Each question is worth the same number of points.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **Give reasoning and justify all your answers**, quoting any theorems you use.

1. (a) Solve the following linear program: If it is unbounded, give a direction of unboundedness.

   \[
   \begin{align*}
   \text{min } & -2x_1 + 3x_2 + x_3 \\
   \text{subject to } & x_1 + x_3 \geq 2, \\
   & 4x_1 - x_2 = -1 \\
   & (x_1, x_2, x_3) \geq 0.
   \end{align*}
   \]

   (b) Formulate a linear program that finds the Chebyshev approximate solution of the system \(Ax = b\), where

   \[
   A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 4 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ 10 \end{bmatrix}.
   \]

   Write the problem in tableau form. *Do not solve* it, but indicate which element of the tableau would be your first pivot.

2. Solve the following linear program for all values of the parameter \(t\) in the interval \((-\infty, \infty)\). For each piece of the solution indicate clearly:
parameter range, solution $x(t)$, and optimal objective value $z(t)$.

$$\begin{align*}
\min & \quad x_1 + 2x_2 \\
\text{subject to} & \quad -x_1 \geq 4 + t, \\
& \quad x_1 + x_2 \geq -2t, \\
& \quad x_1, x_2 \geq 0.
\end{align*}$$

3. Consider the following quadratic program:

$$\begin{align*}
\min & \quad x_1^2 - 2x_1x_2 + 4x_2^2 + 2x_1 \\
\text{subject to} & \quad 2x_1 + x_2 \geq 4, \\
& \quad x_1, x_2 \text{ both free}.
\end{align*}$$

(a) Write down the KKT conditions for this problem.
(b) Solve the problem using Lemke’s method.
(c) Does the solution change if we change the constraint to an equality constraint: $2x_1 + x_2 = 4$? Explain.

4. Suppose that in a two-player zero-sum game, the loss matrix for Player 1 is as follows:

$$A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}.$$ 

(a) Write down the expected loss for Player 1, when Player 1 plays a randomized strategy $x = (x_1, x_2, x_3)$ and Player 2 plays a randomized strategy $y = (y_1, y_2, y_3)$.
(b) Show that Player 1 can achieve a negative expected loss (i.e. an expected gain) if if Player 2 plays any strategy other than $(y_1, y_2, y_3) = (1/3, 1/3, 1/3)$.
(c) Show that $\bar{x} = (1/3, 1/3, 1/3)$ and $\bar{y} = (1/3, 1/3, 1/3)$ form a Nash equilibrium pair.
(d) Let $\bar{x} = (1/3, 1/3, 1/3)$ as in part (c). Is it possible for $(\bar{x}, \hat{y})$ to be a Nash equilibrium pair, for some strategy vector $\hat{y}$ not equal to $(1/3, 1/3, 1/3)$? Explain.