1. Consider the following matrix and vector:

\[
A = \begin{bmatrix}
-1 & 0 & 3 & 4 \\
2 & 1 & -1 & 6 \\
4 & 1 & -7 & -2
\end{bmatrix}, \quad b = \begin{bmatrix}
3 \\
1 \\
-5
\end{bmatrix}.
\]

(a) Find all solutions to the linear system of equations \(Ax = b\).

(b) Are the rows of the matrix \(A\) linearly independent or linearly dependent? If linearly dependent, state the linear dependence relationship between them.

(c) Suppose that in addition to \(Ax = b\), we impose the additional constraint on the solution that \(x_2 \geq 5\). By modifying your solution to (a), write down the solution set of this expanded system.

(d) In geometric terms, what kinds of shapes do your solution sets in (a) and (c) represent? (A line, a ray, a plane, a point, or some other shape?)

2. Consider the following linear program in standard form:

\[
\min_{x_1, x_2, x_3} 2x_2 + 5x_3
\]

subject to \(- x_1 + 2x_2 + 6x_3 \geq 5,\)

\(-x_2 - x_3 \geq 3,\)

\(x_1, x_2, x_3 \geq 0.\)
Find all solutions of this problem. Say which are the vertex solutions.

3. Consider the following linear program:

\[
\begin{align*}
\min_{x_1, x_2, x_3} & \quad x_1 + x_2 - x_3 \\
\text{subject to} & \quad x_1 + x_2 \leq 1, \\
& \quad 5x_2 + x_3 \geq 3, \\
& \quad x_1 + x_2 + x_3 = -1, \\
& \quad x_1, x_2 \geq 0. \quad (x_3 \text{ free})
\end{align*}
\]

(a) Write down the dual of this problem and find (by “eyeballing”) a feasible point for the dual.

(b) Write down the KKT (optimality) conditions for this problem.

(c) Without constructing any tableaus, find a lower bound on the optimal objective value for the linear program above.

(d) Solve the problem above. (Hint: You might find Scheme II and the dual simplex method helpful.)

(e) Use the KKT conditions to construct a solution to the dual.

4. Consider the following linear program in standard form:

\[
\begin{align*}
\min_x & \quad p^T x \\
\text{subject to} & \quad Ax \geq b, \quad x \geq 0.
\end{align*}
\]

Suppose that this problem is an unbounded linear program.

(a) If we change \( p \) to some other vector \( \tilde{p} \), could the problem become infeasible rather than unbounded? Is it possible to choose \( \tilde{p} \) so that the problem has a solution?

(b) Define \( \hat{p} = A^T \hat{u} + e \), where \( e = (1, 1, \ldots, 1)^T \) and \( \hat{u} \) is some given vector with nonnegative entries. If we replace \( p \) in (1) by the vector \( \hat{p} \) defined here, can we guarantee that this modified linear program is unbounded, that is is infeasible, or that it has a solution? Or can we not say for sure whether it is in any one of these three categories?

(c) Suppose we take the unbounded linear program (1) and replace \( b \) by the zero vector. Is the modified problem unbounded, infeasible, or does it have a solution? Or can we not say for sure if it is any of these?