1. Define the function \( f \) to be the following strictly convex quadratic:
\[
f(x) = \frac{1}{2} x^T Ax + b^T x + c,
\]
where \( x \in \mathbb{R}^n \) and \( A \) is an \( n \times n \) positive definite matrix.
(a) Find an explicit formula for the exact minimizing \( \alpha \) of the function
\[
t(\alpha) \overset{\text{def}}{=} f(x + \alpha p),
\]
where \( x \) and \( p \) are vectors such that \( p \neq 0 \) and \( x \) is not a minimizer of \( f \).
(b) For what values of \( c_1 \) is the first Wolfe condition satisfied by the minimizing \( \alpha \) from part (a)? (The first Wolfe condition is that \( f(x + \alpha p) \leq f(x) + c_1 \alpha \nabla f(x)^T p \).)

2. Let \( \{x_k\} \) be a sequence of vectors in \( \mathbb{R}^n \) and let \( f \) be a twice continuously differentiable function.
(a) If \( \{\nabla f(x_k)\} \) has an accumulation point at 0, does it follow that the sequence \( \{x_k\} \) must have a stationary accumulation point?
(b) Suppose that \( \lim_{k \to \infty} x_k = x^* \) for some \( x^* \), that \( \lim_{k \to \infty} \nabla f(x_k) = 0 \), and that there is a constant \( \beta > 0 \) such that matrices \( \nabla^2 f(x_k) \) are positive definite with
\[
\|\nabla^2 f(x_k)\|_{(\nabla^2 f(x_k))^{-1}} \leq \beta, \text{ for all } k > 0.
\]
Are the second-order sufficient conditions for \( x^* \) to be a local minimizer of \( f \) satisfied at \( x^* \)?

3. (a) The BFGS quasi-Newton updating formula for the approximate inverse Hessian \( H_k \) can be written as follows:
\[
H_{k+1} = (I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho_k s_k s_k^T,
\]
where
\[
\rho_k = \frac{1}{y_k^T s_k}.
\]
Show that if \( H_k \) is positive definite and the curvature condition \( y_k^T s_k > 0 \) holds, then \( H_{k+1} \) is also positive definite.
(b) If \( y_k^T s_k \leq 0 \), is it still possible for \( H_{k+1} \) to be positive definite?

4. (a) Consider the function \( r : \mathbb{R} \to \mathbb{R} \) defined by \( r(x) = x^q \), where \( q \) is an integer greater than 2. (Note that \( x^* = 0 \) is the sole root of this function and that it is degenerate, that is, \( r'(x^*) \) is singular.) Show that Newton’s method converges Q-linearly, and find the value of the convergence ratio (the limiting bound on \( \|x_{k+1} - x^*\|/\|x_k - x^*\| \)).
(b) Show that Newton’s method applied to the function $r(x) = -x^5 + x^3 + 4x$
starting from $x_0 = 1$ generates a sequence of iterates that alternates between $+1$ and $-1$.
(c) Find the roots of the function in (b), and check that they are nondegenerate.