Final Examination

CS 525 - Fall 2009

Monday, December 21, 2009, 10:05a-12:05p.

Each question is worth the same number of points.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. You need to give reasoning and justify all your answers, quoting any theorems you use.

1. Solve the following linear program: If it is unbounded, give a direction of unboundedness.

   \[ \begin{align*}
   \min & \quad 4x_1 - 12x_2 - x_3 \\
   \text{subject to} & \quad 2x_1 + x_3 \geq 2, \\
   & \quad 3x_2 + x_3 = 1, \\
   & \quad x_1 \text{ free}, \ x_2, x_3 \geq 0.
   \end{align*} \]

2. Solve the following linear program for all values of the parameter \( t \) in the interval \((-\infty, \infty)\). For each piece of the solution indicate clearly: parameter range, solution \( x(t) \), and optimal objective value \( z(t) \).

   \[ \begin{align*}
   \min & \quad -x_1 + t(x_1 + x_2) \\
   \text{subject to} & \quad -x_1 + x_2 \geq -1, \\
   & \quad -x_2 \geq -3, \\
   & \quad x_1, x_2 \geq 0.
   \end{align*} \]
3. Consider the following quadratic program:

\[
\begin{align*}
\text{min } & \quad x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 \\
\text{subject to } & \quad x_1 + 2x_2 \geq 3, \\
& \quad x_1 - x_2 \geq -2, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(a) Write down KKT conditions for this problem.
(b) Solve the problem using Lemke’s method.
(c) Does the solution change if we change the coefficient of \(x_1\) in the objective from 1 to 4? Explain.

4. Given a matrix \(A\), show that exactly one of the following two statements is true:

I. There is a vector \(x\) such that \(Ax > 0\) (that is, all components of \(Ax\) are strictly positive);

II. There is a vector \(u\) such that \(A' u = 0, u \geq 0, \text{ and } u \neq 0\).

5. A husband and wife are deciding how to spend their evening. The wife prefers to go to a baseball game, while the husband would prefer to visit an art gallery, but in either case, they would like to go out together. In formulating their decision as a bimatrix game, the loss matrix entry for each person is 4 if they go to separate events, 1 if they go to their preferred event together, and 2 if they go together to their less favored event together.

(a) Write down the loss matrix \(A\) for the wife and \(B\) for the husband. Let strategy 1 be “attend baseball” and strategy 2 be “visit art gallery”.

(b) If the randomized strategy vectors for the husband and wife are denoted by \(x\) and \(y\), respectively, show that \(\bar{x} = (1, 0)'\) and \(\bar{y} = (1, 0)'\) is a Nash equilibrium pair.

(c) Show that \(\bar{x} = (0.6, 0.4)'\) and \(\bar{y} = (0.4, 0.6)'\) is also a Nash equilibrium pair.