1. For the following choice of $A$ and $b$, solve the system of equations $Ax = b$ by using tableaus.

$$A = \begin{bmatrix} 0 & -1 & 3 & 3 \\ 2 & 5 & -1 & 0 \\ 2 & 2 & 8 & 9 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. $$

(a) If there are multiple solutions, describe the full solution set. If there are no solutions, say so. If there is a unique solution, write it down.

(b) Are the rows of $A$ linearly independent or linearly dependent? If linearly dependent, write down the relationship between the rows of $A$.

2. Suppose that the following linear program has a solution $x^*$:

$$\min_x c^T x \text{ subject to } Ax \geq b, \ x \geq 0.$$

(a) Write down the dual of this problem and explain why there exists a solution $u^*$ to the dual.
(b) Suppose that the cost vector is tripled in the linear program, that is, \( c \) is replaced by \( 3c \) in the formulation above. For this modified problem give solutions to both the primal and the dual. (Explain your answers.)

(c) Suppose that we negate the cost vector, that is, we replace \( c \) by \(-c\) in the formulation above, and thus seek the minimizer of \(-c^T x\) over the original feasible set. What can you say about the solutions of the modified primal and dual problems?

3. Consider the following linear program, in general form:

\[
\begin{align*}
\text{min} & \quad 2x_1 - x_2 \\
\text{subject to} & \quad x_1 - 2x_2 \geq 5, \\
& \quad 3x_1 + x_2 = 6, \\
& \quad x_1 \geq 0, \quad x_2 \text{ free}.
\end{align*}
\]

(a) Write down the dual of this problem.

(b) Write down the KKT conditions for this problem.

(c) The solution of this problem is \( x^* = (\frac{17}{7}, -\frac{9}{7})' \). Using the KKT conditions, or by any other means, find a solution of the dual.

4. Solve the following linear programs, using any of the techniques we have learnt about in class. Show your working for each. (Each question can be solved using at most two Jordan exchanges.)

(a) \[
\begin{align*}
\text{min} & \quad 2x_1 + 2x_3 \\
\text{subject to} & \quad 4x_1 - x_2 + x_3 \geq 10, \\
& \quad x_1 + x_2 + x_3 = 5, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

(Note that the second constraint is an equality.)

(b) \[
\begin{align*}
\text{min} & \quad 2x_1 + 3x_2 + 2x_3 \\
\text{subject to} & \quad x_1 - x_2 + 3x_3 \geq 10, \\
& \quad x_1 + x_3 \geq 5, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]