Answer all FOUR questions below. One handwritten sheet of notes (written front and back) is allowed. EXPLAIN ALL YOUR ANSWERS.

1. (a) Consider the unconstrained minimization problem \( \min f(x) \), where \( f \) is a smooth function. Using the motivation from Taylor’s Theorem and least-squares, derive a Barzilai-Borwein formula for the line search parameter \( \alpha_k \) in the iteration \( x_{k+1} = x_k - \alpha_k \nabla f(x_k) \).

(b) When \( f(x) = \frac{1}{2} x^T A x \), for symmetric positive definite \( A \), express the formula from part (a) as a function of the latest step \( s_k := x_k - x_{k-1} \) and the Hessian \( A \).

(c) Consider the steepest descent method with exact line search applied to the convex quadratic function \( f \) from part (b). The iterations have the form \( x_{k+1} = x_k - \gamma_k \nabla f(x_k) \), where \( \gamma_k \) is chosen to minimize the function \( f \) along the direction \(-\nabla f(x_k)\). Express \( \gamma_k \) explicitly in terms of \( s_{k+1} := x_{k+1} - x_k \) and \( A \) (using the fact that \( s_{k+1} = \gamma_k \nabla f(x_k) \)).

(d) Comment on the relationship between \( \gamma_k \) from (c) and \( \alpha_k \) from (b).

2. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function and suppose that \( \{x_k\} \) is a sequence of iterates in \( \mathbb{R}^n \). Suppose further that \( \liminf \|\nabla f(x_k)\| = 0 \) and that \( \bar{x} \) and \( \tilde{x} \) are the only two accumulation points of the sequence \( \{x_k\} \).

(a) Must at least one of \( \bar{x} \) and \( \tilde{x} \) be stationary points of \( f \)? Must both of \( \bar{x} \) and \( \tilde{x} \) be stationary points of \( f \)?

(b) How does your answer to part (a) change if \( \{x_k\} \) is a bounded sequence?

3. A fundamental first-order necessary condition for optimality of \( x^* \) in the problem \( \min_{x \in \Omega} f(x) \), where \( \Omega \) is closed and convex, is that

\[
x^* \in \Omega \quad \text{and} \quad \nabla f(x^*)^T(z - x^*) \geq 0 \quad \text{for all} \quad z \in \Omega.
\]

Find the specialization of the first-order optimality conditions to the following two definitions of \( \Omega \), where \( v \in \mathbb{R}^n \) is a fixed vector:

(a) \( \Omega = \{v \mid \gamma \in \mathbb{R}\} \).

(b) \( \Omega = \{v \mid \gamma \geq 0\} \).

4. Consider the direction set \( D = \{p_1, p_2, \ldots, p_{n+1}\} \), where all \( p_i \) are in \( \mathbb{R}^n \) with

\[
p_i = e_i, \quad i = 1, 2, \ldots, n, \quad p_{n+1} = -\frac{1}{n}(1, 1, \ldots, 1)^T,
\]

where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \) with the 1 in position \( i \). Find a positive value of \( \delta > 0 \) such that for all possible \( v \in \mathbb{R}^n \), we have

\[
\max_{i=1,2,\ldots,n+1} p_i^T v \geq \delta \|v\|_1.
\]