

The hybrid proximal decomposition method applied to the computation of a Nash equilibrium for hydrothermal electricity markets

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Abstract In this work, a decomposition method for computing a solution of a short-term hydrothermal scheduling problem is presented. An oligopolistic electricity market composed of two types of power generation units, thermal and hydroelectric, is considered. The hydroelectric units have also the possibility of pumping back water, paying in that case for the electricity consumed. The Nash equilibrium analytic condition is stated as a variational inclusion and it is shown that the associated operator has a structure suitable for decomposition, in particular by applying the variable metric hybrid proximal decomposition technique (VMHPDM). The application of VMHPDM is illustrated in several examples and numerical results for each example are presented.

Keywords Electric power market · Nash-Cournot equilibria · Variational inclusions · Decomposition

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Abbreviations

t	each time period, $t = 1, \dots, T$.
i	each thermal unit, $i = 1, \dots, \mathcal{I}$.
j	each hydroelectric unit, $j = 1, \dots, \mathcal{J}$.
m	each thermal company, $m = 1, \dots, M$.
n	each hydroelectric company, $n = 1, \dots, N$.
C_m^{Th}	index set for thermal units owned by company m .
C_n^H	index set for hydroelectric units owned by company n .
y_{jt}	production at hydroelectric unit j for time period t (or consumption, if this quantity is negative).
y	hydroelectric production vector in $\mathbb{R}^{\mathcal{J}T}$.
x_{it}	production at thermal unit i for the time period t .
x	thermal production vector in $\mathbb{R}^{\mathcal{I}T}$.
y_j^{LOW}, y_j^{UP}	hydroelectric production bounds for unit j .
y_j^{TOT}	total hydroelectric generation for unit j .
α_j	efficiency coefficient of hydroelectric unit j .
$y^{LOW}, y^{UP}, y^{TOT}, \alpha$	corresponding vectors in $\mathbb{R}^{\mathcal{J}}$.
x_i^{LOW}, x_i^{UP}	thermal productions bounds for unit i .
ϕ_i, ω_i, ψ_i	thermal cost coefficients associated to the quadratic, linear, and independent term for unit i .
$x^{LOW}, x^{UP}, \phi, \omega, \psi$	corresponding vectors in $\mathbb{R}^{\mathcal{I}}$.
a_t, D_t	inverse demand coefficients for time t .
a, D	corresponding vectors in \mathbb{R}^T .
Ben_n^H	benefit obtained by the hydroelectric company n .
Ben_m^T	benefit obtained by the thermal company m .
p_t	market price for period t .
$f_j(\cdot)$	non-smooth function representing the efficiency gap between pumping and generation for unit j .
$c_i^T(\cdot)$	quadratic function representing the thermal production cost for unit i .
\mathcal{K}_n^H	feasible set for the hydroelectric company n .
\mathcal{K}_m^{Th}	feasible set for the thermal company m .
I_n	identity matrix in $\mathbb{R}^{n \times n}$.
$\mathbf{1}_n$	vector of ones in \mathbb{R}^n .
$\lambda_{max}(M)$	maximal eigenvalue of the symmetric positive definite (SPD) matrix M .
$\lambda_{min}(M)$	minimal eigenvalue of the SPD matrix M .
$\text{diag}(v)$	diagonal matrix with $(\text{diag}(v))_{n,n} = v_n$, for $v \in \mathbb{R}^n$.
$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$	operator that maps points in \mathbb{R}^n to subsets of \mathbb{R}^m .
$\text{dom}F$	domain of $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, i.e., $\{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$.
$N_{\mathcal{K}}(x)$	normal cone of a convex set \mathcal{K} at x , i.e. $N_{\mathcal{K}}(x) = \begin{cases} \{\eta : \eta^\top (y - x) \leq 0, \forall y \in \mathcal{K}\}, & \text{if } x \in \mathcal{K}, \\ \emptyset, & \text{if } x \notin \mathcal{K}. \end{cases}$
$\text{rint } \mathcal{K}$	relative interior of the set \mathcal{K} .
$\text{Proj}_{\mathcal{K}}(x)$	(orthogonal) projection of the point x onto the set \mathcal{K} .

1 Introduction

Deregulation in electricity production markets occurred in the last century in the USA and extended thereafter (in some different degrees) to most of the countries. It thoroughly changed the behavior of the electricity production companies and, accordingly, the computation of the price of electricity provided by those companies.

Nash-Cournot models are often encountered in the technical literature representing producer behavior in real-world electricity markets, and it had also been considered in hydrothermal coordination problems. One of the first works in this field is Scott and Read (1996), which is applied to the New Zealand's electricity market. The model of Scott and Read (1996) makes use of Dual Dynamic Programming, and at each stage an optimization problem is superimposed on a Cournot market equilibrium. In Rivier et al. (2001) the market equilibrium is stated using certain equations expressing the optimal behavior of generation companies. This approach takes advantage of the fact that the optimality conditions can be directly solved due to the Mixed Complementarity Problem (MCP) structure, which allows for the use of special complementarity methods to solve realistically sized problems. In Baldick (2002) Cournot and supply function equilibria are numerically compared under the addition of transmission constraints. In Ruiz et al. (2010) some theoretical results pertaining to the Cournot model applied to short-term electricity markets are presented. Price, quantities and profits are first obtained, and then results related to sensitivities and limit values are derived and discussed.

A hydrothermal coordination problem considering pumped storage units is presented in Moitre et al. (2005) and defined as a bilevel optimization problem. It is solved using dynamic programming and some heuristics, and applied to a real case of the Argentinian electricity market. In Hobbs and Pang (2010) the authors make a very deep mathematical study of an extended problem formulated as a nonmonotone multivalued variational inequality problem, that can include nonsmooth functions like concave piecewise demand functions. This formulation allows a more realistic model but it needs some simplifications to be solvable.

In this work, a model for the short-term optimal scheduling of hydraulic and thermal electricity generation units based on Nash-Cournot equilibrium theory (as in Ruiz et al. (2010)) is presented. It is also assumed that the hydraulic units have the ability of pumping water back in order to reuse it considering an approach similar to Moitre et al. (2005). This last assumption introduces non-differentiability in the formulation, but the presented numerical algorithm is able to deal with it. The analytical formulation of the problem is based on variational inequalities as in Hobbs and Pang (2010), but the algorithm for solving the problem is different (it is based on the general methodology presented in Parente et al. (2008)).

The mathematical conditions of the Nash-Cournot equilibrium are stated in terms of a variational inclusion of the form $0 \in T(\cdot)$, where T is a maximal monotone operator that has certain special structure suitable for applying the decomposition method developed in Solodov (2004). In order to enhance numerical efficiency, a variable metric version of the algorithm (Lotito et al. 2009) is applied.

Section 2 describes the electricity production market introducing the problem formulation as a coupled-in-time Nash equilibrium problem. In Sect. 3 the problem is

cast as a variational inclusion, and it is shown that it can be tackled in the framework of the decomposition methods of Lotito et al. (2009). A more technical description of the method is given in the appendix. In Sect. 4 some examples and numerical results obtained with this methodology are presented. Finally, in the last section some conclusions are stated.

2 Electricity production markets

Electricity generation markets used to be government-driven monopolies, with a national company owning every segment of the system. In the last two decades, there has been a worldwide trend leading to the deregulation of these markets, introducing different kinds of competition in some segments of this industry. That is why competitive market concepts are used nowadays in a traditionally monopolist industry (Rubiales et al. 2007).

To model monopolistic electric systems, only technical aspects of the system need to be considered. On the other hand, when oligopolistic electricity markets are modeled, economical aspects of pool agents must also be taken into account. The latter strongly influence the electricity price because power generation is distributed among the market actors (Moitre et al. 2005). The energy spot price is an essential instrument which must be known in this kind of models. It represents the cost of the next Megawatt (MW) of charge to be provided, considering restrictions on several aspects, such as transport and maintenance of security and quality of service of the system.

In this work, a production system composed by thermal and hydroelectric power units is considered. The thermal production unit burns some fossil fuel (coal, natural gas, etc.) producing a high pressure fluid flow (steam or gas) that will drive the generator turbines. These units are ready to produce as long as fuel is available, and can be started relatively fast. Their disadvantage is the use of non-renewable resources and the consequent pollutant emissions (like sulfur and nitrogen oxides from coal burning). In contrast, hydroelectric generation units use the kinetic energy of falling or flowing water to rotate the turbine blades. The problem with these plants is that the availability of water is not guaranteed.

A distinctive characteristic of this work is the fact that a special type of hydroelectric generation units, called *pumped storage*, is considered. This kind of hydroelectric units allow a more rational use of the hydraulic resources of a country. They have two different reservoirs at different levels connected by a penstock and a reversible turbine. When the system demand reaches its maximum level (peak hours), water flows from the upper to the lower reservoir through the hydro-power plant, generating electricity as a conventional hydro-power plant. Conversely, when the demand is lower (off-peak hours), the second reservoir refills the upper one by pumping water back. By means of this procedure, the plant has more water to generate electricity during periods of peak consumption. This operation allows the plant to control the load of the system, increasing the load in off-peak hours and supplying power in peak hours.

The main problem when scheduling hydroelectric plants is to obtain the best strategy to manage the available water to generate electricity. Hydroelectric plants with reservoirs have the ability to administrate the usage of water among different periods.

Thus thermal generation during the present period, and the possibility to store water to spend it in the future, as well as vice versa, must be considered. The capacity storage units should move energy blocks on time—spending water now should introduce more thermal generation in the future. The aim of this type of optimization is to find the point in which the sum of the future profit and the current profit is maximized. Therefore, the optimization problem is coupled through different periods.

As mentioned before, each unit operator acting in an oligopolistic market seeks to maximize its own profit arriving to a Nash equilibrium point. In order to set the analytical model, the benefits of the thermal and the hydroelectric unit operators are defined. At each time period t , the operators will be paid a market price p_t for the quantity of energy produced. We shall call y_{jt} the hydroelectric generation (or consumption, if this quantity is negative) at hydroelectric unit j for time period t . The production at thermal unit i for the time period t is denoted as x_{it} . The hydroelectric benefit for company n , assuming no production costs, is computed as

$$Ben_n^H = \sum_{j \in C_n^H} \sum_{t=1}^T f(y_{jt}) p_t, \tag{1}$$

where C_n^H is the index set for the different hydroelectric units for the company n and

$$f(s) = \begin{cases} s, & \text{if } s \geq 0, \\ \alpha s, & \text{if } s < 0, \end{cases}$$

is used to represent the difference between pumping ($s < 0$) and generating ($s > 0$). The efficiency coefficient $\alpha > 1$ indicates that the energy used to pump water is higher than the energy generated by the same volume of water. In an efficient pumped storage plant, this coefficient will be close to one. The coefficient α may depend on the unit j , but it is reasonable to assume that all of them are equal.

The main issue is to know the optimal total volume of water to be spent in the planning horizon. In the literature, there are two methods for dealing with this issue (Wood and Wollenberg 1996). The first one considers that the total amount of water in the reservoir is available in the short term, but a value to the amount of water that is not spent is assigned to motivate hydroelectric plants to keep water beyond the horizon of analysis. The second approach considers that a known fixed volume of water is available to be used in the planning horizon (obviously, less than the total volume of water in the reservoir) as a result of long-term programming that takes into account other modeling aspects (uncertainty in weather, demand, etc.). In this work, the second approach is adopted, avoiding the need to assign the value of water.

As it is assumed that the hydroelectric plant must spend the total amount of water available in the short-term planning horizon, the value of water will be the same for each period, and its price is calculated in a long-term scheduling algorithm considering other modeling aspects, as uncertainty in weather, demand, etc. The water available to be used by unit j during the planning horizon is represented by the total power y_j^{TOT} that the plant can generate with it, so the production schedule for each unit must satisfy the restriction

$$\sum_t y_{jt} = y_j^{TOT}. \tag{2}$$

On the other hand, the thermal benefit for company m is computed as

$$Ben_m^{Th} = \sum_{i \in C_m^{Th}} \sum_{t=1}^T (x_{it} p_t - c_i^{Th}(x_{it})), \tag{3}$$

where C_m^{Th} is the index set for the different thermal units for the company m and c_i^{Th} is the thermal production cost (assumed quadratic).

In order to reduce and simplify the problem of thermal unit scheduling, any on-off restrictions in the thermal unit operation are ignored.

The market price is given by the inverse demand function

$$p_t(d) = \frac{1}{a_t}(D_t - d), \tag{4}$$

obtained inverting the demand function (assumed affine)

$$d_t(p) = D_t - a_t p, \tag{5}$$

whose coefficients D_t and a_t , in turn, are obtained given the value of the elasticity around a demand-price anchor point. The hypothesis of Nash equilibrium means that the demand is supplied by all the players. In order to compute the equilibrium price in formula (4), the total production of energy at time t , stated as d_t , is composed of thermal and hydroelectric production. Thus,

$$p_t = \frac{1}{a_t} \left(D_t - \sum_{j=1}^{\mathcal{J}} y_{jt} - \sum_{i=1}^{\mathcal{I}} x_{it} \right), \tag{6}$$

and the benefits for each type of plants would depend on their own production and the productions of the other plants.

Defining the feasible sets for the thermal and hydroelectric production variables as \mathcal{K}_m^{Th} for each thermal company m and \mathcal{K}_n^H for each hydroelectric company n , respectively, the conditions for the Nash equilibrium considering the production vector (x^*, y^*) are defined in the following form:

$$Ben_m^{Th}(x^*, y^*) = \max_{x_m \in \mathcal{K}_m^{Th}} Ben_m^{Th}(x_m, x_{/m}^*, y^*), \quad m = 1, \dots, M, \tag{7}$$

and, simultaneously,

$$Ben_n^H(x^*, y^*) = \max_{y_n \in \mathcal{K}_n^H} Ben_n^H(x^*, y_n, y_{/n}^*), \quad n = 1, \dots, N, \tag{8}$$

where $x_m = (x_{it})_{i \in C_m^{Th}}$, $x_{/m} = (x_{it})_{i \notin C_m^{Th}}$, $y_n = (y_{jt})_{j \in C_n^H}$ and $y_{/n} = (y_{jt})_{j \notin C_n^H}$.

The optimality conditions for (8) and (7) can be stated as a variational inclusion of the form

$$0 \in T(x^*, y^*) = \begin{pmatrix} \partial_{x_1} Ben_1^{Th}(x^*, y^*) + \mathcal{N}_{\mathcal{K}_1^{Th}}(x^*) \\ \vdots \\ \partial_{x_M} Ben_M^{Th}(x^*, y^*) + \mathcal{N}_{\mathcal{K}_M^{Th}}(x^*) \\ \partial_{y_1} Ben_1^H(x^*, y^*) + \mathcal{N}_{\mathcal{K}_1^H}(y^*) \\ \vdots \\ \partial_{y_N} Ben_N^H(x^*, y^*) + \mathcal{N}_{\mathcal{K}_N^H}(y^*) \end{pmatrix}, \tag{9}$$

where $\partial(\cdot)$ is the subdifferential operator (with respect to the variables $(x_i)_{i \in \mathcal{C}_m^{Th}}$ or the variables $(y_j)_{j \in \mathcal{C}_n^H}$ appearing as a subscript) and $\mathcal{N}_{\mathcal{K}}(x)$ is the normal cone to the set \mathcal{K} at the point x . Equivalent optimality conditions are stated in Rockafellar and Wets (1997).

In the next section, a more precise version of this model that justifies the use of the decomposition method of Lotito et al. (2009) is given.

3 Hydrothermal Nash-Cournot Equilibrium Problem

In this section, the analytic model for the Hydrothermal Nash-Cournot Equilibrium Problem is presented. It is also shown that it can be solved applying the variable metric hybrid proximal decomposition method (VMHPDM) developed in Lotito et al. (2009). A detailed description of VMHPDM in its general form is given in the appendix. In short, for the present implementation, we need to show that the inclusion (9) has the following structure:

$$T = F \times [G + H], \tag{10}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is maximal monotone, $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is continuous, and $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is maximal monotone.

The proposed model consists of an oligopoly where \mathcal{I} thermal plants are owned by M companies, and \mathcal{J} pumped storage hydroelectric plants are owned by N companies, i.e., there are M sets \mathcal{C}_m^{Th} of indices representing \mathcal{I}_m thermal plants each, and N sets \mathcal{C}_n^H of indices representing \mathcal{J}_n hydroelectric units each, respectively, with $\sum_{m=1}^M \mathcal{I}_m = \mathcal{I}$ and $\sum_{n=1}^N \mathcal{J}_n = \mathcal{J}$. The Nash equilibrium derived from the intention to maximize the individual benefit of all the companies is studied considering T periods. The variables for the thermal and hydroelectric production in T periods are respectively $x = (x_{it}) \in \mathbb{R}^{\mathcal{I}T}$ and $y = (y_{jt}) \in \mathbb{R}^{\mathcal{J}T}$.

Following (1), and taking into account (6), the hydroelectric benefit functions for $n = 1, \dots, N$, are defined as

$$Ben_n^H(x, y) = \sum_{j \in \mathcal{C}_n^H} \sum_{t=1}^T \frac{f(y_{jt})}{a_t} \left[D_t - \sum_{j=1}^{\mathcal{J}} y_{jt} - \sum_{i=1}^{\mathcal{I}} x_{it} \right].$$

From (3) and (6), considering the quadratic cost

$$c_i^{Th}(x_{it}) = \frac{\phi_i}{2} x_{it}^2 + \omega_i x_{it} + \psi_i,$$

the benefit functions are redefined as

$$Ben_m^{Th}(x, y) = \sum_{i \in \mathcal{C}_m^{Th}} \sum_{t=1}^T \left(\frac{x_{it}}{a_t} \left[D_t - \sum_{j=1}^{\mathcal{J}} y_{jt} - \sum_{i=1}^{\mathcal{I}} x_{it} \right] - \frac{\phi_i}{2} x_{it}^2 - \omega_i x_{it} - \psi_i \right).$$

The constraints for each hydroelectric unit are given by the production bounds y_j^{LOW} , y_j^{UP} and the restriction (2), so the n company constraints set is defined as

$$\mathcal{K}_n^H = \left\{ y_j \in \mathbb{R}^{\mathcal{J}_n T} : y_j^{LOW} \leq y_{jt} \leq y_j^{UP}, t = 1, \dots, T, \sum_t y_{jt} = y_j^{TOT}, j \in \mathcal{C}_n^H \right\}. \tag{11}$$

For each thermal unit only the restrictions given by production bounds x_i^{LOW} , x_i^{UP} are considered, so the constraints set for company m is the box

$$\mathcal{K}_m^{Th} = \{x_m \in \mathbb{R}^{\mathcal{I}_m T} : x_i^{LOW} \leq x_{it} \leq x_i^{UP}, t = 1, \dots, T, i \in \mathcal{C}_m^{Th}\}.$$

Then the sets $\mathcal{K}^H = \otimes_{n=1}^N \mathcal{K}_n^H$ and $\mathcal{K}^{Th} = \otimes_{m=1}^M \mathcal{K}_m^{Th}$ are defined.

In order to write the programs in matricial form, it will be useful to define the vectors

$$\begin{aligned} Z^{UP} &= (y^{UP \top}, \dots, y^{UP \top}, -y^{LOW \top}, \dots, -y^{LOW \top})^\top \in \mathbb{R}^{2\mathcal{J}T}, \\ X^{LOW} &= (x^{LOW \top}, \dots, x^{LOW \top})^\top \in \mathbb{R}^{\mathcal{I}T}, \\ X^{UP} &= (x^{UP \top}, \dots, x^{UP \top})^\top \in \mathbb{R}^{\mathcal{I}T}, \end{aligned}$$

as well as the matrices

$$\begin{aligned} S^H &= \begin{pmatrix} \mathbf{1}_{\mathcal{J}}^\top & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \mathbf{1}_{\mathcal{J}}^\top \end{pmatrix} \in \mathbb{R}^{T \times \mathcal{J}T}, & S^{Th} &= \begin{pmatrix} \mathbf{1}_{\mathcal{I}}^\top & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \mathbf{1}_{\mathcal{I}}^\top \end{pmatrix} \in \mathbb{R}^{T \times \mathcal{I}T}, \\ E &= \begin{pmatrix} 0 & I_{\mathcal{J}T} \\ I_{\mathcal{J}T} & 0 \end{pmatrix} \in \mathbb{R}^{2\mathcal{J}T \times 2\mathcal{J}T}, & A &= \begin{pmatrix} \text{diag } \alpha & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \text{diag } \alpha \end{pmatrix} \in \mathbb{R}^{\mathcal{J}T \times \mathcal{J}T} \end{aligned}$$

and

$$T^H = (I_{\mathcal{J}}^\top, \dots, I_{\mathcal{J}}^\top) \in \mathbb{R}^{\mathcal{J} \times \mathcal{J}T}.$$

Matrices S_n^H and S_n^{Th} are defined by

$$(S_n^H)_{t,jt} = \begin{cases} (S^H)_{t,jt}, & \text{if } j \in \mathcal{C}_n^H, \\ 0, & \text{if } j \notin \mathcal{C}_n^H, \end{cases} \quad (S_n^{Th})_{t,jt} = \begin{cases} (S^{Th})_{t,jt}, & \text{if } j \notin \mathcal{C}_n^H, \\ 0, & \text{if } j \in \mathcal{C}_n^H, \end{cases}$$

so

$$S_{/n}^H = \sum_{k \neq n} S_k^H. \tag{12}$$

An analogous notation is used for defining matrices S_n^{Th} , E_n , A_n , T_n^H , etc. Also, the vectors $y_n \in \mathbb{R}^{\mathcal{J}nT}$ and $x_m \in \mathbb{R}^{\mathcal{I}mT}$ are identified with the vectors in $\mathbb{R}^{\mathcal{J}T}$ and $\mathbb{R}^{\mathcal{I}T}$, respectively, obtained by completing the indices with zeros, concluding that $y_{/n} = y - y_n$ and $x_{/m} = x - x_m$.

Observe that $p(x, y) = (p_1, \dots, p_T)$ turns out to be

$$p(x, y) = \text{diag}(a)^{-1} [D - S^H y - S^{Th} x].$$

The first step should be dealing with the minimization in \mathcal{K}_n^H of the opposite hydroelectric benefit functions $(-Ben_n^H)$, which are non-smooth at zero. Since $\alpha_j > 1$ for every j , the functions $(-Ben_n^H)$ are convex and one possible option is to minimize them using constrained bundle methods (see Bonnans et al. 2006). In this work a different approach is adopted by writing y in the form $y = y^+ - y^-$, where

$$(y^+)_{jt} = \begin{cases} y_{jt}, & \text{if } y_{jt} \geq 0, \\ 0, & \text{if } y_{jt} < 0, \end{cases} \quad (y^-)_{jt} = \begin{cases} 0, & \text{if } y_{jt} \geq 0, \\ -y_{jt}, & \text{if } y_{jt} < 0, \end{cases}$$

for $j = 1, \dots, \mathcal{J}$ and $t = 1, \dots, T$. And $z = (y^{+\top}, y^{-\top})$, so $y = Rz$ where $R = (I_{\mathcal{J}T}, -I_{\mathcal{J}T})$.

With this notation, the constraints set (11) has the form

$$\bar{\mathcal{K}}_n^H = \left\{ z_n : 0 \leq z_n \leq Z_n^{UP}, (T_n, -T_n)z_n = y_n^{TO\top}, z_n^\top E_n z_n = 0 \right\}.$$

Observe that the equality $z_n^\top E_n z_n = 0$ implies that $y_n^+ \perp y_n^-$.

Then, each hydroelectric company n has a quadratic program of the form

$$\min_{z_n \in \bar{\mathcal{K}}_n^H} z_n^\top Q_n^H z_n + (b_n^z + b_n^x)^\top z_n, \tag{13}$$

where

$$\begin{aligned} Q_n^H &= \left(S_n^H, -S_n^H A_n \right)^\top \text{diag}(a)^{-1} \left(S_n^H, -S_n^H \right) \\ &= \begin{pmatrix} (S_n^H)^\top \text{diag}(a)^{-1} S_n^H & -(S_n^H)^\top \text{diag}(a)^{-1} S_n^H \\ (-S_n^H A_n)^\top \text{diag}(a)^{-1} S_n^H & (S_n^H A_n)^\top \text{diag}(a)^{-1} S_n^H \end{pmatrix}, \end{aligned} \tag{14}$$

$$b_n^z = (S_n^H, -S_n^H A_n)^\top \text{diag}(a)^{-1} (S_{/n}^H, -S_{/n}^H) z_{/n}, \tag{15}$$

$$b_n^x = (S_n^H, -S_n^H A_n)^\top \text{diag}(a)^{-1} (S^{Th} x - D). \tag{16}$$

Then the optimality condition for z_n^* to be a solution of (13) is

$$0 \in \left(Q_n^H{}^\top + Q_n^H \right) z_n^* + (b_n^z + b_n^x) + N_{\bar{\mathcal{K}}_n^H}(z_n^*). \tag{17}$$

Proceeding, for each thermal company m the following optimization problem should be solved

$$\min_{x_m \in \mathcal{K}_m^{Th}} x_m^\top Q_m^{Th} x_m + (d_m^x + d_m^z)^\top x_m. \tag{18}$$

Here the constraints set is the box

$$\mathcal{K}_m^{Th} = \{x_m : X_m^{LOW} \leq x_m \leq X_m^{UP}\},$$

and Q_m^{Th} , d_m^x and d_m^z are given by

$$Q_m^{Th} = (S_m^{Th})^\top \text{diag}(a)^{-1} S_m^{Th} + \frac{1}{2} \Phi_m, \tag{19}$$

$$d_m^x = (S_m^{Th})^\top \text{diag}(a)^{-1} S_{/m}^{Th} x_{/m}, \tag{20}$$

$$d_m^z = (S_m^{Th})^\top \text{diag}(a)^{-1} \left((S^H, -S^H)_z - D \right) + \Omega_m, \tag{21}$$

where

$$\Phi = \begin{pmatrix} \text{diag}(\phi) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \text{diag}(\phi) \end{pmatrix} \in \mathbb{R}^{IT \times IT} \quad \text{and} \quad \Omega = \begin{pmatrix} \omega \\ \vdots \\ \omega \end{pmatrix} \in \mathbb{R}^{IT}.$$

The corresponding optimality condition is

$$0 \in 2Q_m^{Th} x_m^* + (d_m^x + d_m^z) + N_{\mathcal{K}_m^{Th}}(x_m^*). \tag{22}$$

Now the Nash-Cournot Equilibrium Problem can be formulated as the problem of finding (x^*, z^*) which satisfies all the optimality conditions for x_m and z_n , respectively. In particular, a solution of the equilibrium problem must be a feasible point of the constraint set $\hat{\mathcal{K}} = \mathcal{K}^{Th} \times \hat{\mathcal{K}}^H$, where

$$\hat{\mathcal{K}}^H = \bigotimes_{n=1}^N \hat{\mathcal{K}}_n^H = \left\{ z : 0 \leq z \leq Z^{UP}, (T, -T)z = y^{TOT}, z^\top E z = 0 \right\}. \tag{23}$$

The next proposition shows that the non-linear constraint $z^\top E z = 0$ in (23) can be eliminated.

Proposition 1 *Let (x^*, z^*) be a solution of the Nash-Cournot Equilibrium Problem with the relaxed constraints set $\hat{\mathcal{K}} = \mathcal{K}^{Th} \times \hat{\mathcal{K}}^H$, where*

$$\hat{\mathcal{K}}^H = \left\{ z : 0 \leq z \leq Z^{UP}, (T, -T)z = y^{TOT} \right\}, \tag{24}$$

i.e., (x^, z^*) satisfies the equations*

$$Ben_n^H(x^*, z^*) = \max_{z_n \in \hat{\mathcal{K}}_n^H} Ben_n^H(x^*, z_n, z_{/n}^*), \quad n = 1, \dots, N, \tag{25}$$

and

$$Ben_m^{Th}(x^*, z^*) = \max_{x_m \in \mathcal{K}_m^{Th}} Ben_m^{Th}(x_m, x_{/m}^*, z^*), \quad m = 1, \dots, M. \tag{26}$$

Then z^* satisfies the non-linear constraint $z^{*\top} E z^* = 0$.

Proof Suppose that (x^*, z^*) verifies (25) and (26), but $z^{*\top} E z^* \neq 0$. Then, since $z^* \geq 0$, there exist indices j_0 and t_0 such that

$$\begin{aligned} z_{j_0 t_0}^* &= (y^*)_{j_0 t_0}^+ > 0, \\ z_{j_0 t_0 + \mathcal{J}T}^* &= (y^*)_{j_0 t_0}^- > 0. \end{aligned}$$

Define the new point \bar{z} by

$$\begin{aligned} \bar{z}_{jt} &= z_{jt}^*, && \text{if } jt \neq j_0 t_0 \text{ and } jt \neq j_0 t_0 + \mathcal{J}T, \\ \bar{z}_{j_0 t_0} &= z_{j_0 t_0}^* - z_{j_0 t_0 + \mathcal{J}T}^*, && \text{if } z_{j_0 t_0}^* \geq z_{j_0 t_0 + \mathcal{J}T}^*, \\ \bar{z}_{j_0 t_0 + \mathcal{J}T} &= 0, && \text{if } z_{j_0 t_0}^* \geq z_{j_0 t_0 + \mathcal{J}T}^*, \\ \bar{z}_{j_0 t_0} &= 0, && \text{if } z_{j_0 t_0}^* < z_{j_0 t_0 + \mathcal{J}T}^*, \\ \bar{z}_{j_0 t_0 + \mathcal{J}T} &= z_{j_0 t_0 + \mathcal{J}T}^* - z_{j_0 t_0}^*, && \text{if } z_{j_0 t_0}^* < z_{j_0 t_0 + \mathcal{J}T}^*. \end{aligned}$$

Note that $0 \leq \bar{z} \leq Z^{UP}$ and $\bar{z}_{jt} - \bar{z}_{jt+\mathcal{J}T} = z_{jt}^* - z_{jt+\mathcal{J}T}^*$ for all $j = 1, \dots, \mathcal{J}$ and $t = 1, \dots, T$. Then,

$$\sum_{t=1}^T (\bar{z}_{jt} - \bar{z}_{jt+\mathcal{J}T}) = \sum_{t=1}^T (z_{jt}^* - z_{jt+\mathcal{J}T}^*) = y_j^{TOT}, \quad \forall j,$$

and hence, $(T, -T)\bar{z} = y^{TOT}$ and $\bar{z} \in \hat{\mathcal{K}}^H$.

Also, for all $t = 1, \dots, T$ the prices p_t satisfy

$$\begin{aligned} p_t(x^*, \bar{z}) &= \frac{1}{a_t} \left(D_t - \sum_{j=1}^{\mathcal{J}} (\bar{z}_{jt} - \bar{z}_{jt+\mathcal{J}T}) - \sum_{i=1}^{\mathcal{I}} x_{it}^* \right) \\ &= \frac{1}{a_t} \left(D_t - \sum_{j=1}^{\mathcal{J}} (z_{jt}^* - z_{jt+\mathcal{J}T}^*) - \sum_{i=1}^{\mathcal{I}} x_{it}^* \right) = p_t(x^*, z^*). \end{aligned}$$

Hence, for all $m \in \{1, \dots, M\}$ and all $n \in \{n : 1 \leq n \leq N, j_0 \notin C_n^H\}$, the benefits satisfy

$$\begin{aligned} Ben_m^{Th}(x_m^*, x_{/m}^*, \bar{z}) &= Ben_m^{Th}(x_m^*, x_{/m}^*, z^*), \\ Ben_n^H(x^*, \bar{z}_n, \bar{z}_{/n}) &= Ben_n^H(x^*, z_n^*, z_{/n}^*). \end{aligned}$$

If $z_{j_0 t_0}^* \geq z_{j_0 t_0 + \mathcal{J}T}^*$ then

$$\begin{aligned} f(\bar{y}_{j_0 t_0}) p_{t_0} &= (\bar{z}_{j_0 t_0} - \alpha_{j_0} \bar{z}_{j_0 t_0 + \mathcal{J}T}) p_{t_0} = (z_{j_0 t_0}^* - z_{j_0 t_0 + \mathcal{J}T}^*) p_{t_0} \\ &> (z_{j_0 t_0}^* - \alpha_{t_0} z_{j_0 t_0 + \mathcal{J}T}^*) p_{t_0} = f(y_{j_0 t_0}^*) p_{t_0}, \end{aligned}$$

where the fact that $\alpha_{j_0} > 1$ has been used. Similarly, if $z_{j_0 t_0}^* < z_{j_0 t_0 + \mathcal{J}T}^*$ then

$$\begin{aligned} f(\bar{y}_{j_0 t_0}) p_{t_0} &= (\bar{z}_{j_0 t_0} - \alpha_{j_0} \bar{z}_{j_0 t_0 + \mathcal{J}T}) p_{t_0} = \alpha_{t_0} (z_{j_0 t_0}^* - z_{j_0 t_0 + \mathcal{J}T}^*) p_{t_0} \\ &> (z_{j_0 t_0}^* - \alpha_{t_0} z_{j_0 t_0 + \mathcal{J}T}^*) p_{t_0} = f(y_{j_0 t_0}^*) p_{t_0}. \end{aligned}$$

Therefore, for n_0 such that $j_0 \in \mathcal{C}_{n_0}^H$, the benefit for company n_0 verifies

$$Ben_{n_0}^H(x^*, \bar{z}_{n_0}, \bar{z}/n_0) > Ben_m^H(x^*, z_{n_0}^*, z^*/n_0),$$

which contradicts (25). This establishes the claim. □

Coupling the inclusions (17) and (22), for $n = 1, \dots, N$ and $m = 1, \dots, M$, and taking into account Proposition 1 and the definitions (14), (15), (16), (19), (20) and (21), the following variational inclusion is obtained:

$$\begin{aligned} 0 \in & (2Q^{Th} + \Delta)x^* + d(z^*) + N_{\mathcal{K}^{Th}}(x^*) \\ & \times \left(Q^{H\top} + Q^H + B \right) z^* + b(x^*) + N_{\hat{\mathcal{K}}^H}(z^*), \end{aligned} \tag{27}$$

where

$$\begin{aligned} Q^H &= \sum_{n=1}^N Q_n^H, & Q^{Th} &= \sum_{m=1}^M Q_m^{Th}, \\ B &= \sum_{n=1}^N (S_n^H, -S_n^H A_n)^\top \text{diag}(a)^{-1} (S_n^H, -S_n^H), \\ \Delta &= \sum_{m=1}^M (S_m^{Th})^\top \text{diag}(a)^{-1} S_m^{Th}, \\ b(x) &= \sum_{n=1}^N b_n^x = (S^H, -S^H A)^\top \text{diag}(a)^{-1} (S^{Th} x - D), \\ d(z) &= \sum_{m=1}^M d_m^z = (S^{Th})^\top \text{diag}(a)^{-1} ((S^H, -S^H)z - D) + \Omega. \end{aligned}$$

Note that Δ is symmetric, since from (12),

$$\begin{aligned} \Delta^\top &= \left(\sum_{m=1}^M (S_m^{Th})^\top \text{diag}(a)^{-1} \left(\sum_{k \neq m} S_k^{Th} \right) \right)^\top \\ &= \sum_{m=1}^M \sum_{k=1}^M ((S_m^{Th})^\top \text{diag}(a)^{-1} S_k^{Th})^\top - (S_m^{Th})^\top \text{diag}(a)^{-1} S_m^{Th} \end{aligned}$$

$$= \sum_{k=1}^M (S_k^{Th})^\top \text{diag}(a)^{-1} \left(\sum_{m \neq k} S_m^{Th} \right) = \Delta.$$

Therefore, since Q^{Th} is trivially symmetric by its definition, the matrix $2Q^{Th} + \Delta$ is symmetric. Now, defining

$$F(x, z) = (2Q^{Th} + \Delta)x + d(z) + N_{\mathcal{K}^{Th}}(x), \tag{28}$$

$$G(x, z) = \left(Q^H{}^\top + Q^H + B \right) z + b(x), \tag{29}$$

$$H(z) = N_{\hat{\mathcal{K}}^H}(z), \tag{30}$$

the inclusion (27) takes the form of (10). Therefore, VMHPDM can be applied. The algorithm for this particular implementation is outlined below, and it is shown that it is well defined. Some features of the general algorithm are also described, to justify that the present scheme is indeed a special case of VMHPDM. References on details refer to the appendix and can be skipped at first reading. For full description and further discussion, see Lotito et al. (2009).

Algorithm 1 (VMHPDM)

Initialization: Choose $(x^0, z^0) \in \mathbb{R}^{IT} \times \mathbb{R}^{JT}$. Set $k := 0$.

Forward-Backward Splitting Step: Choose a scalar $c_k > 0$ and compute

$$\hat{z}^k = \text{Proj}_{\hat{\mathcal{K}}^H} \left(z^k - c_k G(x^k, z^k) \right). \tag{31}$$

Proximal Step: For adequate choices of the scalar $\beta > 0$ and the SPD matrix U , compute

$$\hat{x}^k = \text{Proj}_{\mathcal{K}^{Th}} \left\{ \frac{1}{\beta} \left(Ux^k - d(\hat{z}^k) \right) \right\}. \tag{32}$$

Approximation Condition Test: Choose the error tolerance parameter $\sigma_k \in (0, 1)$. If the inequality

$$\|s^k\|^2 \leq \sigma_k^2 (c_k \|\hat{x}^k - x^k\|_U^2 + \|\hat{z}^k - z^k\|^2), \tag{33}$$

with

$$s^k = c_k \left(G(\hat{x}^k, \hat{z}^k) - G(x^k, z^k) \right) \tag{34}$$

is not satisfied, decrease c_k , compute a new \hat{z}^k by (31) and repeat the **Proximal Step** with this new \hat{z}^k . (Here, $\|\cdot\|_U$ is the norm induced by the matrix U , i.e. $\|\cdot\|_U = \langle \cdot, U \cdot \rangle$.)

Iterates Update: Stop if $\hat{x}^k = x^k$ and $\hat{y}^k = y^k$. Otherwise, define

$$\begin{aligned} x^{k+1} &= \hat{x}^k, \\ z^{k+1} &= \hat{z}^k - s^k. \end{aligned} \tag{35}$$

Set $k := k + 1$ and go to **Forward-Backward Splitting Step**.

The projection step (31) is a *forward-backward* splitting step because it comes from the equation

$$\hat{z}^k = (I + cH(\cdot))^{-1} \left(I - c_k G(x^k, \cdot) \right) (z^k), \tag{36}$$

which splits the sum $G + H$ “forward” in G and “backward” in H . From (30), equation (36) gives

$$\hat{z}^k + N_{\hat{\mathcal{K}}_H}(\hat{z}^k) \ni z^k - c_k G(x^k, z^k),$$

and this inclusion is equivalent to (31). Using (29), this projection is given by

$$\hat{z}^k = \text{Proj}_{\hat{\mathcal{K}}_H} \left(z^k - c_k \left[(Q^{H^\top} + Q^H + B)z + b(x) \right] \right).$$

This is an orthogonal projection onto the intersection of a hyperplane and a box, and there are $O(\mathcal{J}T)$ -time algorithms to approximate it with any desired precision (see Lotito 2006; Maculan et al. 2003). This justifies the assumption that the splitting step can be computed “exactly”.

Remark 1 As can be seen in the Appendix, equation (36) is a particular case of the general forward-backward splitting step (44), with the choices $\varepsilon_k^z = 0$, $Q_k = I_{2\mathcal{J}T}$ and $G_1^k \equiv 0$. Since it is justified in our setting that it can be computed exactly, we take $e^k = 0$. Note that in this case, the vector \hat{h}^k defined in (44) is given by $\hat{h}^k = \frac{z^k - \hat{z}^k}{c_k} + (Q^{H^\top} + Q^H + B)z^k + b(x^k)$.

The next proposition shows that the projection step (32) is indeed a proximal step.

Proposition 2 *There exist SPD matrices U and U_k and a positive scalar β such that the projection step (32) is equivalent to the exact proximal step consisting in solving the inclusion*

$$0 \in \frac{1}{c_k} U_k (\hat{x}^k - x^k) + F(\hat{x}^k, \hat{z}^k). \tag{37}$$

Proof From (28), the inclusion (37) is equivalent to

$$0 \in \left(2Q^{Th} + \Delta + \frac{1}{c_k} U_k \right) \hat{x}^k + \left(d(\hat{z}^k) - \frac{1}{c_k} U_k x^k \right) + N_{\mathcal{K}^{Th}}(\hat{x}^k). \tag{38}$$

Let $\beta \in \mathbb{R}$ be such that

$$\beta > \max_{k=1}^{\mathcal{I}T} \left\{ \sum_{j=1}^{\mathcal{I}T} |(2Q^{Th} + \Delta)_{kj}| \right\}. \tag{39}$$

Take $U_k = c_k U$ with $U = \beta I_{\mathcal{I}T} - 2Q^{Th} - \Delta$. From (39), U is symmetric and diagonally dominant. Hence, it is positive definite (and so is U_k). Also,

$$2Q^{Th} + \Delta + \frac{1}{c_k} U_k = \beta I_{\mathcal{I}T}.$$

Since $N_{\mathcal{K}^{Th}}(\hat{x}^k) = \beta N_{\mathcal{K}^{Th}}(\hat{x}^k)$ for all $\beta > 0$, dividing by β in (38), the proximal step takes the form

$$\frac{1}{\beta} \left(Ux^k - d(\hat{z}^k) \right) \in \hat{x}^k + N_{\mathcal{K}^H}(\hat{z}^k).$$

Applying the projection operator on both sides of this inclusion, (32) is obtained. This completes the proof. \square

Note that since the set \mathcal{K}^{Th} is a box, the projection (32) can be computed explicitly by the formula

$$\hat{x}^k = \min \left\{ \max \left\{ \frac{1}{\beta} (Ux^k - d(\hat{z}^k)), X^{LOW} \right\}, X^{UP} \right\}. \tag{40}$$

Remark 2 The proximal step (37) is a special case of the general inexact proximal step (46), with the choices $\varepsilon_k^x = 0$ and $P_k^{-1} = U_k$. Since it can be computed exactly by the formula (40), it is justified that $r^k = 0$ can be taken in (46).

The next step checks that inequality (33) is satisfied. This condition guaranties the convergence of the algorithm to a solution of the problem when the iterates are updated by (35). Furthermore, for the given operator T and the choice of matrices $U_k = c_k U$, linear local rate of convergence is obtained if the parameters c_k are bounded away from zero (see Lotito et al. 2009, Theorem 2). In order to prove that the algorithm is well defined, it is necessary to show that (33) can be satisfied by using sufficiently small scalars c_k . This is the content of the next proposition.

Proposition 3 *With the choices of β and U as in the proof of Proposition 2, there exists a scalar $\bar{c} > 0$ such that if the forward-backward splitting step (31) and the proximal step (32) are computed with any positive scalar $c_k < \bar{c}$, then inequality (33) holds.*

Proof From (29) and (34) we obtain that

$$s^k = c_k \left(M_1(\hat{x}^k - x^k) + M_2(\hat{z}^k - z^k) \right),$$

where $M_1 = (S^H, -S^H A)^\top \text{diag}(a)^{-1} S^{Th}$ and $M_2 = (Q^{H^\top} + Q^H + B)$.

Let $L = \max\{\|M_1^\top M_1\|, \|M_2^\top M_2\|\}$ and $\gamma = \max(\|U\|, 1)$. Then

$$\begin{aligned} \|s^k\|^2 &= c_k^2 \|M_1(\hat{x}^k - x^k) + M_2(\hat{z}^k - z^k)\|^2 \\ &\leq 2c_k^2 L (\|\hat{x}^k - x^k\|^2 + \|\hat{z}^k - z^k\|^2) \\ &\leq c_k L (c_k \|U\| \|\hat{x}^k - x^k\|_U^2 + \|\hat{z}^k - z^k\|^2) \\ &\leq c_k L \gamma (c_k \|\hat{x}^k - x^k\|_U^2 + \|\hat{z}^k - z^k\|^2). \end{aligned} \tag{41}$$

Hence, condition (33) is satisfied if $c_k \leq \bar{c} = \sigma_k^2 / (L\gamma)$. This establishes the claim. \square

Although the existence of the bound \bar{c} defined in the above proof ensures that Algorithm 1 is well defined, this bound may be too conservative. In practice, larger values of c_k can be tried. This is an important issue because the speed of convergence is connected to the choice of those parameters: larger values give faster convergence. For this reason, Algorithm 1 allows scalars c_k larger than \bar{c} , which are decreased only if (33) does not hold.

Remark 3 The inequality (33) corresponds to the inequality (50) (i.e., the strong version of (47)) in the appendix, for the particular case where $Q_k = I$, $r^k = 0$ and $\varepsilon_k^x = \varepsilon_k^z = 0$. This justifies taking the step-size $\tau_k a_k = c_k$ in the update, following (49), thus generating the new iterates (35) as in (51).

Let $\nu \in (0, 1)$ be the parameter in the linear convergence estimate (which holds under natural assumptions, see Lotito et al. 2009, Theorem 2), and let (x^*, z^*) be a solution of the problem. Then, for all k sufficiently large, it holds that

$$\begin{aligned} \left\| \begin{pmatrix} x^k - x^* \\ z^k - z^* \end{pmatrix} \right\| &\leq \left\| \begin{pmatrix} x^k - x^{k+1} \\ z^k - z^{k+1} \end{pmatrix} \right\| + \left\| \begin{pmatrix} x^{k+1} - x^* \\ z^{k+1} - z^* \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} x^k - \hat{x}^k \\ z^k - \hat{z}^k \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 \\ s^k \end{pmatrix} \right\| + \nu \left\| \begin{pmatrix} x^k - x^* \\ z^k - z^* \end{pmatrix} \right\| \\ &\leq (1 + c_k \sqrt{2L}) \left\| \begin{pmatrix} x^k - \hat{x}^k \\ z^k - \hat{z}^k \end{pmatrix} \right\| + \nu \left\| \begin{pmatrix} x^k - x^* \\ z^k - z^* \end{pmatrix} \right\|, \end{aligned}$$

where the triangular inequality and (41) have been used. Therefore,

$$\left\| \begin{pmatrix} x^k - x^* \\ z^k - z^* \end{pmatrix} \right\| \leq \frac{1 + c_k \sqrt{2L}}{1 - \nu} \left\| \begin{pmatrix} x^k - \hat{x}^k \\ z^k - \hat{z}^k \end{pmatrix} \right\|. \tag{42}$$

Thus, $\| (x^k - \hat{x}^k, z^k - \hat{z}^k) \|$ measures how far is an iterate from the solution. This justifies the stopping test used in the implementation of the algorithm. In the next section, this implementation is illustrated on some examples.

4 Numerical experiments

In this section, applications to some problems with different production structures are presented.

When a short-term optimization problem is considered, usually the planning horizon varies from 12 hours to 2 days or even a week. In the presented applications the 24 hours planning horizon, i.e., 24 time periods of one hour each, is taken into account. Only for the first example a 12 hours planning horizon is considered.

Each period’s (quantity, price) anchor point is set equal to the associated average load and the price paid by final consumers (in this case it is approximated by the marginal cost of supplying the power demand using only thermal units). The anchor point values for each period are defined in Table 1. In order to estimate the price elasticity

Table 1 Inverse demand function coefficients

Period	Anchor price	Anchor demand	D	a
1	5.32	160.00	213.33	10.02
2	4.91	150.00	200.00	10.18
3	4.67	135.00	180.00	9.63
4	3.05	90.00	120.00	9.82
5	2.56	80.00	106.67	10.41
6	2.99	85.00	113.33	9.46
7	3.41	100.00	133.33	9.76
8	4.88	145.00	193.33	9.89
9	6.25	190.00	253.33	10.13
10	9.18	265.00	353.33	9.63
11	9.97	295.00	393.33	9.86
12	11.34	320.00	426.67	9.41
13	11.49	350.00	466.67	10.16
14	12.78	375.00	500.00	9.78
15	12.73	380.00	506.67	9.95
16	12.71	375.00	500.00	9.83
17	13.77	410.00	546.67	9.92
18	13.96	420.00	560.00	10.03
19	12.22	350.00	466.67	9.55
20	9.61	290.00	386.67	10.06
21	7.12	220.00	293.33	10.30
22	7.01	200.00	266.67	9.51
23	6.31	195.00	260.00	10.31
24	6.34	180.00	240.00	9.46

of demand (ε), the approach defined in Arellano (2004) is followed. In particular, the market demand will be estimated for $\varepsilon = -1/3$ measured at the anchor point. Accordingly, the slope parameter of the demand function is calculated considering that the elasticity at the demand level is equal to ε and the intercept is calculated so as to fit anchor quantity and anchor price at each demand level. The inverse demand function has the form $p = \frac{D}{a} - \frac{d}{a}$, where the obtained coefficient values are given in Table 1.

As already mentioned, in the first problem the 12 hours planning horizon is considered, so only the first 12 anchor points are taken into account. In the other problems, for an increasing number of plants, the parameters introduced in Table 1 are multiplied by an adequate coefficient η , in order to balance the demand with the increase of production offer. The generation structure for problems 1 and 2, is shown in Tables 2 and 3, respectively. For problem 3, the structure is given by adding Tables 3 and 4.

In problem 4, the number of thermal companies was increased to 6 and the hydraulic companies to 4, with 20 thermal plants and 10 hydraulic plants, respectively. In this problem, the value of η is 2.7. In problem 5, there are 30 thermal plants and 15 hydraulic plants, owned by 8 thermal companies and 5 hydraulic companies, respectively. In this case, $\eta = 3.8$. The details of these problems are not specified because

Table 2 Generation structure for problem 1 (12 periods, $\eta = 1$)

Company	Plant	Min. power	Max. power	α	y_j^{TOT}	ϕ_i	ω_i	ψ
1	1	-50	100	1.1	250	-	-	-
2	2	0	80	-	-	0.035	1.75	0
	3	0	50	-	-	0.125	1	0
3	4	0	55	-	-	0.0166	3.15	0
	5	0	30	-	-	0.05	3	0
	6	0	40	-	-	0.05	3	0

Table 3 Generation structure for problem 2 (24 periods, $\eta = 1$)

Company	Plant	Min. power	Max. power	α_j	y_j^{TOT}	ϕ_i	ω_i	ψ
1	1	-50	100	1.1	500	-	-	-
2	2	0	80	-	-	0.035	1.75	0
	3	0	50	-	-	0.125	1	0
3	4	0	55	-	-	0.0166	3.15	0
	5	0	30	-	-	0.05	3	0
	6	0	40	-	-	0.05	3	0
4	7	-20	50	1.15	250	-	-	-
	8	-30	80	1.12	350	-	-	-

Table 4 Additional generation structure for problem 3 (24 periods, $\eta = 1.8$)

Company	Plant	Min. power	Max. power	α_j	y_j^{TOT}	ϕ_i	ω_i	ψ
5	9	0	70	-	-	0.045	2.75	0
	10	0	55	-	-	0.065	2	0
6	11	0	60	-	-	0.0366	2.15	0
	12	0	35	-	-	0.04	2.9	0
	13	0	45	-	-	0.07	2.4	0
7	14	-25	55	1.16	300	-	-	-
	15	-55	110	1.09	650	-	-	-

the purpose of introducing them is to study the algorithm performance on large-scale problems.

The algorithm has been coded by using Scilab 4.0 (INRIA-ENPC, see www.scilab.org) and ran on an Intel i7 CPU 920 2.67GHz with 8Gb of RAM.

Since Algorithm 1 has the theoretical stopping criterion $\hat{x}^k = x^k$ and $\hat{y}^k = y^k$, the natural stopping criterion in our implementation is $\|\hat{x}^k - x^k\| < tol$ and $\|\hat{y}^k - y^k\| < tol$, where tol is a tolerance parameter. Note that, by (42), this stopping test gives a measure of quality of the obtained iterate in terms of proximity to the exact solution. In the numerical experiences, the parameter tol was set equal to 10^{-4} .

Table 5 Numerical performance for the 5 problems

Problem	Size	Number of iterations	Computational time (s)
1	84	484	1.53
2	264	567	16.19
3	480	810	71.64
4	960	1062	464.99
5	1440	1313	989.83

Table 6 Benefits

Th. Comp	<i>BenTh</i> Prob 1	<i>BenTh</i> Prob 2	<i>BenTh</i> Prob 3	Hyd. Comp.	<i>BenH</i> Prob 1	<i>BenH</i> Prob 2	<i>BenH</i> Prob 3
2	4805.86	18001.83	12223.54	1	3204.49	8579.86	7501.82
3	3975.99	15469.36	10250.08	4	–	9324.62	7793.28
5	–	–	11659.71	7	–	–	9779.34
6	–	–	7497.05	–	–	–	–

Table 7 Problem 1

<i>t</i>	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	y^*	y^{*+}	y^{*-}	Price	<i>D</i>
1	13.89	50.00	0.00	25.68	25.68	16.65	16.65	0.00	8.13	213.33
2	10.73	50.00	0.00	24.00	24.00	12.73	12.73	0.00	7.71	200.00
3	5.14	50.00	0.00	21.55	21.55	9.75	9.75	0.00	7.48	180.00
4	0.00	44.43	0.00	12.39	12.39	–3.47	0.00	3.47	5.52	120.00
5	0.00	41.96	0.00	10.57	10.57	–8.79	0.00	8.79	5.03	106.67
6	0.00	42.23	0.00	11.65	11.65	–3.91	0.00	3.91	5.46	113.33
7	0.00	47.70	0.00	14.09	14.09	0.00	0.00	0.00	5.89	133.33
8	8.76	50.00	0.00	23.19	23.19	12.12	12.12	0.00	7.69	193.33
9	24.00	50.00	0.00	30.00	31.34	26.26	26.26	0.00	9.06	253.33
10	48.84	50.00	15.36	30.00	40.00	53.46	53.46	0.00	12.02	353.33
11	59.09	50.00	25.29	30.00	40.00	62.61	62.61	0.00	12.81	393.33
12	66.93	50.00	33.76	30.00	40.00	72.59	72.59	0.00	14.18	426.67

The number of iterations and the computational time for all the problems are reported in Table 5, while Table 6 shows the obtained benefits for Problems 1, 2 and 3.

For the sake of brevity, the thermal and hydraulic production (x^* and y^+) and prices in each period are only reported in Problems 1, 2 and 3 (Tables 7 to 10). The last column shows the *D* parameter used for each problem.

In Problem 1, the obtained z^* in the form $(z)_1^T = y^+$, $(z)_{T+1}^{2T} = y^-$ is also displayed (Table 7), in order to show the complementarity between the positive and negative parts of y^* . Additionally, the hydraulic and thermal productions for this case are shown in Figs. 1 and 2, while the evolution of the benefit functions along iterations is shown in Fig. 3.

Table 8 Problem 2

t	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	y_1^*	y_2^*	y_3^*	Price	D
1	19.14	50.00	0.00	28.31	28.31	0.00	0.01	0.89	8.65	213.33
2	15.15	50.00	0.00	26.21	26.21	-0.53	0.00	0.00	8.15	200.00
3	9.32	50.00	0.00	23.64	23.64	-2.79	0.00	0.00	7.91	180.00
4	0.00	50.00	0.00	17.15	17.15	-16.78	0.00	-11.29	6.49	120.00
5	0.00	50.00	0.00	15.98	15.98	-22.16	0.00	-16.34	6.07	106.67
6	0.00	50.00	0.00	16.01	16.01	-17.19	0.00	-11.90	6.38	113.33
7	0.00	50.00	0.00	18.77	18.77	-13.24	0.00	-7.78	6.84	133.33
8	13.05	50.00	0.00	25.34	25.34	-0.77	0.00	0.00	8.12	193.33
9	28.17	50.00	0.00	30.00	35.51	4.54	0.03	9.18	9.47	253.33
10	47.10	50.00	13.62	30.00	40.00	27.11	11.20	20.35	11.84	353.33
11	55.74	50.00	21.94	30.00	40.00	34.05	14.73	23.88	12.47	393.33
12	61.16	50.00	27.99	30.00	40.00	42.77	18.98	28.13	13.57	426.67
13	71.22	50.00	37.00	30.00	40.00	47.38	21.46	30.61	13.69	466.67
14	76.86	50.00	43.16	30.00	40.00	55.74	25.55	34.71	14.72	500.00
15	78.65	50.00	44.72	30.00	40.00	56.32	25.88	35.03	14.68	506.67
16	77.00	50.00	43.23	30.00	40.00	55.51	25.45	34.60	14.67	500.00
17	80.00	50.00	54.33	30.00	40.00	66.08	30.76	39.91	15.68	546.67
18	80.00	50.00	55.00	30.00	40.00	69.97	32.73	41.88	16.00	560.00
19	69.54	50.00	36.18	30.00	40.00	50.14	22.70	31.85	14.27	466.67
20	54.96	50.00	20.87	30.00	40.00	31.81	13.65	22.81	12.18	386.67
21	36.95	50.00	2.53	30.00	40.00	12.06	3.84	12.99	10.19	293.33
22	29.16	50.00	0.00	30.00	37.27	10.02	2.63	11.78	10.07	266.67
23	29.99	50.00	0.00	30.00	37.10	5.06	0.34	9.49	9.51	260.00
24	23.70	50.00	0.00	30.00	31.87	4.89	0.06	9.21	9.54	240.00

Note that in all problems, as expected, the hydraulic units pump water during low price periods (off peak hours).

5 Conclusion

A mathematical model for the behavior of electricity generating companies acting in an oligopolistic market is presented. The methodology applied to solve the associated variational inclusion is a variable metric proximal decomposition method. The inclusion represents mathematically the coupled-in-time Nash-Cournot equilibrium for the scheduling of hydroelectricity production considering all the market players. The numerical results are at this stage preliminary but show that this methodology could be applied both for the scheduling of large and heterogeneous network of generation units and for the analysis of the market and the regulation policies of the regulatory agencies.

Table 9 Problem 3 (x variables)

t	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*	x_7^*	x_8^*	x_9^*	x_{10}^*
1	30.33	50.00	0.00	28.89	28.89	0.00	55.00	18.12	14.59	45.00
2	27.54	50.00	0.00	27.32	27.32	0.00	55.00	15.21	11.46	45.00
3	21.08	50.00	0.00	24.71	24.71	0.00	55.00	9.15	6.15	45.00
4	5.87	50.00	0.00	16.89	16.89	0.00	51.45	0.00	0.00	44.38
5	4.59	50.00	0.00	15.59	15.59	0.00	49.91	0.00	0.00	42.42
6	3.31	50.00	0.00	16.01	16.01	0.00	49.06	0.00	0.00	42.24
7	9.02	50.00	0.00	18.53	18.53	0.00	54.63	0.00	0.00	45.00
8	25.12	50.00	0.00	26.43	26.43	0.00	55.00	13.00	9.64	45.00
9	41.93	50.00	0.00	30.00	39.14	0.00	55.00	29.63	25.96	45.00
10	65.50	50.00	21.24	30.00	40.00	0.00	55.00	53.57	35.00	45.00
11	77.42	50.00	32.58	30.00	40.00	0.00	55.00	60.00	35.00	45.00
12	80.00	50.00	42.11	30.00	40.00	3.89	55.00	60.00	35.00	45.00
13	80.00	50.00	55.00	30.00	40.00	18.07	55.00	60.00	35.00	45.00
14	80.00	50.00	55.00	30.00	40.00	28.89	55.00	60.00	35.00	45.00
15	80.00	50.00	55.00	30.00	40.00	31.82	55.00	60.00	35.00	45.00
16	80.00	50.00	55.00	30.00	40.00	29.05	55.00	60.00	35.00	45.00
17	80.00	50.00	55.00	30.00	40.00	46.14	55.00	60.00	35.00	45.00
18	80.00	50.00	55.00	30.00	40.00	51.27	55.00	60.00	35.00	45.00
19	80.00	50.00	54.39	30.00	40.00	16.27	55.00	60.00	35.00	45.00
20	76.28	50.00	30.92	30.00	40.00	0.00	55.00	60.00	35.00	45.00
21	52.60	50.00	6.64	30.00	40.00	0.00	55.00	40.18	35.00	45.00
22	43.59	50.00	0.00	30.00	40.00	0.00	55.00	31.75	28.91	45.00
23	44.28	50.00	0.00	30.00	40.00	0.00	55.00	31.86	27.94	45.00
24	36.75	50.00	0.00	29.99	35.46	0.00	55.00	24.94	22.16	45.00

Appendix: General decomposition method

Consider a variational inclusion of the form $0 \in T(\cdot)$, where

$$T(x, z) = F(x, z) \times [G(x, z) + H(z)], \tag{43}$$

with $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, and $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$.

This kind of operators have many practical applications; they have been considered, for example, in Tseng (1997), Solodov (2004), Lotito et al. (2009). Assume that (see Rockafellar and Wets (1997) for definitions)

- A1 G is a (single-valued) continuous function.
- A2 H is maximal monotone.
- A3 The mapping $(x, z) \mapsto F(x, z) \times G(x, z)$ is maximal monotone.
- A4 $\text{dom}H \subset \text{rint}\{z \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ s.t. } F(x, z) \times G(x, z) \neq \emptyset\}$.

Under these assumptions, it follows from Rockafellar (1970) that T is maximal monotone, and that the mapping $x \rightarrow F(x, z)$ is maximal monotone for any fixed

Table 10 Problem 3 (cont.)

t	y_1^*	y_2^*	y_3^*	y_4^*	y_5^*	Price	D
1	-3.45	0.00	0.00	0.08	4.64	6.20	383.99
2	-7.57	0.00	-1.61	0.08	0.64	5.98	360.00
3	-9.42	0.00	-3.77	0.00	0.00	5.85	324.00
4	-26.25	0.00	-20.49	0.00	-9.56	4.91	216.00
5	-32.42	0.00	-26.31	0.00	-14.74	4.66	192.00
6	-25.79	0.00	-20.25	0.00	-9.72	4.88	203.99
7	-22.60	0.00	-16.87	0.00	-6.02	5.11	239.99
8	-7.59	0.00	-1.79	0.09	0.39	5.97	347.99
9	0.00	0.00	0.00	0.02	15.49	6.79	455.99
10	23.94	6.14	21.91	10.37	32.50	8.42	635.99
11	33.64	11.04	26.81	15.45	37.58	8.93	707.99
12	46.35	17.30	33.06	21.35	43.49	9.77	768.00
13	54.76	21.66	37.43	26.30	48.43	10.03	840.00
14	68.46	28.43	44.20	32.78	54.92	10.92	900.00
15	70.10	29.29	45.06	33.77	55.90	10.95	912.00
16	68.23	28.33	44.09	32.71	54.85	10.89	900.00
17	84.62	36.54	52.31	41.00	63.13	11.77	984.00
18	88.94	38.73	54.49	43.26	65.40	11.96	1008.00
19	57.66	22.98	38.75	27.15	49.28	10.39	840.00
20	30.59	9.56	25.32	14.12	36.25	8.72	696.00
21	4.63	0.00	9.03	1.38	23.51	7.28	527.99
22	3.15	0.00	7.21	0.06	21.79	7.22	480.00
23	0.00	0.00	0.66	0.02	16.51	6.83	468.00
24	0.00	0.00	0.78	0.02	15.33	6.84	432.00

$z \in \text{dom } H$ (Tseng 1997, Lemma 2.1). Inclusions with this structure can be solved numerically via decomposition techniques in Tseng (1997), Solodov (2004), Lotito et al. (2009). Among these, the Variable Metric Hybrid Proximal Decomposition Method (VMHPDM) of Lotito et al. (2009) is particularly attractive. VMHPDM is derived from the Variable Metric Hybrid Inexact Proximal Point Method presented in Parente et al. (2008), and both schemes are extensions to the variable metric setting of the methods introduced in Solodov (2004) and Solodov and Svaiter (2001), respectively.

We proceed to explain the method. Given $(x^k, z^k) \in \mathbb{R}^n \times \mathbb{R}^m$, first an inexact *forward-backward splitting* step is performed with the x -part fixed. This step consists in finding a triplet $(\hat{h}^k, \hat{z}^k, \varepsilon_k^z) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}_+$ such that

$$\begin{cases} \hat{h}^k \in (H^{\varepsilon_k^z} + G_1^k)(\hat{z}^k), \\ e^k = c_k Q_k \hat{h}^k + \hat{z}^k - (z^k - c_k Q_k [(G(x^k, z^k) - G_1^k(z^k))]), \end{cases} \tag{44}$$

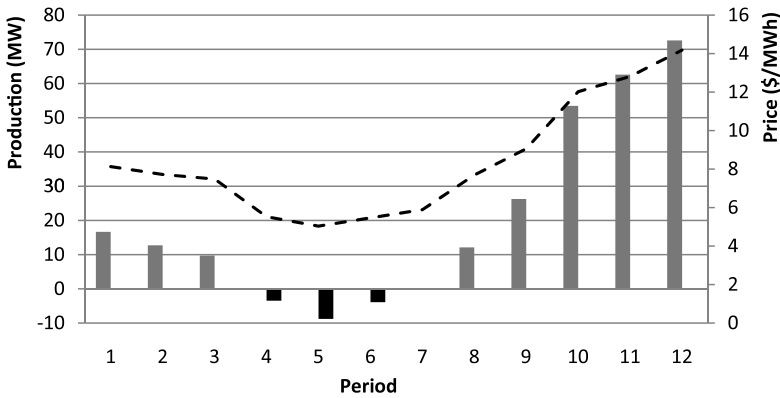


Fig. 1 The computed hydraulic production for Problem 1 is represented by bars with negative values for the pumping. The dashed line represented the market price

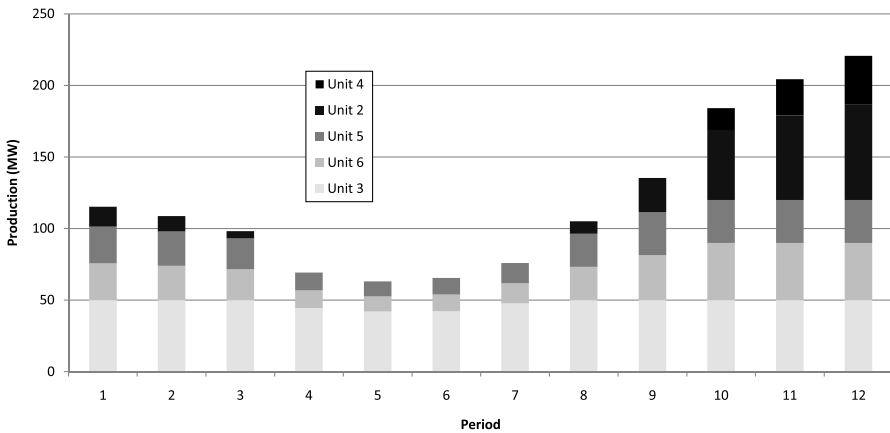


Fig. 2 Computed thermal production for Problem 1

where G_1^k is some adequate Lipschitz-continuous splitting function for G , $c_k > 0$, Q_k is a SPD matrix, and $H^{\varepsilon_k^z}$ is the ε_k^z -enlargement of the maximal monotone operator H (the ε -enlargement of T (see Burachik et al. 1997, 1999) is given by

$$T^\varepsilon(z) := \{v \in \mathbb{R}^n \mid \langle w - v, y - z \rangle \geq -\varepsilon, \forall y \in \mathbb{R}^n, \forall w \in T(y)\}, \varepsilon \geq 0.$$

To clarify the nature of this step and some options concerning the choice of G_k^1 , suppose that H is the normal cone mapping associated to a closed convex set $C \subset \mathbb{R}^m$ and that $Q_k = I_m$, $\varepsilon_k^z = e^k = 0$, for all k . In that case, (44) gives

$$\hat{z}^k = \text{Proj}_C(z^k - c_k(G_1^k(\hat{z}^k) - G_1^k(z^k) + G(x^k, z^k))).$$

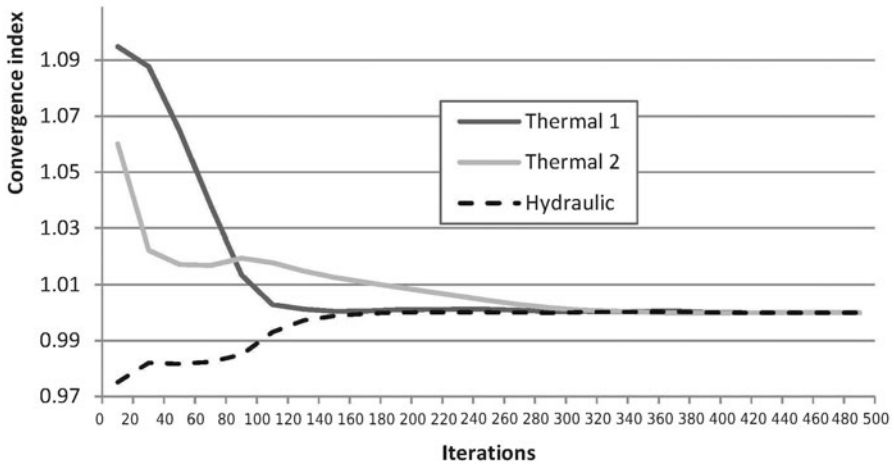


Fig. 3 Convergence of the benefits along iterations for Problem 1

If we take $G_1^k \equiv 0$, then

$$\hat{z}^k = \text{Proj}_C \left(z^k - c_k G(x^k, z^k) \right), \tag{45}$$

which is the standard projection step. If we take $G_1^k \equiv G(x^k, \cdot)$, then

$$\hat{z}^k = \text{Proj}_C \left(z^k - c_k G(x^k, \hat{z}^k) \right),$$

which is an implicit (proximal) step. See Tseng (1997) for a more detailed discussion.

The splitting step above is followed by a hybrid inexact proximal step with the z -part fixed. It consists in finding $(\hat{u}^k, \hat{x}^k, \varepsilon_k^x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ such that

$$\begin{cases} \hat{u}^k \in F^{\varepsilon_k^x}(\hat{x}^k, \hat{z}^k), \\ r^k = c_k P_k \hat{u}^k + \hat{x}^k - x^k, \end{cases} \tag{46}$$

where the enlargement $F^{\varepsilon_k^x}$ is in x for \hat{z}^k fixed, P_k is a SPD matrix, and

$$\begin{aligned} & \|r^k\|_{P_k^{-1}}^2 + \|s^k\|_{Q_k^{-1}}^2 + 2c_k(\varepsilon_k^x + \varepsilon_k^z) \\ & \leq \sigma_k^2 \left(\|c_k \hat{u}^k\|_{P_k^{-1}}^2 + \|c_k \hat{w}^k\|_{Q_k^{-1}}^2 + \|\hat{x}^k - x^k\|_{P_k^{-1}}^2 + \|\hat{z}^k - z^k\|_{Q_k^{-1}}^2 \right), \end{aligned} \tag{47}$$

for $\sigma_k \in (0, 1)$, with

$$\hat{w}^k = G(\hat{x}^k, \hat{z}^k) + \hat{h}^k - G_1^k(\hat{z}^k), \quad s^k = c_k Q_k \hat{w}^k + \hat{z}^k - z^k, \tag{48}$$

where \hat{h}^k is the element of $H(\hat{z}^k)$ computed in the inexact forward-backward splitting step. Condition (47) can be always satisfied provided the splitting step has been solved with sufficient accuracy for a sufficiently small c_k .

The next iterates are obtained by setting

$$\begin{aligned} x^{k+1} &= x^k - \tau_k a_k P_k \hat{u}^k, \\ z^{k+1} &= z^k - \tau_k a_k Q_k \hat{w}^k, \end{aligned} \tag{49}$$

where

$$a_k = \frac{\langle (\hat{u}^k, \hat{w}^k), (x^k - \hat{x}^k, z^k - \hat{z}^k) \rangle - (\varepsilon_k^x + \varepsilon_k^z)}{\|P_k \hat{u}^k\|_{P_k^{-1}}^2 + \|Q_k \hat{w}^k\|_{Q_k^{-1}}^2}, \quad \tau_k \in (0, 2).$$

If a stronger than (47) approximation is used, i.e.,

$$\|r^k\|_{P_k^{-1}}^2 + \|s^k\|_{Q_k^{-1}}^2 + 2c_k(\varepsilon_k^x + \varepsilon_k^z) \leq \sigma_k^2 \left(\|\hat{x}^k - x^k\|_{P_k^{-1}}^2 + \|\hat{z}^k - z^k\|_{Q_k^{-1}}^2 \right), \tag{50}$$

then in (49) we can use the stepsize $\tau_k a_k = c_k$. Suppose that this is the case and (44) and (46) are computed with $e^k = 0$ and $r^k = 0$, respectively. Then, taking into account (48), the iterates updates take the form

$$\begin{aligned} x^{k+1} &= \hat{x}^k, \\ z^{k+1} &= \hat{z}^k - s^k. \end{aligned} \tag{51}$$

The decomposition framework outlined above contains some instances of the schemes described in Tseng (1997) and Pennanen (2002), as well as the proximal-based decomposition for convex minimization of Chen and Teboulle (1994) and the proximal alternating directions method for variational inequalities of He et al. (2002).

With some mild assumptions on the operator and the parameters involved, the described algorithm converges globally to a solution of (43), with linear local rate of convergence, (see Lotito et al. 2009, Theorem 2). The use of variable metrics can be useful in some instances for obtaining simpler subproblems, or for the acceleration of convergence (see Parente et al. (2008), He et al. (2002)).

References

Arellano MS (2004) Market power in mixed hydro-thermal electric systems. In: *Econometric society Latin American meetings 211*. Econometric Society

Baldick R (2002) Electricity market equilibrium models: the effect of parameterization. *IEEE Trans Power Syst* 17(4), 1170–1176

Bonnans JF, Gilbert JCh, Lemaréchal C, Sagastizábal C (2006) *Numerical optimization: theoretical and practical aspects*. Springer, Berlin, pp 177–180

Burachik RS, Iusem AN, Svaiter BF (1997) Enlargement of monotone operators with applications to variational inequalities. *Set-Valued Anal* 5:159–180

Burachik RS, Sagastizábal CA, Svaiter BF (1999) ε -Enlargements of maximal monotone operators: Theory and applications. In: Fukushima M, Qi L (eds) *Reformulation—nonsmooth, piecewise smooth, semismooth and smoothing methods*. Kluwer Academic, Norwell, pp 25–44

Chen X, Teboulle M (1994) A proximal-based decomposition method for convex minimization problems. *Math Program* 64:81–101

Hobbs BF, Pang JS (2010) Nash-Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints

- He B, Liao LZ, Han D, Yang H (2002) A new inexact alternating directions method for monotone variational inequalities. *Math Program* 92:103–118
- Lotito PA (2006) Issues in the implementation of the DSD algorithm for the traffic assignment problem. *Eur J Oper Res* 175:1577–1587
- Lotito PA, Parente LA, Solodov MV (2009) A class of variable metric decomposition methods for monotone variational inclusions. *J Convex Anal* 16, 857–880. Special Issue Dedicated to Stephen Simons on the occasion of his 70th birthday
- Maculan N, Santiago CP, Macambira EM, Jardim MHC (2003) An $O(n)$ algorithm for projecting a vector on the intersection of a hyperplane and a box in \mathbb{R}^n . *J Optim Theory Appl* 117:553–574
- Moitre D, Sauchelli V, García G (2005) Optimización Dinámica Binivel de Centrales Hidroeléctricas de bombeo en un Pool Competitivo—Parte I: Modelo y Algoritmo. *Revista IEEE América Latina* 3(2), 62–67
- Parente LA, Lotito PA, Solodov MV (2008) A class of inexact variable metric proximal point algorithms. *SIAM J Optim* 19:240–260
- Pennanen T (2002) A splitting method for composite mappings. *Numer Funct Anal Optim* 23:875–890
- Rivier M, Ventosa M, Ramos A (2001) A generation operation planning model in deregulated electricity markets based on the complementarity problem. In: Ferris MC, Mangasarian OL, Pang J-S (eds.) *Applications and algorithms of complementarity*. Kluwer Academic, Boston, pp 273–298
- Rockafellar RT (1970) On the maximality of sums of nonlinear monotone operators. *Trans Am Math Soc* 149:75–88
- Rockafellar RT, Wets J-B (1997) *Variational analysis*. Springer, New York
- Rubiales A, Mayorano F, Lotito P (2007) Optimización aplicada a la coordinación hidrotérmica del mercado eléctrico argentino. *Mec Comput XXVI*, 3343–3359
- Ruiz C, Conejo AJ, García-Bertrand R (2010) Some analytical results pertaining to Cournot models for short-term electricity markets
- Scott TJ, Read EG (1996) Modelling hydro reservoir operation in a deregulated electricity market. *Int Trans Oper Res* 3:243–253
- Solodov MV (2004) A class of decomposition methods for convex optimization and monotone variational inclusions via the hybrid inexact proximal point framework. *Optim Methods Softw* 19:557–575
- Solodov MV, Svaiter BF (2001) A unified framework for some inexact proximal point algorithms. *Numer Funct Anal Optim* 22:1013–1035
- Tseng P (1997) Alternating projection-proximal methods for convex programming and variational inequalities. *SIAM J Optim* 7:951–965
- Wood AJ, Wollenberg BF (1996) *Power generation, operation, and control*, 2nd edn. Wiley-Interscience/Wiley, New York