(b) Linearly dependent: $A_3 = -2A_1 + A_2$. 

Set $y = 0$, choose $x_3 = \alpha$, $x_4 = \beta$, obtain solution: 

$$
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} &= \begin{bmatrix}
-3 \\
-14 \\
\alpha \\
\beta \\
\end{bmatrix} + \begin{bmatrix}
3 \\
-5 \\
0 \\
0 \\
\end{bmatrix} \alpha + \begin{bmatrix}
4 \\
1 \\
0 \\
0 \\
\end{bmatrix} \beta \quad \text{where } \alpha, \beta \in \mathbb{R}
\end{align*}
$$
(c) \[
\begin{pmatrix}
-3 \\
7 \\
0
\end{pmatrix} + \begin{pmatrix}
-3 \\
-5 \\
0
\end{pmatrix} x + \begin{pmatrix}
4 \\
0 \\
-1
\end{pmatrix} \beta \geq \begin{pmatrix}
7 \\
-5 \\
0
\end{pmatrix}
\]
\( x \in \mathbb{R}, \beta \in \mathbb{R} \)

(d) Solution of (c) is a (two-dimensional) plane.
Solution of (d) is a (two-dimensional) half-plane.

(2) This problem is clearly infeasible, since the constraint \(-x_2 - x_3 \geq 23\) cannot hold when \(x_2 \geq 0\) and \(x_3 \geq 0\).

We can see this too by setting up the tableau with dual labels:

\[
\begin{array}{cccc}
 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
 x_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 x_2 & 0 & 1 & 0 & 0 & 0 & 0 \\
 x_3 & 0 & 0 & 1 & 0 & 0 & 0 \\
 x_4 & 0 & 0 & 0 & 1 & 0 & 0 \\
 x_5 & 0 & 0 & 0 & 0 & 1 & 0 \\
 x_6 & 0 & 0 & 0 & 0 & 0 & 1 \\
 \end{array}
\]

\[
\begin{array}{cccc}
 & -1 & 2 & 6 & -5 \\
 \epsilon & 0 & -1 & -1 & -3 \\
 \end{array}
\]

The tableau is dual feasible, so we can proceed to apply dual simplex without a "Phase I". But if we select row 2 as a pivot row, we see that there is no suitable pivot column – the dual variable \(u_2\) can go to \(\epsilon\) while remaining dual feasible and driving the dual objective to \(\epsilon\). Hence the dual is unbounded, so by strong duality, the primal is infeasible.
3. (a) First write the primal in "general" form.

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 - x_3 \\
\text{st} & \quad -x_1 - x_2 - x_3 = -1 \\
& \quad 5x_2 + x_3 \geq 3 \\
& \quad x_1 \geq 0, x_2 \geq 0 \quad (x_3 \text{ free})
\end{align*}
\]

Dual (b)

\[
\begin{align*}
\text{max} & \quad -u_1 + 3u_2 - u_3 \\
\text{st} & \quad -u_1 + u_2 \leq 1 \\
& \quad -u_1 + 5u_2 + u_3 \leq 1 \\
& \quad u_2 + u_3 = -1 \\
& \quad u_1 \geq 0, u_2 \geq 0 \quad (u_3 \text{ free})
\end{align*}
\]

Feasible point: \( u = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) giving objective 1

(b) KKT:

\[
\begin{align*}
0 & \leq -x_1 - x_2 + 1 \perp u_1 \geq 0 \\
0 & \leq 5x_2 + x_3 - 3 \perp u_2 \geq 0 \\
x_1 + x_2 - x_3 = -1 \\
0 & \leq u_1 - u_2 + 1 \perp x_1 \geq 0 \\
0 & \leq u_1 - 5u_2 - u_3 + 1 \perp x_2 \geq 0 \\
u_1 - u_3 = -1
\end{align*}
\]

(c) Lower bound on primal obj given by value of dual objective at a dual feasible point (for weak duality). The point \( u = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) gives a dual objective of 1, so this is a lower bound.
(c) Start with the general form of the problem:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{1} )</td>
<td>( -1 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( x_{2} )</td>
<td>( 0 )</td>
<td>( 5 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( x_{3} )</td>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( x_{4} )</td>
<td>( 1 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( x_{5} )</td>
<td>( -1 )</td>
<td>( -1 )</td>
<td>( -1 )</td>
<td>( -1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( x_{6} )</td>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( -1 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

**Scheme II**

- Delete \( x_4 \) column (equality constraint).
- Move \( x_3 \) to bottom (free variable).

**DUAL Simplex**:

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{4} )</td>
<td>( -\frac{1}{4} )</td>
</tr>
<tr>
<td>( x_{6} )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( x_{3} )</td>
<td>( \frac{3}{4} )</td>
</tr>
<tr>
<td>( x_{2} )</td>
<td>( -\frac{3}{4} )</td>
</tr>
</tbody>
</table>

Optimal!
Solution is \( x^* = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \) with optimal objective 3.

Thus when we plug \( x^* \) into the KKT conditions, we find that the following conditions on \( u \) must be satisfied:

\[ u_1 \geq 0 \]
\[ u_2 \geq 0 \]
\[ u_1 - u_2 + 1 \geq 0 \]
\[ u_1 - 5u_2 - u_3 + 1 = 0 \]
\[ u_2 + u_3 = -1 \]

Solve these for \( u_1, u_2, u_3 \).

Setting \( u_1 = 0 \), we need \( u_2 \) and \( u_3 \) to satisfy:

\[ u_2 \geq 0 \]
\[ -u_2 + 1 \geq 0 \]
\[ -5u_2 - u_3 \geq -1 \]
\[ u_2 + u_3 = -1 \]

Solving the last two equations, we obtain:

\[ -7u_2 = -2 \Rightarrow u_2 = \frac{1}{7} \]

\[ u_2 = -1 - \frac{1}{7} = -\frac{8}{7} \]

Thus a dual solution is \( u^* = \begin{pmatrix} 0 \\ \frac{1}{7} \\ -\frac{8}{7} \end{pmatrix} \) with dual objective 3.
(4) The problem can't become infeasible because its constraints don't change, and the original form is known to be feasible.

Yes, it is possible to choose \( f \) so that the problem has a solution. e.g. \( f(x) = 0 \), then the solution is any feasible point.

(5) Writing the dual of the LP as:

\[
\begin{align*}
\text{max} & \quad b' u \\
\text{st} & \quad A^t u \leq f, \quad u \geq 0
\end{align*}
\]

we see that a subset \( A^t u \leq p \cdot A^t u + e \), so that \( u \) is dual feasible. Since we know that the primal is also feasible, strong duality tells us that both primal and dual have a solution.

(6) When we set \( b = 0 \), the problem becomes

\[
\begin{align*}
\text{min} & \quad p^t x \\
\text{st} & \quad A x \leq 0, \quad x \geq 0
\end{align*}
\]

while its dual becomes

\[
\begin{align*}
\text{max} & \quad 0^t u \\
\text{st} & \quad A^t u \leq p, \quad u \geq 0
\end{align*}
\]

Since the original primal was unbounded, its dual must be infeasible, by strong duality.

But the constraints of \( D' \) are the same as for the original dual, so \( D' \) is also infeasible.

Hence \( P' \) is either infeasible or unbounded (strong duality). Since \( x = 0 \) is a feasible point for \( P' \), \( P' \) must be unbounded.