1 Lecture Outline

The following lecture covers Section 3.3 and 3.4 from the textbook [1]

- Review of the Phase II Procedure
- Review Example 3-3-1 to review unboundedness.
- So far, we always assume that $x_1, x_2...x_n = 0$ is feasible but it might not always be the case.
- Introduce the idea of adding an artificial variable to soak up any infeasibility that we might get by setting $x_1, x_2...x_n = 0$.
- State the phase I problem of minimizing $x_0$.
- If phase I terminates with $x_0 > 0$, the problem is infeasible.
- If phase I terminates with $x_0 = 0$, the problem is feasible and the optimal tableau for the Phase I problem is the starting tableau of the Phase II procedure.
- Go through setup on p.61 and Algorithm 3.2 on p.62.
- Do example 3-4-1.
- Do another made-up example that terminates in infeasibility.
  Constraints
  $\begin{align*}
  x_1 + 2x_2 - x_3 &\geq 5 \\
  -x_1 - 3x_2 + x_3 &\geq 10.
  \end{align*}$
  (Ends with a phase-I optimal objective of 15/2.)

2 Review Phase II

The simplex method is usually comprised of two phases. Phase I finds a feasible point while Phase II starts out with a feasible tableau and applies repeated pivots to move it to an optimal tableau.

2.1 Phase II Algorithm

1. Construct an initial tableau. (Make sure that the problem is in standard form before doing so)

2. If this initial tableau is not feasible, apply Phase I to generate a feasible tableau. For now let us assume that $x_N = 0$ is feasible.

3. Use the pricing scheme to determine a pivot column $s$ which has a negative element in the bottom row. If none exists, the tableau is optimal/unbounded.

4. Use the ratio test to determine the pivot row $r$ such that

$$\min_i \left\{-\frac{h_i}{H_{is}} | H_{is} < 0 \right\}.$$

If there exists no row $i$ such that $H_{is} < 0$, then tableau is unbounded.

5. Exchange $x_{B(r)}$ and $x_{N(s)}$ using Jordan exchange.
2.2 Illustrative Examples

We illustrate example 3-3-1 of the textbook using MATLAB.

\[
\begin{align*}
\min_x & \quad -2x_1 - 3x_2 + x_3 \\
\text{subject to} & \quad x_1 + x_2 + x_3 \geq -3 \\
& \quad -x_1 + x_2 - x_3 \geq -4 \\
& \quad x_1 - x_2 - 2x_3 \geq -1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Refer to handout to enumerate the steps involved.

In the final tableau, we can see that there is a pivot column but no pivot row. Hence the problem is unbounded. We can read the solution off the tableau as:

\[
x_4(\lambda) = 2\lambda + 4 \quad x_5(\lambda) = 5 \quad x_2(\lambda) = \lambda + 1,
\]

The solution to the problem, i.e the original three variables of the problem can be written as

\[
x(\lambda) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\] (1)

Hence, we can set the vector \( u = (1, 0, 0)^t \) and \( v = (1, 1, 0)^t \) to obtain the direction of unboundedness. The objective function \( z(\lambda) = -5\lambda - 3 \).

3 Simple Phase I

In all the problems we have studied so far, the linear program has been stated in the standard form and the tableau constructed from the initial problem data has been feasible. Essentially this means that the origin i.e \( x_1, x_2, ..., x_n = 0 \) is a feasible point for the LP. The Phase 1 procedure is the process of finding a feasible point for the LP so that we can proceed with the Phase 2 procedure.

Goal of Phase 1: Identify if a linear program has a feasible point. If it does, it should be easy to construct a feasible tableau resulting out of the Phase 1 procedure. If there is no feasible point, we should be able to say that with certainty that the problem is infeasible.

Example
3.1 Phase 1 Problem

Let us consider the following problem (P).

\[
\begin{align*}
\min_x & \quad 2x_1 + 3x_2 + x_1 \\
\text{subject to} & \quad x_1 + x_2 + x_3 \geq -1 \\
& \quad x_1 + x_3 \geq 5 \\
& \quad x_1 + 2x_2 \geq 7 \\
& \quad x \geq 0
\end{align*}
\]

The origin for this problem is infeasible \((x_1, x_2, x_3 = 0)\). So we need to add an artificial variable that soaks up the infeasibility. In this particular example, let us call that artificial variable \(x_0\). Add the artificial variable to all the constraints where the origin is infeasible. Hence we get the following

\[
\begin{align*}
\min_{x_0, x} & \quad x_0 \\
\text{subject to} & \quad x_1 + x_2 + x_3 \geq -1 \\
& \quad x_1 + x_3 + x_0 \geq 5 \\
& \quad x_1 + 2x_2 + x_0 \geq 7 \\
& \quad x, x_0 \geq 0
\end{align*}
\]

Note that we could add \(x_0\) to all the constraints but it’s a wasteful exercise to do so. The new formulation with the artificial variable asks the following question.

**What is the the minimum value of \(x_0\) for which the original problem is feasible?**

Note that this problem always has a feasible solution. We can do so by simple setting \(x_0\) to the maximum value of the right hand side. In this particular example, \((x_1, x_2, x_3 = 0, x_0 = 7)\) is a feasible point.

**Important Note**

If this phase 1 problem has an optimal value 0, then the original problem is feasible. If it has an optimal value of something strictly greater than zero, then the original problem is strictly infeasible.

3.1 Phase 1 Problem
Let us now formally define the Phase 1 problem. (Let us call this P1).

\[
\begin{align*}
\min_{x_0, x} & \quad x_0 \\
\text{subject to} & \quad x_{n+i} = A_i x - b_i + x_0 & b_i > 0 \\
& \quad x_{n+i} = A_i x - b_i & b_i \leq 0 \\
& \quad x, x_0 \geq 0
\end{align*}
\]

Note that the above formulation has been written in canonical form. P1 always has a feasible solution, which is obtainable by setting \( x_0 = \max(\max_i b_i, 0) \). \( x_i = 0 \quad \forall i \in N = \{1, 2...n\} \). This fact can be formally proved as follows.

\[
\begin{align*}
\text{if } b_i \leq 0 \Rightarrow x_{n+i} &= A_i x - b_i = -b_i \geq 0 \\
\text{if } b_i \geq 0 \Rightarrow x_{n+i} &= A_i x - b_i + x_0 = -b_i + \max_i b_i \geq 0
\end{align*}
\]

Using the above two equations, we can assert that every \( x_b \geq 0 \), \( \forall b \in B = \{n + 1, n + 2,...,n + m\} \)

Since the phase one problem P1 is always feasible. We can always apply the simplex algorithm on P1 to find out the minimum \( x_0 \) for which the original problem P is feasible.

**Claim 1** The original problem P has a feasible point iff P1 has an optimal solution of \( x_0 = 0 \).

Claim 1 is easy to prove in one direction because if there is a feasible point \( \bar{x} \) for P, then clearly, \((\bar{x}, 0)\) is feasible point for P1. Since \( x_0 \geq 0 \) and the objective \( z_0 = x_0 \), the point \((\bar{x}, 0)\) must be the optimal solution for P1.

For the other direction: Let \((\tilde{x}, 0)\) be the optimal solution for P1. Since \((\tilde{x}, 0)\) is optimal for P1, it is also feasible for P1. Therefore

\[
\begin{align*}
x_{n+i} &= A_i \tilde{x} - b_i + x_0 & b_i > 0 \\
x_{n+i} &= A_i \tilde{x} - b_i & b_i \leq 0 \\
\tilde{x}, x_0 &\geq 0
\end{align*}
\]

Hence \( \tilde{x} \) is feasible for P. This also means that the optimal tableau for the Phase 1 Procedure is the starting point of the Phase 2 procedure.

### 3.2 Phase 1 Algorithm

We summarize the algorithm for Phase 1 in the following section (Algorithm 3.2 of the textbook)

1. If \( b \leq 0 \), then \( x_B = -b \), \( x_N = 0 \) is a feasible for the initial tableau and no Phase 1 is needed. Skip to Phase 2.
2. If \( b \not\leq 0 \), then introduce artificial variable \( x_0 \) (and a new objective) for the Phase 1 problem. Set up the initial tableau of the Phase 1 Problem.
3. Perform a **special pivot** of the \( x_0 \) column, which corresponds to the action of setting \( x_0 = \max_i b_i \). This is done by pivoting on the column corresponding to \( x_0 \) with the pivot row as the most negative entry in the last column of the problem.
4. Apply simplex algorithm to solve the Phase 1 problem until an optimal tableau is reached. If the optimal solution $x_0^* > 0$, then the problem is infeasible. STOP. Else proceed to the next step.

5. Strike off the columns and rows corresponding to the Phase 1 problem and proceed to Phase 2.

### 3.3 Illustrative Examples

We illustrate example 3-4-1 of the textbook using MATLAB.

\[
\begin{align*}
\min_{x} & \quad 4x_1 + 5x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq -1 \\
& \quad x_1 + 2x_2 \geq 1 \\
& \quad 4x_1 + 2x_2 \geq 8 \\
& \quad -x_1 - x_2 \geq -3 \\
& \quad -x_1 + x_2 \geq 1 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Refer to handout to enumerate the steps involved.

#### Infeasibility Illustration

\[
\begin{align*}
\min_{x} & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad x_1 + 2x_2 - x_3 \geq 5 \\
& \quad -x_1 - 3x_2 + x_3 \geq 10 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Refer to handout to enumerate the steps involved.

### References