Answer all FOUR questions below. One handwritten sheet of notes (written front and back) is allowed. EXPLAIN ALL YOUR ANSWERS.

1. Consider the unconstrained problem \( \text{min}_x f(x) \), where \( f : \mathbb{R}^n \to \mathbb{R} \) is smooth.
   (a) One form of the Barzilai-Borwein method takes steps of the form \( x_{k+1} = x_k - \gamma_k \nabla f(x_k) \), where
   \[
   \gamma_k := \frac{s_k s_k^T}{s_k^T y_k}, \quad s_k := x_k - x_{k-1}, \quad y_k := \nabla f(x_k) - \nabla f(x_{k-1}).
   \]
   Write down an explicit formula for \( \gamma_k \) in terms of \( s_k \) and \( A \), for the special case in which \( f \) is strictly convex quadratic, that is, \( f(x) = (1/2)x^T A x \), where \( A \) is symmetric positive definite.
   (b) Considering the steepest descent method \( x_{k+1} = x_k - \alpha_k \nabla f(x_k) \), applied to the strictly convex quadratic, write down an explicit formula for the exact minimizing \( \alpha_k \).
   (c) Show that the steplengths obtained in parts (a) and (b) are related as follows: \( \gamma_{k+1} = \alpha_k \).

2. Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) be a twice continuously differentiable function and suppose that \( \{x_k\} \) is a sequence of iterates in \( \mathbb{R}^n \).
   (a) Suppose that \( \lim \inf \|\nabla f(x_k)\| = 0 \). Is it true that all accumulation points of \( \{x_k\} \) are stationary (that is, satisfy first-order necessary conditions)?
   (b) Suppose that \( \lim \nabla f(x_k) = 0 \). Is it true that all accumulation points of \( \{x_k\} \) are stationary?
   (c) Suppose that the sequence \( \{x_k\} \) converges to a point \( x^* \), that the gradients \( \nabla f(x_k) \) converge to zero, and that the Hessians \( \nabla^2 f(x_k) \) at all these points are positive definite. Show that second-order necessary conditions are satisfied at the limit \( x^* \).
   (d) For the situation described in part (c), can we say that second-order sufficient conditions will be satisfied at \( x^* \)? Explain. (Hint: The eigenvalues of a matrix depend continuously on the elements of the matrix.)

3. (a) The BFGS quasi-Newton updating formula for the approximate inverse Hessian \( H_k \) can be written as follows:
   \[
   H_{k+1} = (I - \rho_k y_k s_k^T) H_k (I - \rho_k y_k s_k^T) + \rho_k y_k s_k^T,
   \]
   where
   \[
   \rho_k = \frac{1}{y_k^T s_k}.
   \]
   Show that if \( H_k \) is positive definite and the curvature condition \( y_k^T s_k > 0 \) holds, then \( H_{k+1} \) is also positive definite.
   (b) If \( y_k^T s_k \leq 0 \), is it still possible for \( H_{k+1} \) to be positive definite?
4. Consider the following form of the conjugate gradient method for solving $Ax = b$ (or, equivalently, minimizing $f(x) = (1/2)x^T Ax - b^T x$, where $A$ is symmetric positive definite.

Given $x_0$; Set $r_0 ← Ax_0 - b$, $p_0 ← -r_0$, $k ← 0$; 
while $r_k ≠ 0$

\[
\alpha_k ← -\frac{r_k^T p_k}{p_k^T Ap_k};
\]
\[
x_{k+1} ← x_k + \alpha_k p_k;
\]
\[
r_{k+1} ← Ax_{k+1} - b;
\]
\[
\beta_{k+1} ← \frac{r_{k+1}^T p_{k+1}}{p_k^T Ap_k};
\]
\[
p_{k+1} ← -r_{k+1} + \beta_{k+1} p_k;
\]
\[
k ← k + 1;
\]
end (while)

Show that
\[
r_k^T p_j = 0, \quad \text{for all } j = 0, 1, \ldots, k - 1. \tag{0.1}
\]

You may assume that the vectors $p_j$ are conjugate, that is, $p_j^T Ap_i = 0$ when $i ≠ j$.

(Hint: Prove by induction. Show first that $r_k^T p_k = 0$ for all $k$, which establishes (0.1) for $k = 1$. Then show that if (0.1) holds for some $k$, it continues to hold for $k + 1$, that is, $r_{k+1}^T p_j = 0$ for all $j = 0, 1, \ldots, k$.)