Final Examination

CS 730 - Spring 2010

Thursday, May 13, 2010, 5:05pm-7:05pm

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. Give reasoning and justify all your answers.

The number of points is given at the start of each question. There are 75 points in total.

1. (15 points)

(a) Given any two matrices $A$ and $B$ with the same number of columns, show that the following two statements are equivalent:

I. The rows of $A$ are linearly independent, and there is a vector $d \neq 0$ such that $Ad = 0$ and $Bd > 0$.

II. There is no vector pair $(\mu, \lambda)$ (not all zero) such that $\lambda \geq 0$ and $A^T \mu + B^T \lambda = 0$.

(b) Consider the constraint system

$$c_i(x) \geq 0 \ (i \in I), \ c_i(x) = 0 \ (i \in E),$$

at the point $x^* \in \mathbb{R}^n$, and let $A(x^*)$ denote the active set at $x^*$. One way to state the Mangasarian-Fromovitz constraint qualification (MFCQ) is: The gradients $\nabla c_i(x^*)$, $i \in E$ are linearly independent and there exists a vector $d \neq 0$ such that

$$\nabla c_i(x^*)^T d = 0 \ \text{for all} \ i \in E, \ \nabla c_i(x^*)^T d > 0 \ \text{for all} \ i \in A(x^*) \cap I.$$

Use part (a) to derive an alternative statement of MFCQ.
2. (20 points)

Consider the following nonlinear program:

\[
\min_x f(x) \text{ subject to } c(x) \geq 0,
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( c : \mathbb{R}^n \to \mathbb{R}^m \) are smooth functions, and its \( \ell_1 \)-penalized counterpart

\[
\min_{(x,t)} f(x) + \mu e^T t \text{ subject to } c(x) + t \geq 0, \ t \geq 0,
\]

where \( \mu > 0 \) is a penalty parameter and \( e = (1, 1, \ldots, 1)^T \).

(a) Write down the KKT conditions for both problems.

(b) Suppose that \( x^* \) is a KKT point for the first problem with optimal multipliers \( \lambda^* \). Under what condition on \( \mu \) is the same \( x^* \) together with \( t^* = 0 \) a KKT point for the penalized formulation? If this condition holds, what are the optimal multipliers for the constraints \( c(x) + t \geq 0 \) and \( t \geq 0? \)

(c) Suppose in addition to the assumptions of (b) that strict complementarity is satisfied for the first problem by \( x^* \) and \( \lambda^* \). Under what additional condition on \( \mu \) will strict complementarity also be satisfied at the corresponding KKT point for the penalized formulation?

3. (15 points)

Consider the following convex quadratic programming problem:

\[
\min \frac{1}{2} x^T Dx + c^T x \text{ subject to } x \geq 0, \ e^T x = 1,
\]

where \( D \) is an \( n \times n \) positive definite diagonal matrix and \( e = (1, 1, \ldots, 1)^T \).

(a) Write down the Wolfe dual of this problem, and eliminate variables as necessary to express it in the following form:

\[
\max_{\lambda, \mu} \frac{1}{2} \begin{bmatrix} \lambda^T \\ \mu \end{bmatrix} P \begin{bmatrix} \lambda \\ \mu \end{bmatrix} + t^T \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \text{ subject to } \lambda \geq 0,
\]

where \( \lambda \in \mathbb{R}^n \) is the vector of multipliers for the constraints \( x \geq 0 \) and \( \mu \) is the (scalar) multiplier for \( e^T x = 1 \). Give explicit formulas for \( P \) and \( t \).
(b) Is it always possible to reformulate the dual as a problem involving \( \lambda \) alone (that is, to eliminate \( \mu \))? Explain your answer.

4. (25 points)

(a) Solve the following semidefinite program in the symmetric 2 \( \times \) 2 matrix \( X \):

\[
\min_{X \succeq 0} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot X \text{ s.t. } \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \cdot X = -1, \begin{bmatrix} 0 & 0.5 \\ 0.5 & -1 \end{bmatrix} \cdot X = -1.
\]

(b) Write down the dual of the problem in (a), formulated in terms of two real variables (say \( y_1 \) and \( y_2 \)). (You do not need to solve it.)

(c) For the barrier function \( f : \mathbb{SR}^{n \times n} \to \mathbb{R} \) defined by \( f(X) = -\ln \det X \), we know that the second derivative operator \( f''(X) \) is defined by

\[
f''(X)UV = (X^{-1}UX^{-1}) \cdot V = \text{trace}(X^{-1}UX^{-1}V),
\]

where \( U \) and \( V \) are any two matrices in \( \mathbb{SR}^{n \times n} \). Show that

\[
f''(X)UV = f''(X)VU.
\]