Try solving for too first. Can use duality simply.

Choose \((1, 2)\) to define

\[
\begin{array}{ccc}
\frac{1}{2} & 1 & \frac{1}{2} \\
& -3 & 3 \\
& 2 & 3 & 0 \\
& 0 & 1 & 0 \\
\end{array}
\]

Optimal for \(\frac{1}{2} - \frac{1}{2} = 0, \ -2 + t = 0\)

\(10\), \(t \leq 1\), \(t \geq -3\)

On interval \(t \in [-3, 1]\), solution is \(x(t) = (y_t)\) with objective \(z(t) = \frac{3 + t}{2}\).

When \(t\) decreases through \(-3\), the reduced cost in column 2 becomes negative, first on \((2, 2)\) element.

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_3)</td>
<td>(\frac{3}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(x_4)</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(z)</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>(z_0)</td>
<td>(\frac{1}{2})</td>
<td>-(\frac{1}{2})</td>
</tr>
</tbody>
</table>
On \( t \in [-9, -2] \) solution is \( x(t) = (0) \)
with objective value \( z(t) = 3 + t \).

As \( t \) decreases through \(-9\), first reduced cost becomes negative but there is no suitable pivot - problem is unbounded for \( t \leq -9 \).

Returning to tableau \( * \), as \( t \) increases through \(-9\), reduced cost in column 1 becomes negative. Pivot in (1,1) element to tableau:

\[
\begin{array}{ccc|c}
 x_2 & x_3 & 1 & \\
 x_2 & -2 & 2 & 1 \\
x_4 & -5 & 2 & 4 \\
 2 & -1 & 4 & 2 \\
 2_0 & 1 & 0 & 0 \\
\end{array}
\]

After when \(-1 + t > 0 \Rightarrow t > 1 \)
and \( 10 > 0 \Rightarrow \text{all } t \).

So on interval \( t \in [1, 2) \), solution is \( x(t) = (0) \)
with objective \( z \).

Summary:

<table>
<thead>
<tr>
<th>( [t_1, t_2] )</th>
<th>( x(t) )</th>
<th>( z(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [-9, -9] )</td>
<td>unbounded</td>
<td></td>
</tr>
<tr>
<td>( [-9, -3] )</td>
<td>((0))</td>
<td>(3 + t)</td>
</tr>
<tr>
<td>( [-3, 1] )</td>
<td>((0))</td>
<td>(\frac{3}{2} + \frac{1}{2}t)</td>
</tr>
<tr>
<td>( [1, \infty) )</td>
<td>((0))</td>
<td>(2)</td>
</tr>
</tbody>
</table>
Consider the following LP and its dual:

\begin{align*}
(\text{P}) \quad \min \quad & -a^T y \\
\text{subject to} \quad & -B^T y = 0, \\
& y - a \in \mathbb{R}^n
\end{align*}

\begin{align*}
(\text{D}) \quad \max \quad & \sigma^T v \\
\text{subject to} \quad & -Bu + \nu = 0, \\
& \sigma^T v = 1, \\
& \nu \in \mathbb{R}^n.
\end{align*}

Can write \( \text{D} \) equivalently as:

\begin{align*}
(\text{D}^{'}) \quad \max \quad & \sigma^T u \\
\text{subject to} \quad & Bu = 0, \\
& \nu = 0, \\
& e^T (Bu) = 1.
\end{align*}

\text{II is true} \implies \ (\text{P}) \text{ is feasible but unbounded} \implies \ (\text{D}^{'}) \text{ is infeasible (by strong duality)} \implies \ I \text{ is false}.

\text{II is false} \implies \ (\text{P}) \text{ is feasible with solution } (y, x) = (0, 0) \implies \ (\text{D}^{'}) \text{ is feasible (with } \nu = 0, \text{ satisfying)} \text{ by strong duality} \implies \ I \text{ is true}.
(2) (Alternative) This answer uses the facts that (a) any primal-dual solution pair satisfies KKT and (b) there exists a strictly complementary primal-dual solution pair (Thm 4.9.3).

Consider this primal-dual pair:

\[ \begin{align*}
(\text{P}) \quad & \text{min } \mathbf{c}^\top \mathbf{x} s.t. \mathbf{A} \mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0, \\
(\text{D}) \quad & \text{max } \mathbf{c}^\top \mathbf{y} s.t. \mathbf{B} \mathbf{y} = \mathbf{d}, \mathbf{y} \geq 0.
\end{align*} \]

Clearly both are feasible with trivial solutions. The KKT conditions satisfied by any primal-dual pair, are

\[ \begin{align*}
0 & \leq \mathbf{y} \perp \mathbf{B} \mathbf{x} \geq 0, \\
\mathbf{B}^\top \mathbf{y} & = 0.
\end{align*} \]

Theorem 4.9.2 says that there exist vectors \( \mathbf{y}, \mathbf{z} \) satisfying KKT and in addition

\[ \mathbf{z} + \mathbf{B} \mathbf{y} \geq 0. \quad (\ast) \]

We reason as follows:

I. \( \ast \) is true  \( \Rightarrow \exists \) vector \( \mathbf{y} \) with \( \mathbf{B}^\top \mathbf{y} = 0, \mathbf{y} \geq 0 \), which solves (P)

- \( \Rightarrow \) any vector \( \mathbf{y} \) with \( \mathbf{B} \mathbf{x} \geq 0 \) solves (D) and must have \( \mathbf{B} \mathbf{x} = 0 \)
  - by complementarity

- \( \Rightarrow \) I is false

II. \( \ast \) false  \( \Rightarrow \) any \( \mathbf{y} \) with \( \mathbf{B}^\top \mathbf{y} = 0, \mathbf{y} \geq 0 \) has \( \mathbf{y}_i = 0 \) for at least one index \( i \)

- \( \Rightarrow \) given (P),(D) that satisfy the KKT conditions and (I)

we have \( \mathbf{y}_i = 0 \) and thus \( \mathbf{B} \mathbf{y} \geq 0 \) for this \( i \)

\( \Rightarrow \mathbf{z} \) satisfies \( \mathbf{B} \mathbf{z} \geq 0 \) with \( \mathbf{B} \mathbf{y} = 0 \)

\( \Rightarrow \) I is true
### Scheme I: Pivot $x_3$ in $t_3$ and delete its column

\[
\begin{array}{ccc|c}
\text{x}_1 & \text{x}_2 & \text{x}_3 & 1 \\
\hline
\text{x}_1 & 2 & 0 & 1 & -2 \\
\text{x}_2 & 0 & 3 & 1 & -1 \\
\text{t}_3 & 4 & -5 & -1 & 0 \\
\end{array}
\]

### Scheme II: Pivot $x_3$ to side $r$

\[
\begin{array}{ccc|c}
\text{x}_1 & \text{x}_2 & \text{x}_3 & 1 \\
\hline
\text{x}_1 & 2 & -3 & 1 & -1 \\
\text{x}_3 & 0 & -3 & 1 \\
\text{t}_3 & 4 & -2 & -1 & -1 \\
\end{array}
\]

### Scheme III: Pivot $x_2$ to side $r$

\[
\begin{array}{ccc|c}
\text{x}_1 & \text{x}_2 & \text{x}_3 & 1 \\
\hline
\text{x}_1 & 1 & -2 & 1 \\
\text{x}_3 & 0 & -3 & 1 \\
\text{t}_3 & 4 & -2 & -1 \\
\end{array}
\]

Optimal Solution: \( x = \begin{pmatrix} y_2 \\ -1 \end{pmatrix} \)
\[ \text{(3) (b)} \]
\[ \max \ 2u_1 + u_2 \]
\[ u, u_1 \geq 0, \quad 2u_1 \leq 4 \]
\[ 3u_1 \leq -5 \]
\[ u_1 + u_2 \leq -1 \]
\[ u_1 \geq 0 \quad (u_1 \text{ free}) \]

\[ \text{(c)} \quad \text{KKT:} \]
\[ 2u_1 = 4 \]
\[ 0 \leq -5 - 3u_2 + x_2 \leq 0 \]
\[ 0 \leq -1 - u_1 - u_2 + x_2 \leq 0 \]
\[ 0 \leq 2x_1 + x_2 - 2 + u_1 \leq 0 \]
\[ 2x_1 + x_2 = 1 \]

(d) Clearly from first KKT condition we have \( u_1 = 2 \).

Since \( x_2 \geq 0 \) in primal solution we must have
\[ -1 - u_1 - u_2 = 0 \Rightarrow u_2 = -1 - u_1 = -3. \]

Checking other conditions:
\[ 0 \leq -5 - 3u_2 + 4 \quad \checkmark \]
\[ u_1 \geq 0 \quad \checkmark \]

So primal solution is \( u = (-3) \).

(e) Changing from \(-5\) to \(-8\) changes the second KKT condition to
\[ 0 \leq -8 - 3u_2 + x_2 \leq 0 \]

This and all other KKT conditions continue to be satisfied by the original primal and dual solutions, so primal solution does not change.
(4) \[ A = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix} \]

(b) no, because \( A + B \neq 0 \).

(c) Note that for \( \bar{x} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}, \bar{y} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \) we have
\[ \bar{x}^T A \bar{y} = 1, \quad \bar{x}^T B \bar{y} = 2 \]

To show Nash equilibrium, we have
\[ \bar{x}^T A \bar{y} = (x_1, x_2) \begin{bmatrix} 1/6 & 5/6 \end{bmatrix} = x_1 + x_2 + 3/2 x_2 = 1 + 3 x_2 \geq 2 \quad \checkmark \]
\[ \bar{x}^T B \bar{y} = (x_1, x_2) \begin{bmatrix} 2/3 & 1/3 \end{bmatrix} = 2 x_1 + 2 x_2 \geq 2 + 2 x_2 \geq 2 \quad \checkmark \]

(d) for \( \bar{x} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}, \bar{y} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} \) we have
\[ \bar{x}^T A \bar{y} = \begin{bmatrix} 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 1/6 & 5/6 \\ 5/6 & 1/6 \end{bmatrix} \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \frac{70}{25} \geq \frac{14}{5} \]
\[ \bar{x}^T B \bar{y} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/5 & 3/5 \\ 3/5 & 2/5 \end{bmatrix} \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \frac{70}{25} \geq \frac{14}{5} \]

Checking Nash equilibrium properties.
\[ \bar{x}^T A \bar{y} = (x_1, x_2) \begin{bmatrix} 2/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix} = x_1 x_2 + 3/5 x_2 = \frac{14}{5} \geq \frac{14}{5} \quad \checkmark \]
\[ \bar{x}^T B \bar{y} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{14}{5} \geq \frac{14}{5} \quad \checkmark \]

(c) \( \bar{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \bar{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is also a Nash eq. strategy.

(reasoning is the same as in (c)).