Midterm Examination

CS 525 - Spring 2009

Wednesday, March 11, 2009, 7:15-9:15pm

Each question is worth the same number of points.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. (a) For the following choice of $A$ and $b$, solve the system of equations $Ax = b$. If there are multiple solutions, describe the full solution set. If there are linear dependence relations between the rows of the coefficient matrix, state them.

   \[
   A = \begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.
   \]

   (b) Repeat part (a) for the following choice of $A$ and $b$:

   \[
   A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ -3 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.
   \]

2. Solve the following linear program. If it is infeasible, say so. If it is unbounded, give a direction of unboundedness. If there are multiple solutions, describe the full set of solutions.

   \[
   \begin{array}{l}
   \text{min} & x_1 - 2x_2 + 2x_3 \\
   \text{subject to} & -x_1 + x_2 - 3x_3 \geq 1, \\
   & x_1 + 4x_2 + 4x_3 \leq 2, \\
   & 4x_1 + x_2 - 6x_3 \leq 5, \\
   & x_1, x_2, x_3 \geq 0.
   \end{array}
   \]
3. Solve the following linear program. (Hint: Use Scheme II.) If it infeasible, say so. If it is unbounded, give a direction of unboundedness. If there are multiple solutions, describe the full set of solutions.

\[ \begin{align*}
\text{min} & \quad 4x_1 + 6x_2 + 2x_3 \\
\text{subject to} & \quad 2x_1 - 3x_2 + x_3 = 4, \\
& \quad 3x_1 - 5x_2 + x_3 \geq 9, \\
& \quad x_1, x_2 \geq 0, \quad x_3 \text{ free.}
\end{align*} \]

4. (a) Explain why the following linear program cannot be infeasible, and give an upper bound on its optimal objective value:

\[ \begin{align*}
\text{min} & \quad p'x \\
\text{subject to} & \quad Ax \geq 0, \quad x \geq 0.
\end{align*} \]

(b) Write down conditions under which the problem in part (a) has a solution. (Hint: Use the dual.) Assuming these conditions hold, write down a solution to this problem without doing any computations at all. (Justify your answers by citing the appropriate theorems.)

(c) Write down the dual of the following linear program in \( n \) unknowns, and find the solution of the dual:

\[ \begin{align*}
\text{max} & \quad x_1 + 2x_2 + 3x_3 + \ldots + nx_n \\
\text{subject to} & \quad x_1 + x_2 + x_3 + \ldots + x_n \leq 1, \\
& \quad x_1, x_2, x_3, \ldots, x_n \geq 0.
\end{align*} \]