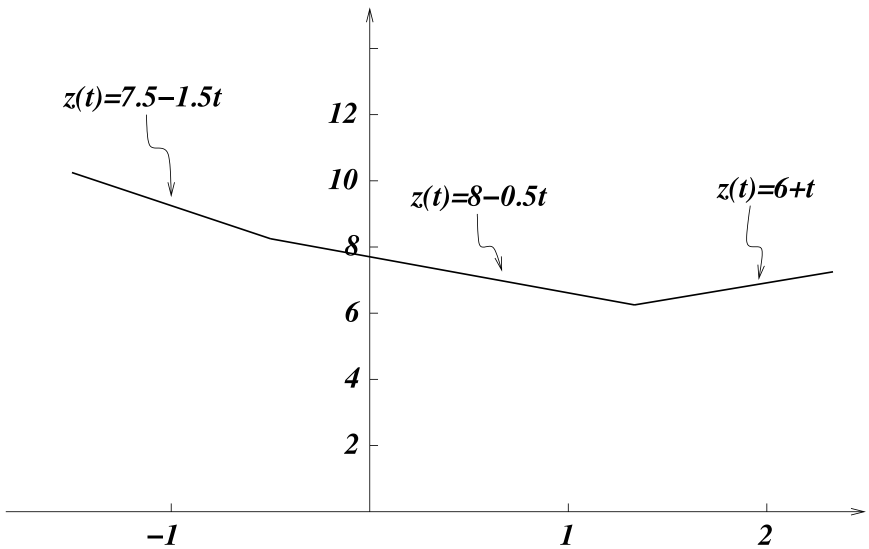

Figure 6.3: Minimum parametric objective value for parametrized right-hand side

3. Return to the lower limit t_L identified at the initial value of t.

4. If $t_L > -\infty$, determine which components of the right-hand side $b + th$ become negative as t decreases through t_L. Perform the pivots needed to restore optimality or establish infeasibility. If infeasible, set $z(t) = +\infty$ for all $t < t_L$ and stop. Otherwise, determine the new range (t_L, t_U) on which this basis is optimal, and determine the solution and the parametrized objective value $z(t)$ on this range. Repeat this step as necessary, until $t_L = -\infty$.

To justify the statement in Step 1 that if the problem is unbounded for some t, it is unbounded for all t, we make use again of strong duality. If (6.7) is feasible but unbounded for some t, then by Theorem 4.4.2 (ii) the dual of (6.7) is infeasible for this t. But since the dual constraints do not depend on t, the dual is infeasible for *all* t. Hence, by using Theorem 4.4.2 (ii) again, we conclude that (6.7) is unbounded for all t.

In Step 1, It is possible to find a value of t for which the problem (6.7) is feasible by means of a Phase-I problem, rather than trial and error. We introduce the artificial variable x_0 and solve the following problem:

$$z(t) = \min_{x,t,x_0} x_0 \text{ subject to } Ax + x_0 e \geq b + th, \ \ x \geq 0, \ x_0 \geq 0, \tag{6.9}$$

where $e = (1, 1, \ldots, 1)'$. Note that the parameter t is a *free variable* in this problem. If we have $x_0 = 0$ at the solution of this problem, we can choose the value of t from its optimum as the starting value in Step 1, and proceed. If on the other hand the problem (6.9) is infeasible, or if $x_0 > 0$ at the solution, then the original problem (6.7) is infeasible for all t.

Exercise 6-4-1. Consider the parametric linear program (6.7) with the following data:

$$p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad h = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Determine $z(t)$ for all t. (Include the ranges of t, if any, on which the problem is infeasible ($z(t) = \infty$) or unbounded ($z(t) = -\infty$).)

Exercise 6-4-2. Consider the linear programming problem

$$\max_u \; b'u \text{ subject to } A'u \le c, \;\; u \ge 0,$$

where

$$A' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Let

$$\Delta c = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \text{ and } \Delta b = \begin{bmatrix} 1 \\ -8 \\ 4 \end{bmatrix}.$$

Let $f(\theta)$ be the optimal value of the LP when b is replaced by $b + \theta\Delta b$, and $g(\theta)$ be the optimal value of the LP when c is replaced by $c + \theta\Delta c$. Evaluate $f(\theta)$ and $g(\theta)$ as functions of θ. (Hint: The original problem can be thought of as the dual to a standard form problem.)

Exercise 6-4-3. Consider the problem

$$\begin{array}{lrcl} \min & & 3x_1 - 2x_2 & \\ & x_1 - x_2 & \ge & 7 + t, \\ \text{subject to} & -x_1 & \ge & -5 + t \\ & x_1, x_2 & \ge & 0. \end{array}$$

1. Set up and solve the problem (6.9) to find a value of t for which this problem is feasible.

2. Starting from this value of t, find the solutions of this problem for all values of t.