppose that the assumptions in (ii) hold. By strict copositiveness of $MN^\top$ on $(0^+\mathcal{P})^*$, and Property II, it follows from (15) that $0 = v = G^\top(-GMx + GNy + g)_+$. Then the stationarity conditions (13), (14) for problem (12) reduce to

$$\nabla_x \psi(x,y) = 0 \quad \text{and} \quad \nabla_y \psi(x,y) = 0,$$

that is $\nabla_{(x,y)} \psi(x,y) = 0$. By Property IV, we conclude that $\psi(x,y) = 0$. Therefore, by Property I, x and y are complementary. The proof that $Mx - Ny \in \mathcal{P}$ follows as before.

Assume now that (iii) is satisfied. We have

$$\begin{aligned} 0 &= \langle x, -M^\top v + \nabla_x \psi(x,y)\rangle + \langle y, N^\top v + \nabla_y \psi(x,y)\rangle \\ &= \langle x, \nabla_x \psi(x,y)\rangle + \langle y, \nabla_y \psi(x,y)\rangle + \langle -Mx + Ny, v\rangle \\ &= 2\psi(x,y) + \langle -GMx + GNy + g, (-GMx + GNy + g)_+\rangle \\ &\quad -\langle g, (-GMx + GNy + g)_+\rangle \\ &= 2\psi(x,y) + \|(-GMx + GNy + g)_+\|^2 - \langle g, (-GMx + GNy + g)_+\rangle, \end{aligned}$$

where the third equation follows from Property V. Since $(0,0) \in \mathcal{P}$, we have that $g \leq 0$. It now follows that $(-GMx + GNy + g)_+ = 0$ and $\psi(x,y) = 0$. By Property I, (x,y) is a solution of XLCP. ■

5. Concluding Remarks

We have studied several optimization reformulations for the extended linear complementarity problem. For the bilinear programming reformulation, some new conditions were established which guarantee that every Karush-Kuhn-Tucker point of the reformulation is a solution of the XLCP. We also proposed some new unconstrained and bound constrained reformulations, and established the equivalence of their stationary points to the solutions of XLCP under the same assumptions used for the bilinear program.

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