1. (10 points) Solve the following linear program. (Hint: Use Scheme II.) If it infeasible, say so. If it is unbounded, give a direction of unboundedness. If there are multiple solutions, describe the full set of solutions.

\[
\begin{align*}
\text{min} & \quad 2x_1 + 3x_2 + x_3 \\
\text{subject to} & \quad 2x_1 - 3x_2 = 4, \\
& \quad 3x_1 - 5x_2 + x_3 \geq 9, \\
& \quad x_1, x_2 \geq 0, \quad x_3 \text{ free}.
\end{align*}
\]

2. (10 points)

(a) Find all solutions of the linear system \(Ax = b\), where \(A\) and \(b\) are given below. If the rows of \(A\) are linearly dependent, write out the dependence relation.

\[
A = \begin{bmatrix}
1 & -3 & 0 \\
2 & 4 & 1 \\
0 & -10 & -1
\end{bmatrix}, \quad b = \begin{bmatrix}
-2 \\
3 \\
-7
\end{bmatrix}.
\]

(b) Using Jordan exchanges, find the inverse of the following matrix:

\[
A = \begin{bmatrix}
0 & 1 & -3 \\
1 & -1 & 2 \\
-2 & 3 & 0
\end{bmatrix}.
\]
3. (10 points) Solve the following linear program by the simplex method:

\[
\begin{align*}
\min & \quad 2x_1 + 3x_2 - 7x_3 \\
\text{subject to} & \quad x_1 + x_2 + x_3 \geq 2, \\
& \quad 2x_1 - 5x_2 + x_3 \geq -1, \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

If the problem is infeasible, say so. If it is unbounded, find vectors \(u, v \in \mathbb{R}^3\) such that \(u + \lambda v\) is a direction of unboundness, for \(\lambda \geq 0\).

4. (15 points) Consider the following linear program:

\[
\begin{align*}
\min & \quad x_1 + 2x_2 + 3x_3 + \cdots + nx_n \\
\text{subject to} & \quad a_1x_1 + a_2x_2 + a_3x_3 + \cdots a_nx_n \geq 1, \\
& \quad x_1 + x_2 + x_3 + \cdots + x_n \geq 1, \\
& \quad x_1, x_2, x_3, \ldots, x_n \geq 0.
\end{align*}
\]

where \(a_1, a_2, \ldots, a_n\) are strictly positive constants that satisfy

\[a_1 > a_2 > \cdots > a_n > 1.\]

(a) Write down the dual of this problem.

(b) Show that only the first general constraint in the dual could possibly be active. (The “general constraints” are all constraints other than the nonnegativity constraints on the dual variables.)

(c) By inspection, find the solution of the dual. (Hint: Use part (b).)

(d) Write down the KKT conditions for this problem.

(e) Using the KKT conditions, or by any other means, find the solution of the primal.