Relaxations for Production Planning Problems with Increasing By-products

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Problem Description

- Production process involves desirable & undesirable products.
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- Ratio of by-products to total production increases monotonically.
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- New discrete time MINLP formulation.
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- MIP Approximation & Relaxation schemes.
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- Ratio of by-products to total production increases monotonically.
- Non-convex problem.

**Contributions**
- New discrete time MINLP formulation.
- MIP Approximation & Relaxation schemes.

**Performance evaluation**
Problem Description
The production process creates a mixture of useful products $\mathcal{P}^+$ and by-products $\mathcal{P}^-$. 
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Decisions span a planning horizon $\mathcal{T}$. 

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• Decisions span a planning horizon $\mathcal{T}$. 

• *Discrete* decisions determine the start time of the production process.
The production process creates a mixture of useful products $\mathcal{P}^+$ and by-products $\mathcal{P}^−$.

Decisions span a planning horizon $\mathcal{T}$.

Discrete decisions determine the start time of the production process.

Continuous decisions determine the production profile evaluated by production functions $f(\cdot)$ and $g_p(\cdot)$. 
Production functions

- Production function $f(\cdot)$ is a concave function that determines the maximum production rate as a function of total production.

![Graph showing the relationship between total production and maximum production rate.]
Production functions

- **Production function** \( f(\cdot) \) is a concave function that determines the **maximum** production rate as a function of total production.

- **Product fraction functions** \( g_p(\cdot) \) evolve **monotonically** as a function of the total production.

![Graph showing production function and product fraction functions](image-url)
Production functions

- **Production function** $f(\cdot)$ is a concave function that determines the **maximum** production rate as a function of total production.

- **Product fraction functions** $g_p(\cdot)$ evolve **monotonically** as a function of the total production.

![Graph showing maximum production rate and product fraction as functions of total production](image)
Production function $f(\cdot)$ is a concave function that determines the maximum production rate as a function of total production.

Product fraction functions $g_p(\cdot)$ evolve monotonically as a function of the total production.
Cumulative production $v(t)$ is calculated using production rate $x(t)$

$$v(t) = \int_0^t x(s) \, ds$$
Continuous time formulation

Cumulative production $v(t)$ is calculated using production rate $x(t)$

$$v(t) = \int_{0}^{t} x(s)ds$$

Mixture production rate is limited by production function $f(\cdot)$

$$x(t) \leq f(v(t))$$
Cumulative production $\nu(t)$ is calculated using production rate $x(t)$

$$\nu(t) = \int_{0}^{t} x(s)ds$$

Mixture production rate is limited by production function $f(\cdot)$

$$x(t) \leq f(\nu(t))$$

Product production rates $y_p(t)$ calculated by fraction functions $g_p(\cdot)$

$$y_p(t) = x(t) g_p(\nu(t))$$
Continuous time formulation

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Product production rates $y_p(t)$ calculated by fraction functions $g_p(\cdot)$

$$y_p(t) = x(t) g_p(v(t))$$

Production profiles are active only after the start time $z(t)$

$$v(t) = 0 \quad \forall t < z(t)$$
Discrete time MINLP formulations
Past models have proposed a natural discretization of this continuous time model.

Continuous time formulation

\[
\begin{align*}
\nu(t) &= \int_{0}^{t} x(s) \, ds \\
x(t) &\leq f(\nu(t)) \\
y_p(t) &= x(t) \, g_p(\nu(t)) \\
\nu(t) &= 0 \quad \forall t < z(t) \\
z(t) &\in \mathcal{T} \rightarrow \{0, 1\}, \text{ increasing}
\end{align*}
\]
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\[ v_t \quad \text{Cumulative production up to time period } t \in \mathcal{T}. \]
Past models have proposed a natural discretization of this continuous time model.

Continuous time formulation

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\nu(t) &= \int_0^t x(s) \, ds \\
x(t) &\leq f(\nu(t)) \\
y_p(t) &= x(t) \, g_p(\nu(t)) \\
\nu(t) &= 0 \quad \forall t < z(t) \\
z(t) &: T \to \{0, 1\}, \text{ increasing}
\end{align*}
\]

- \(\nu_t\): Cumulative production up to time period \(t \in T\).
- \(x_t\): Mixture production during time period \(t \in T\).
Discrete time formulations

Past models have proposed a natural discretization of this continuous time model.

Continuous time formulation

\[ F \]

\[
\begin{align*}
\nu(t) &= \int_0^t x(s) \, ds \\
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\( \nu_t \) Cumulative production up to time period \( t \in \mathcal{T} \).

\( x_t \) Mixture production during time period \( t \in \mathcal{T} \).

\( y_{p,t} \) Product \( p \in \mathcal{P} \) production during time period \( t \in \mathcal{T} \).
Discrete time formulations

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Continuous time formulation

\[ v(t) = \int_{0}^{t} x(s) \, ds \]

\[ x(t) \leq f(v(t)) \]

\[ y_{p,t} = x(t) \, g_{p}(v(t)) \]

\[ v(t) = 0 \quad \forall t < z(t) \]

\[ z(t) : \mathcal{T} \rightarrow \{0, 1\} \text{, increasing} \]

\( v_{t} \) Cumulative production up to time period \( t \in \mathcal{T} \).

\( x_{t} \) Mixture production during time period \( t \in \mathcal{T} \).

\( y_{p,t} \) Product \( p \in \mathcal{P} \) production during time period \( t \in \mathcal{T} \).

\( z_{t} \) Facility on/off decision variable.
Past models have proposed a natural discretization of this continuous time model.

**Continuous time formulation** (F)

\[ v(t) = \int_0^t x(s) \, ds \]

\[ x(t) \leq f(v(t)) \]

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Discrete time formulation (F_1)

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\[ v_t = \sum_{s=0}^{t} x_s \]

\[ x_t \leq \Delta_t f(v_{t-1}) \]
Discrete time formulations

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Continuous time formulation \((F)\)

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\[ v_t \leq M \, z_t \]

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How much of product $p$ is produced up to time $t$?

Discrete time formulation ($F_1$)

\begin{align*}
\nu_t &= \sum_{s=0}^{t} x_s \\
x_t &\leq \Delta_t f(v_{t-1}) \\
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How much of product $p$ is produced up to time $t$?

$$w_{p,t} \overset{\text{def}}{=} \sum_{s \leq t} y_{p,s}$$

Discrete time formulation (F1)

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Formulation $F_1$

How much of product $p$ is produced up to time $t$?

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Formulation $F_1$ formulation

How much product is produced up to time $t$?

$$W_{p,t} \overset{\text{def}}{=} \sum_{s \leq t} y_{p,s}$$
$$= \sum_{s \leq t} x_s g_p(v_s - 1)$$

Discrete time formulation ($F_1$)

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Can we do better?

Alternate formulation

Can we calculate exactly how much of product $p \in P$ is produced up to and including time period $t$?

$$w_p, t = \int_{0}^{t} y_p(s) \, ds = \int_{0}^{t} x(s) g_p(v(s)) \, ds = \int v_t g_p(v) \, dv = h_p(v_t)$$

Continuous time formulation \((F)\)

$$v(t) = \int_{0}^{t} x(s) \, ds \
\text{s.t.} \quad x(t) \leq f(v(t)) \
y_p(t) = x(t) g_p(v(t)) \
v(t) = 0 \quad \forall \, t < z(t)$$

$z(t)$ : $T \rightarrow \{0, 1\}$, inc
Can we do better?

Can we calculate \textbf{exactly} how much of product \( p \in \mathcal{P} \) is produced up to and including time period \( t \)?
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w_{p,t} = \int_{0}^{t} y_p(s) \, ds \\
= \int_{0}^{t} x(s) \, g_p(v(s)) \, ds \\
= \int_{0}^{\nu_t} g_p(v) \, dv
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\[
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\]

\[
def = h_{p}(v_{t})
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Alternate formulation

Key Idea

- Integral of a **non-increasing** function is **concave**.
Alternate formulation

Key Idea

- Integral of a non-increasing function is concave.
- Integral of a non-decreasing function is convex.
Alternate formulation

Key Idea

- Integral of a **non-increasing** function is **concave**.
- Integral of a **non-decreasing** function is **convex**.
- Lets deal with $h_p(\cdot)$ instead of $g_p(\cdot)$!

![Graphs showing product fraction and cumulative product](image_url)
Comparing formulations

What have we done so far?

\[ v_t = \sum_{s=0}^{t} x_s \]

\[ x_t \leq \Delta_t f(v_{t-1}) \]

\[ y_{p,t} = x_t \ g_p(v_{t-1}) \]

\[ v_t \leq M \ z_t \]

\[ z_t \geq z_{t-1} \]
Comparing formulations

What have we done so far?

<table>
<thead>
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Comparing formulations

What have we done so far?

**F₁**

\[
\begin{align*}
vt &= \sum_{s=0}^{t} x_s \\
x_t &\leq \Delta t f(v_{t-1}) \\
y_{p,t} &= x_t g_p(v_{t-1}) \\
v_t &\leq M z_t \\
z_t &\geq z_{t-1}
\end{align*}
\]

**F₂**

\[
\begin{align*}
vt &= \sum_{s=0}^{t} x_s \\
x_t &\leq \Delta t f(v_{t-1})
\end{align*}
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Comparing formulations

What have we done so far?

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Comparing formulations

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F_1 & \\
\nu_t &= \sum_{s=0}^{t} x_s \\
x_t &\leq \Delta_t f(\nu_{t-1}) \\
y_{p,t} &= x_t \, g_p(\nu_{t-1}) \\
\nu_t &\leq M \, z_t \\
z_t &\geq z_{t-1}
\end{align*}
\]

\[
\begin{align*}
F_2 & \\
\nu_t &= \sum_{s=0}^{t} x_s \\
x_t &\leq \Delta_t f(\nu_{t-1}) \\
y_{p,t} &= h_p(\nu_t) - h_p(\nu_{t-1}) \\
\nu_t &\leq M \, z_t
\end{align*}
\]
Comparing formulations

What have we done so far?

\[ v_t = \sum_{s=0}^{t} x_s \]

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\[ v_t \leq M \ z_t \]

\[ z_t \geq z_{t-1} \]
Comparing Formulations

Which formulation is better?

\[ F_1 \]

\[ v_t = \sum_{s=0}^{t} x_s \]

\[ x_t \leq \Delta_t f(v_{t-1}) \]

\[ y_{p,t} = x_t g_p(v_{t-1}) \]

\[ v_t \leq M z_t \]

\[ z_t \geq z_{t-1} \]

\[ F_2 \]

\[ v_t = \sum_{s=0}^{t} x_s \]

\[ x_t \leq \Delta_t f(v_{t-1}) \]

\[ y_{p,t} = h_p(v_t) - h_p(v_{t-1}) \]

\[ v_t \leq M z_t \]

\[ z_t \geq z_{t-1} \]

\[ F_2 \] is a more accurate formulation of \( F_1 \) than \( F_1 \).

\( F_2 \) is computationally better because it deals with convex functions while \( F_1 \) deals with bivariate functions.
Comparing Formulations

Which formulation is **better**?

- $F_2$ is a more **accurate** formulation of $F$ than $F_1$.

![Graph showing comparison between $F_1$ and $F_2$ for useful product fraction and by-product fraction.](image)
Comparing Formulations

Which formulation is **better**?

- $F_2$ is a more **accurate** formulation of $F$ than $F_1$.
- $F_2$ is computationally better because it deals with **convex** functions while $F_1$ deals with **bivariate** functions.
MIP Approximations & Relaxations
Mixed Integer Non-Linear Programs (MINLP)

... are slow and hard!
Mixed Integer Non-Linear Programs (MINLP)

... are slow and hard!

Why MINLP is like Cricket

- It goes on forever.
- May not produced a result.
But...the MILP force is here

We only need to approximate or relax univariate convex and concave functions.
Approximate all the nonlinear production functions with piecewise linearizations. [1]
Approximations & Relaxations I

**Piecewise Linear Approximation (PLA)**

Approximate all the nonlinear production functions with piecewise linearizations.[1]

- **Pros**
  - ‘Close’ to a feasible solution of the MINLP formulation.

---

**Total production** ($vt$)

**Maximum production rate** ($f$)

**Cumulative useful product** ($g_p^+$)

**Cumulative by-product** ($h_p^-$)

---

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Approximations & Relaxations I

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- **Pros**
  - ‘Close’ to a feasible solution of the MINLP formulation.

- **Cons**
  - Introduces additional SOS2 variables to branch on.

---

**Graphs:**
- Maximum production rate ($f$)
- Cumulative useful product ($g_u$)
- Cumulative by-product ($h_p$)

**Equations:**
- Total production ($v_t$)
- Maximum production rate ($f$)
- Total production ($v_t$)
- Cumulative useful product ($g_u$)
- Total production ($v_t$)
- Cumulative by-product ($h_p$)
Approximations & Relaxations I

**Piecewise Linear Approximation (PLA)**

Approximate all the nonlinear production functions with piecewise linearizations.[1]

- **Pros**
  - ‘Close’ to a feasible solution of the MINLP formulation.

- **Cons**
  - Introduces additional SOS2 variables to branch on.
  - NOT a relaxation of the original formulation.

---

**Srikrishna Sridhar, Jeff Linderoth, James Leudtke**

SILO Seminars: Feb 1, 2012
Specially ordered sets (SOS)

Total production \((v_t)\)

Cumulative useful product \((g_{p^+})\)

Approximating \(f(v_t)\)

\[ f(v_t) \approx \sum_{o \in O} \lambda_{t,o} f(a_o) \]
Specially ordered sets (SOS)

Cumulative useful product \((g_{p+})\)

Total production \((v_t)\)

\[ a_0, f(a_0) \]
\[ a_1, f(a_1) \]
\[ a_2, f(a_2) \]
\[ a_3, f(a_3) \]

Cumulative useful product

Approximating \(f(v_t)\)

\[ f(v_t) \approx \sum_{o \in \mathcal{O}} \lambda_{t,o} f(a_o) \]

\[ 1 = \sum_{o \in \mathcal{O}} \lambda_{t,o} \]
Specially ordered sets (SOS)

Total production \((v_t)\)
\[
\begin{align*}
[a_0, f(a_0)] & \quad [a_1, f(a_1)] & \quad [a_2, f(a_2)] & \quad [a_3, f(a_3)] \\
\end{align*}
\]

Cumulative useful product \((g_p^+))\)

\[
\sum_{o \in O} \lambda_{t,o} f(a_o)
\]

Structure: Only two adjacent non zeros.

Approximating \(f(v_t)\)

\[
f(v_t) \approx \sum_{o \in O} \lambda_{t,o} f(a_o)
\]

\[
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\]
Specially ordered sets (SOS)

Cumulative useful product \((g_{p^+})\)

Total production \((vt)\)

Approximating \(f(v_t)\)

\[
f(v_t) \approx \sum_{o \in O} \lambda_{t,o} f(a_o)
\]

\[
1 = \sum_{o \in O} \lambda_{t,o}
\]

**Structure**: Only two adjacent non zeros.

\[
\{\lambda_{t,o} | o \in O\} \in \text{SOS2}
\]
Piecewise Linear Approximation (PLA)

### $F_2$

\[ \nu_t = \sum_{s=0}^{t} x_s \]
Piecewise Linear Approximation (PLA)

\[ v_t = \sum_{s=0}^{t} x_s \]
\[ x_t \leq \Delta_t f(v_{t-1}) \]

Piecewise Linear Approximation (PLA)

\[ v_t = \sum_{s=0}^{t} x_s \]
\[ v_t = \sum_{o \in O} B_{o} \lambda_{t,o} \]
\[ x_t \leq \Delta_t \sum_{o \in O} F_{o} \lambda_{t,o} \]
### Piecewise Linear Approximation (PLA)

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_2$</td>
<td>$v_t = \sum_{s=0}^{t} x_s$</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>$y_{p,t} = h_p(v_t) - h_p(v_{t-1})$</td>
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</tbody>
</table>
\[ v_t = \sum_{s=0}^{t} x_s \]
\[ x_t \leq \Delta_t f(v_{t-1}) \]
\[ y_{p,t} = h_p(v_t) - h_p(v_{t-1}) \]
\[ v_t \leq M z_t \]
\[ z_t \geq z_{t-1} \]
Secant Relaxation (1-SEC)

Relax all the nonlinear production functions using inner and outer approximations.[2]
Secant Relaxation (1-SEC)

Relax all the nonlinear production functions using inner and outer approximations.[2]

- **Pros**
  - Relaxation of the original formulation.

- **Cons**
  - May not be 'close' to a feasible solution of the MINLP formulation.
Secant Relaxation (1-SEC)

Relax all the nonlinear production functions using inner and outer approximations.[2]

- **Pros**
  - Relaxation of the original formulation.
  - Does NOT introduce additional SOS2 variables.

- **Cons**
  - May not be ‘close’ to a feasible solution of the MINLP formulation.

\[
\begin{align*}
\text{Total production} & : \sum_j v_{tj} \\
\text{Maximum production rate} & : f \\
\text{Total production} & : \sum_j v_{tj} \\
\text{Cumulative useful product} & : h_{p+} \\
\text{Total production} & : \sum_j v_{tj} \\
\text{Cumulative by-product} & : h_{p-}
\end{align*}
\]
Secant Relaxation (1-SEC)

\[ F_2 \]

\[ v_t = \sum_{s=0}^{t} x_s \]

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\[ v_t = \sum_{s=0}^{t} x_s \]

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\[ y_{p,t} = w_{p,t} - w_{p,t-1} \]

\[ w_{p,t} = \sum_{o \in O} \hat{H}_{p,o} \lambda_{t,o} \]

\[ z_t \geq z_{t-1} \]

\[ z_t = \sum_{o \in O} \lambda_{t,o} \{ \lambda_{t,o} \mid o \in O \} \in \text{SOS2} \]
**Multiple Secant Relaxation (k-SEC)**

*Relax* all the nonlinear production functions using inner and outer approximations but use multiple secants instead of a just a single one.

**Pros**
- "Close" to a feasible solution of the MINLP formulation.
- Relaxation of the original formulation.

**Cons**
- Introduces additional SOS2 variables to branch on.
Multiple Secant Relaxation (k-SEC)

Relax all the nonlinear production functions using inner and outer approximations but use multiple secants instead of just a single one.

- **Pros**
  - ‘Close’ to a feasible solution of the MINLP formulation.
  - Relaxation of the original formulation.

**Diagram:**

- Cumulative useful product ($h_p^-$)
- Cumulative by-product ($h_p^+$)

**Equation:**

\[
\text{Total production } (v_t) = \text{Cumulative useful product } (h_p^-) + \text{Cumulative by-product } (h_p^+)
\]
Multiple Secant Relaxation (k-SEC)

Relax all the nonlinear production functions using inner and outer approximations but use multiple secants instead of a just a single one.

- **Pros**
  - ‘Close’ to a feasible solution of the MINLP formulation.
  - Relaxation of the original formulation.

- **Cons**

![Graphs showing Total production and Cumulative useful product vs Total production and Cumulative by-product](image-url)
Multiple Secant Relaxation (k-SEC)

Relax all the nonlinear production functions using inner and outer approximations but use multiple secants instead of a just a single one.

- **Pros**
  - ‘Close’ to a feasible solution of the MINLP formulation.
  - **Relaxation** of the original formulation.

- **Cons**
  - Introduces additional SOS2 variables to branch on.

**Formulas**

\[
\text{Total production} (v_t) \quad \text{Cumulative useful product} \ (h_{p-}) \quad \text{Cumulative by-product} \ (h_{p+})
\]
Multiple Secant Relaxation (k-SEC)

\[ v_t = \sum_{s=0}^{t} x_s \]

\[ v_t = \sum_{o \in O} \hat{B}_o \lambda_{t,o} \]

\[ x_t \leq \Delta_t \sum_{o \in O} \hat{F}_o \lambda_{t,o} \]

\[ y_{p,t} = w_{p,t} - w_{p,t-1} \]

\[ \sum_{o \in O} H_{p,o} \lambda_{t,o} \leq w_{p,t} \leq \sum_{o \in O} \hat{H}_{p,o} \lambda_{t,o} \quad \forall p \in P^+ \]

\[ \sum_{o \in O} H_{p,o} \lambda_{t,o} \geq w_{p,t} \geq \sum_{o \in O} \hat{H}_{p,o} \lambda_{t,o} \quad \forall p \in P^- \]

\[ z_t \geq z_{t-1} \]

\[ z_t = \sum_{o \in O} \lambda_{t,o} \]
Performance Evaluation
Experiments

Goals

- Impact on formulation **accuracy** in going from $F_1$ to $F_2$
- Impact in **solution time** in going from $F_1$ to $F_2$ as solved by our models.

Sample Application

**Transportation problem** with production facilities manufacturing products for customers.
Transportation problem with production facilities $\mathcal{I}$ manufacturing products $\mathcal{P}^+$ for customers $\mathcal{J}$.
- Transportation problem with production facilities $\mathcal{I}$ manufacturing products $\mathcal{P}^+$ for customers $\mathcal{J}$.
- Demand made by customers are known a priori.
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Facility operations follow known production functions.
Transportation problem with production facilities $\mathcal{I}$ manufacturing products $\mathcal{P}^+$ for customers $\mathcal{J}$.

Demand made by customers are known a priori.

Facility operations follow known production functions.

Facilities incur fixed, operating, transportation and penalty costs.
Comparing formulations: Small instances

Table: Comparing solution quality of the two different MINLP formulations $F_1$ and $F_2$ using BARON

| $|\mathcal{I}|$ | $|\mathcal{T}|$ | $|\mathcal{P}|$ | Formulation | Solution difference $\Delta y^*_{i,p,t}$ (Range: 0 – 30) |
|------|------|------|---------------|----------------|
| 5    | 5    | 2    | $F_1$ | $F_2$ | Maximum ($\forall i, p, t$) | Average ($\forall i, p, t$) |
|      |      |      | Solution Bound | Best $F_1$ Feasible Solution | Repaired $F_1$ Solution Bound | Best $F_2$ Feasible Solution |
| 5    | 5    | 2    | 0.171 | 0.200 | 0.272 | 0.208 | 0.219 | 5.17 | 0.47 |

![Graphs showing Useful product fraction ($g_{p+}$) and By-product fraction ($g_{p-}$) for $F_1$ and $F_2$.]
Comparing formulations: Small instances

**Table:** Comparing solution quality of the two different MINLP formulations $F_1$ and $F_2$ using BARON

| $|I|$ | $|T|$ | $|P|$ | Formulation | Solution difference $\Delta y_{i,p,t}^* (\text{Range } : 0 - 30)$ |
|-----|-----|-----|------------|--------------------------------------------------|
|     |     |     | $F_1$ | $F_2$ | $\Delta y_{i,p,t}^*$ |
|     |     |     | Solution Bound | Best $F_1$ Feasible Solution | Repaired $F_1$ Solution | Solution Bound | Best $F_2$ Feasible Solution | Maximum ($\forall i, p, t$) | Average ($\forall i, p, t$) |
| 5   | 5   | 2   | 0.171 | 0.200 | **0.272** | 0.208 | 0.219 | 5.17 | 0.47 |

**Diagram:**

- **Useful product fraction ($g_{p+}$):**
  - $F_1$ and $F_2$

- **By-product fraction ($g_{p-}$):**
  - $F_1$ and $F_2$
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| 5 | 5 | 2 | F$_1$ | F$_2$ | Maximum $(\forall i, p, t)$ | Average $(\forall i, p, t)$ |
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![Useful product fraction ($g_p^+$)](image1)

![By-product fraction ($g_p^-$)](image2)
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![Graphs showing useful product fraction and by-product fraction](image-url)
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| 5 | 5 | 2 | 0.171 | 0.200 | 0.272 | 0.208 | 0.219 | 5.17 | 0.47 |
| 5 | 5 | 2 | 0.150 | 0.177 | 0.228 | 0.181 | 0.186 | 5.04 | 0.33 |
| 5 | 5 | 2 | 0.157 | 0.175 | 0.243 | 0.190 | 0.198 | 4.68 | 0.40 |
| 5 | 10 | 2 | 0.255 | 0.369 | 0.381 | 0.325 | 0.340 | 0.41 | 0.06 |
| 5 | 10 | 2 | 0.256 | 0.358 | 0.388 | 0.324 | 0.341 | 1.33 | 0.12 |
| 5 | 10 | 2 | 0.303 | 0.377 | 0.464 | 0.385 | 0.399 | 3.14 | 0.34 |
| 10 | 10 | 2 | 0.357 | 0.607 | 0.770 | 0.637 | 0.670 | 4.49 | 0.32 |
| 10 | 10 | 2 | 0.507 | 0.784 | 0.954 | 0.797 | 0.820 | 3.84 | 0.32 |
| 10 | 10 | 2 | 0.377 | 0.692 | 0.754 | 0.645 | 0.675 | 2.60 | 0.13 |
| 15 | 10 | 2 | 0.656 | 1.085 | 1.308 | 1.100 | 1.141 | 3.84 | 0.30 |
| 15 | 10 | 2 | 0.540 | 0.960 | 1.053 | 0.903 | 0.945 | 2.16 | 0.14 |
| 15 | 10 | 2 | 0.552 | 1.033 | 1.090 | 0.901 | 0.940 | 1.01 | 0.08 |

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Comparing MIP schemes: Large instances

**Table:** Comparing gaps of $F_1$ (with BARON) with MIP formulations (with Gurobi) of $F_2$ on large instances with more than 200 binary variables.

| $|I|$ | $|T|$ | $|P|$ | Bounds ( $F_2$ ) | Best $F_2$ feasible solution | Time (sec) / [Optimality gap (%)] |
|-----|-----|-----|----------------|-----------------|---------------------------|
|     |     |     | 1-SEC k-SEC   | PLA 1-SEC k-SEC | $F_1$ PLA 1-SEC k-SEC |
| 15  | 15  | 2   | 1394.13 1392.1| 1412.07 1417.74 1416.98 | [49.5] [0.86] [0.77] [1.01] |

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| |I| |T| |P| |
|---|---|---|---|---|---|---|---|---|---|
|Bounds (F_2)|Best F_2 feasible solution|Time (sec) / [Optimality gap (%)]| |
|1-SEC | k-SEC| PLA | 1-SEC | k-SEC| F_1 | PLA | 1-SEC | k-SEC|
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|-----|-----|-----|--------|----------------|--------------------------------|
|     |     |     | 1-SEC  | 1-SEC | PLA | 1-SEC | k-SEC | $F_1$ | PLA | 1-SEC | k-SEC |
| 15  | 15  | 2   | 1394.13| 1392.1| 1412.07| 1417.74| 1416.98| [49.5] | [0.86] | [0.77] | [1.01] |

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|     |     |     | 1-SEC     | k-SEC           | PLA                | 1-SEC       | k-SEC       | $F_1$  | PLA     | 1-SEC       | k-SEC       |
| 15  | 15  | 2   | 1394.13   | 1392.1          | 1412.07            | 1417.74     | 1416.98    | [49.5] | [0.86]  | [0.77]    | [1.01]    |
| 15  | 15  | 4   | 1391.38   | 1385.82         | 1432.00            | 1431.74     | 1436.59    | [50.2] | [1.60]  | [1.41]    | [1.62]    |
| 15  | 15  | 6   | 1283.03   | 1271.9          | 1326.2             | 1335.69     | 1330.13    | [81.2] | [1.97]  | [1.89]    | [2.23]    |
| 15  | 20  | 2   | 1465.65   | 1465.4          | 1500.92            | 1510.79     | 1498.87    | [53.0] | [1.90]  | [1.67]    | [1.72]    |
| 15  | 20  | 4   | 1573.95   | 1571.02         | 1663.04            | 1665.75     | 1691.03    | [63.9] | [2.56]  | [2.39]    | [2.86]    |
| 15  | 20  | 6   | 1614.51   | 1608.73         | 1691.04            | 1691.4      | 1696.03    | [83.1] | [3.12]  | [2.71]    | [3.09]    |
| 20  | 20  | 2   | 2185.07   | 2184.68         | 2245.19            | 2247.45     | 2254.25    | [58.2] | [1.93]  | [1.98]    | [2.14]    |
| 20  | 20  | 2   | 1865.12   | 1863.33         | 1906.58            | 1906.93     | 1905.17    | [49.1] | [1.24]  | [1.46]    | [1.57]    |
| 20  | 20  | 6   | 2058.69   | 2042.32         | 2163.22            | 2183.31     | 2185.59    | -      | [3.05]  | [3.15]    | [3.60]    |
| 25  | 25  | 2   | 3274.29   | 3270.23         | 3383.73            | 3381.22     | 3383.53    | [83.0] | [3.93]  | [3.60]    | [3.96]    |
| 25  | 25  | 4   | 3222.66   | 3223.06         | 3417.42            | 3413.46     | 3437.34    | [83.2] | [32.2]  | [22.9]    | [33.6]    |
| 25  | 25  | 6   | 2973.45   | 2963.5          | 4465.04            | 3919.11     | 4510.94    | [83.2] | [32.2]  | [22.9]    | [33.6]    |
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| 15  | 15  | 4   | 1391.38 1385.82 | 1432.00 1431.74 1436.59 | [50.2] [1.60] [1.41] [1.62] |
| 15  | 15  | 6   | 1283.03 1271.9 | 1326.2 1335.69 1330.13 | [81.2] [1.97] [1.89] [2.23] |
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| 25  | 25  | 2   | 3274.29 3270.23 | 3383.73 3381.22 3383.53 | - [2.28] [2.35] [2.63] |
| 25  | 25  | 4   | 3222.66 3223.06 | 3417.42 3413.46 3437.34 | [83.0] [3.93] [3.60] [3.96] |
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| 15  | 20  | 2   | 1465.65 1465.4 | 1500.92 1510.79 1498.87 | [53.0] [1.90] [1.67] [1.72] |
| 15  | 20  | 4   | 1573.95 1571.02 | 1663.04 1665.75 1691.03 | [63.9] [2.56] [2.39] [2.86] |
| 15  | 20  | 6   | 1614.51 1608.73 | 1691.04 1691.4 1696.03 | [83.1] [3.12] [2.71] [3.09] |
| 20  | 20  | 2   | 2185.07 2184.68 | 2245.19 2247.45 2254.25 | [58.2] [1.93] [1.98] [2.14] |
| 20  | 20  | 2   | 1865.12 1863.33 | 1906.58 1906.93 1905.17 | [49.1] [1.24] [1.46] [1.57] |
| 20  | 20  | 6   | 2058.69 2042.32 | 2163.22 2183.31 2185.59 | - [3.05] [3.15] [3.60] |
| 25  | 25  | 2   | 3274.29 3270.23 | 3383.73 3381.22 3383.53 | - [2.28] [2.35] [2.63] |
| 25  | 25  | 4   | 3222.66 3223.06 | 3417.42 3413.46 3437.34 | [83.0] [3.93] [3.60] [3.96] |
| 25  | 25  | 6   | 2973.45 2963.5 | 4465.04 3919.11 4510.94 | [83.2] [32.2] [22.9] [33.6] |
Table: Comparing gaps of $F_1$ (with BARON) with MIP formulations (with Gurobi) of $F_2$ on large instances with more than 200 binary variables.

| $|I|$ | $|T|$ | $|P|$ | $\text{Bounds ( } F_2 \text{ )}$ | Best $F_2$ feasible solution | Time (sec) / [Optimality gap (%)] |
|-----|-----|-----|-----------------|-----------------|-----------------------------|
|     |     |     | 1-SEC k-SEC PLA 1-SEC k-SEC | F_1 PLA 1-SEC k-SEC |
| 15  | 15  | 2   | 1394.13 1392.1 | 1412.07 1417.74 1416.98 | [49.5] [0.86] [0.77] [1.01] |
| 15  | 15  | 4   | 1391.38 1385.82 | 1432.00 1431.74 1436.59 | [50.2] [1.60] [1.41] [1.62] |
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