1. Consider the following linear program.

$$\begin{align*}
\min_{x_1, x_2} & \quad -2x_1 + 3x_2 \\
\text{subject to} & \quad x_1 + x_2 \geq -2 \\
& \quad -x_1 + 2x_2 = 1 \\
& \quad 0 \leq x_1 \leq 5, \quad (x_2 \text{ free}).
\end{align*}$$

(a) Solve this problem. (Hint: Use Scheme II.)

(b) Write down the KKT conditions for this problem.

(c) Write down the dual, in any suitable form.

(b) Find the solution of the dual, and check that the optimal objectives are the same for primal and dual.

2. Show that exactly one of the following two systems has a solution:

(I) : \quad Ax = b, \quad 0 \leq x \leq e,

(II) : \quad A^T u - v \leq 0, \quad v \geq 0, \quad b^T u - e^T v = 1,

where $e$ is the vector of 1s, with the same dimension as $x$. 

3. Consider the following problem, which has variables \((x_1, x_2)\), an absolute value term in the objective, and a parameter \(t\) (which appears in both objective and constraints):

\[
\min_{x_1, x_2} |x_1 + t| \quad \text{subject to} \quad 2x_1 + 3x_2 \geq -2 + 3t, \quad (x_1, x_2) \geq 0.
\]

(a) By introducing an auxiliary variable \(x_3\), express this problem equivalently as a linear program, in which the parameter \(t\) appears only in the constraints.

(b) Find solutions of the linear program for all values of the parameter \(t\). Tabulate the values of \((x_1, x_2)\) and the optimal objective as a function of \(t\).

4. Consider the following problem in a single variable \(x_1\):

\[
\min_{x_1} x_1^2 - 2x_1 + \max(0, 5 - x_1) \quad \text{subject to} \quad x_1 \geq 0.
\]

(a) By introducing an auxiliary variable \(x_2\), reformulate this problem as an equivalent quadratic program.

(b) Write down the KKT conditions for this quadratic program.

(c) Use Lemke’s method to find the optimal value of \(x_1\).

(d) Is the solution you found a global solution of the quadratic programming formulation? Explain why or why not.