1 Lecture Outline

The following lecture covers Section 3.2 and 3.5 from the textbook [1]

- Define vertex (Def 3.2.1).
- Discuss how each vertex is defined by a set \( N \subset \{1, 2, \ldots, n + m\} \) with \( n \) elements.
- Degenerate vertices are those than can be expressed by more than one \( N \). Not all sets of \( n \) intersections of linear functions \( x_j \) define a vertex (because of possible infeasibility). Illustrate using example Fig 3.1. Some of the intersections occur outside the feasible region.
- Describe Theorem 3.2.2. Introduce the form \( Ax = b \). Say that \( N \subset \{1, 2, \ldots, m + n\} \) corresponds to a vertex if all these conditions hold: (i) it has \( n \) elements (ii) \( A_B \) is invertible; (iii) \( \mathbf{A}_B^{-1} b \geq 0 \).
- Work through the example in the case of
  \[
  x_3 = x_1 + x_2 - 1 \geq 0, \quad x_4 = 2x_1 - x_2 - 3 \geq 0,
  \]
  Do the cases \( B = \{3, 4\}, N = \{1, 2\} \) (easy) and \( B = \{2, 3\}, N = \{1, 4\} \).
- Repeat definition of a degenerate vertex: It can be defined by more than one index set \( N \). (A nongenerate vertex is defined by a unique set \( N \).)
- Multiple solutions: Show that if any two points are a solution, then all points on the line joining them are also solutions.
- Do example 3-3-3 on the board. Show how we can move from one solution to the next by selecting columns with reduced cost \( c_s = 0 \), and applying the ratio test to find a pivot row. Objective doesn’t change, but we move to another vertex solution.
- Define nondegenerate tableau. They correspond to nondenerate vertices.
- Prove Th 3.5.2: finite termination in the nondegenerate case.
2 Vertex

The idea of vertices is important to the geometric interpretation of the simplex method. Let us consider the following feasible region:

\[ S := \{ x \in \mathbb{R}^n | Ax \geq b, x \geq 0 \} \]  
\[ (1) \]

A vertex is a point in the feasible region represented by the intersection of \( n \) hyperplanes defined by

\[ x_i = 0 \quad \forall i \in N \]

where \( N \) is the set of nonbasic variables for a feasible tableau.

**Definition 3.2.1:** For the feasible region of \( S \) in (1), let us define

\[ x_{n+i} := A_i x - b_i, i \in \{1, 2 \ldots m\} \]

A vertex of the set \( S \) is defined as any point \((x_1, x_2 \ldots x_n) \in S \) which satisfies

\[ x_n = 0 \]

where \( N \) is a subset of \( \{1, 2 \ldots n + m\} \) containing \( n \) elements such that the linear functions defined by \( x_j, j \in N \) are linearly independent.

**Notes:**

- It is important for the \( n \) functions to be linearly dependent. If not, the equation \( x_n = 0 \) has either zero solutions or infinitely many solutions.
- Linear functions defined by \( x_j, j \in N \) refers to the variable \( x_j \) being expressed in terms of the original \( n \) variables \( x_1, x_2, \ldots, x_n \).  
- Constants on the right-hand side should not be included in the determination of linear independence.
- Degenerate vertices are those than can be expressed by more than one \( N \).
- Not all sets of \( n \) functions \( x_j \) define a vertex (because of possible infeasibility). In these cases, some of the intersections occur outside the feasible region.

**Illustration** Example 3-1-1

\[ \min 3x_1 + x_2 \]
\[ x_1 + 2x_2 \geq -1 \]
\[ 2x_1 + x_2 \geq 0 \]
\[ x_1 - x_2 \geq -1 \]
\[ x_1 - 4x_2 \geq -13 \]
\[ -4x_1 + x_2 \geq -23 \]
\[ x_1, x_2 \geq 0 \]

**Theorem 3.2.2:** If \( \bar{x} \) is a vertex of \( S \) with corresponding index set \( N \). Then if we define

\[ A := [A - I] \quad B := \{1, 2, \ldots, n + m\} \backslash N \]
\[ (2) \]
then the vertex $\bar{x}$ satisfies
\[ A_B x_B + A_N x_n = b \quad x_B \geq 0 \quad x_n = 0 \quad (3) \]
and $A_B$ is invertible (the basis matrix).

Also, $\bar{x}$ can be represented by a feasible tableau ($h \geq 0$) of the form
\[
\begin{array}{c|cc}
  \bar{x}_B & x_n & 1 \\
  z & H & h \\
  & c' & \alpha
\end{array}
\]

Notes:
- $A_B$ is an $(m \times m)$ matrix while $A_N$ is an $(m \times n)$ matrix.

**Proof:** Let us start by assuming $A_B^{-1}$ exists. Now, from the definition of vertex, we have that $\bar{x}$ satisfies.
\[ A_B \bar{x}_B + A_N \bar{x}_N = b \quad \bar{x}_B \geq 0 \quad \bar{x}_N = 0 \]
Premultiplying by $A_B^{-1}$, we have
\[ \bar{x}_B = -A_B^{-1} A_N \bar{x}_N + A_B^{-1} b \]
which can be written as a tableau with $H = -A_B^{-1} A_N$ and $h = A_B^{-1} b$.

Now, if $p'x$ denotes the objective function, then we can rewrite this as
\[ z = p' x = p'_B \bar{x}_B + p'_N x_n = p'_B (-A_B^{-1} A_N \bar{x}_N + A_B^{-1} b) + p'_N x_n = (p_N - p'_B A_B^{-1} A_N) x_N + p'_B A_B^{-1} b \]
Hence, we can write $c' = p'_N - p'_B A_B^{-1} A_N$ and $\alpha = p'_B A_B^{-1} b$.

**Question** For a given vertex $\bar{x}$. Why is the tableau defined below feasible?
\[
\begin{array}{c|cc}
  \bar{x}_B & \bar{x}_N & 1 \\
  z & -A_B^{-1} A_N & A_B^{-1} b \\
  & p'_N - p'_B A_B^{-1} A_N & p'_B A_B^{-1} b
\end{array}
\]
By the definition of a vertex, $\bar{x}_B \geq 0$ and $\bar{x}_N = 0$. Hence,
\[ \bar{x}_B = -A_B^{-1} A_N \bar{x}_N + A_B^{-1} b \geq 0 \]
\[ = A_B^{-1} b \geq 0 \quad \text{(since } \bar{x}_N = 0) \]
Examples

Let us consider the following problem

\[
\min_x x_1 + x_2 \\
\text{subject to} \\
x_3 = x_1 + x_2 \geq 1 \\
x_4 = 2x_1 - x_2 \geq 3
\]

For the above problem, we have

\[
A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}, \\
b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\
p' = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}
\]

(a) Let us try to work out the tableau for \(B = \{3, 4\}, N = \{1, 2\}\). We have

\[
A_B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\
A_N = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}
\]

We can compute the tableau using the formulae above:

\[
H = -A_B^{-1}A_N = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \\
h = A_B^{-1}b = [-1 & -3] \\
c' = p'_N - p'_B A_B^{-1} A_N = [1 & 1] \\
\alpha = p'_B A_B^{-1} b = 0
\]

(b) We can also work out a more complicated example with \(B = \{1, 3\}, N = \{2, 4\}\). We have

\[
A_B = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \\
A_N = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}
\]

We can compute the tableau using the formulae above:

\[
H = -A_B^{-1}A_N = \begin{bmatrix} -0.5 & -0.5 \\ -1.5 & -0.5 \end{bmatrix}, \\
h = A_B^{-1}b = [1.5 & 0.5] \\
c' = p'_N - p'_B A_B^{-1} A_N = [1.5 & 0.5] \\
\alpha = p'_B A_B^{-1} b = 1.5
\]

3 Degeneracy

In Definition 3.2.1, we observe that two different values of \(N\) can produce the same vertex. A vertex that is uniquely defined by a single value of the index set \(N\) is called a non-degenerate vertex. Vertices that can be expressed by more than one value of the index set \(N\) are called degenerate vertex.
4 Finite Termination

Definition 3.5.1 A feasible tableau is degenerate if the last column contains zero elements. If the elements in the last column are strictly positive, the tableau is nondegenerate. A linear program is said to be non-degenerate if all feasible tableaus for that linear program are nondegenerate.

In this lecture, we review whether the Simplex Algorithm 1 terminates to either a solution or a finite direction of unboundedness. We demonstrate that finite termination is only possible under some restrictive assumptions.

Algorithm 1 Simplex algorithm: Solves \( \min_x p^T x \text{ s.t. } Ax \geq b \)

1. Construct an initial tableau. (Make sure that the problem is in standard form before doing so)
2. If this initial tableau is not feasible, apply Phase I to generate a feasible tableau.
3. Use a pricing scheme to determine a pivot column \( c \). If none exists, the tableau is optimal.
4. Use the ratio test to determine the pivot row \( r \). If none exists, the tableau is unbounded.
5. Exchange \( x_{B(r)} \) and \( x_{N(s)} \) using a labeled Jordan exchange.

Geometric Interpretation

A tableau is non degenerate if the vertex defined by that tableau is defined by the intersection of exactly \( n \) hyperplanes defined by \( x_N = 0 \). Vertices that lie at the intersection of more than \( n \) hyperplanes are degenerate.

A linear program is feasible and nondegenerate if each of the vertices of the feasible region for that linear program is uniquely defined by a set.

Theorem 3.5.2: If a linear program is feasible and non degenerate, then starting at any feasible tableau, the objective function strictly decreases at each pivot step. After a finite number of steps the method terminates with an optimal point or a finite direction of unboundedness.

Proof: At every iteration we have a tableau which is

1. Optimal
2. Non-optimal
3. Unbounded

If the tableau is optimal or unbounded, then the algorithm terminates. If the tableau is non-optimal, we pivot on \( H_{rs} \) for which \( h_r > 0 \) (non degeneracy) and \( c_s < 0 \) (by pivot selection), and \( H_{rs} < 0 \) (by the ratio test)

\[
\begin{align*}
    x_B &= \begin{bmatrix} x_N & 1 \end{bmatrix} \\
    z &= \begin{bmatrix} c^T & \alpha \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
    x_B &= \begin{bmatrix} \tilde{H} & h \end{bmatrix} \\
    z &= \begin{bmatrix} \tilde{c}^T & \tilde{\alpha} \end{bmatrix}
\end{align*}
\]

Here

\[
\tilde{\alpha} = \alpha - \frac{c_s h_r}{H_{rs}} < \alpha
\]
The strict decrease in the objective function ensures that no two consecutive steps of simplex will have identical objectives. If the size of the nonbasic set is $n$ and the index set is $n + m$ there are only $\binom{n+m}{m}$ possible simplex steps, it results in a finite termination.

References