Final Examination

CS 730 - Spring 2010

Thursday, May 13, 2010, 5:05pm-7:05pm

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **Give reasoning and justify all your answers.**

The number of points is given at the start of each question. There are 75 points in total.

1. (15 points)

   (a) Given any two matrices A and B with the same number of columns, show that the following two statements are equivalent:

      I. The rows of A are linearly independent, and there is a vector $d \neq 0$ such that $Ad = 0$ and $Bd > 0$.

      II. There is no vector pair (μ, λ) (not all zero) such that $\lambda \geq 0$ and $A^T \mu + B^T \lambda = 0$.

   (b) Consider the constraint system

   $$c_i(x) \geq 0 \ \ (i \in \mathcal{I}), \quad c_i(x) = 0 \ \ (i \in \mathcal{E}),$$

   at the point $x^* \in \mathbf{R}^n$, and let $\mathcal{A}(x^*)$ denote the active set at x^*. One way to state the Mangasarian-Fromovitz constraint qualification (MFCQ) is: The gradients $\nabla c_i(x^*)$, $i \in \mathcal{E}$ are linearly independent and there exists a vector $d \neq 0$ such that

   $$\nabla c_i(x^*)^T d = 0 \ \text{ for all } i \in \mathcal{E}, \ \ \nabla c_i(x^*)^T d > 0 \ \text{ for all } i \in \mathcal{A}(x^*) \cap \mathcal{I}.$$

   Use part (a) to derive an alternative statement of MFCQ.

2. (20 points)

   Consider the following nonlinear program:

$$\min_x f(x) \text{ subject to } c(x) \geq 0,$$

   where $f : \mathbf{R}^n \to \mathbf{R}$ and $c : \mathbf{R}^n \to \mathbf{R}^m$ are smooth functions, and its ℓ_1-penalized counterpart

$$\min_{(x,t)} f(x) + \mu e^T t \text{ subject to } c(x) + t \geq 0, \ t \geq 0,$$

   where $\mu > 0$ is a penalty parameter and $e = (1, 1, \ldots, 1)^T$.

   (a) Write down the KKT conditions for both problems.

   (b) Suppose that x^* is a KKT point for the first problem with optimal multipliers λ^*. Under what condition on μ is the same x^* together with $t^* = 0$ a KKT point for the penalized formulation? If this condition holds, what are the optimal multipliers for the constraints $c(x) + t \geq 0$ and $t \geq 0$?

   (c) Suppose in addition to the assumptions of (b) that *strict complementarity* is satisfied for the first problem by x^* and λ^*. Under what additional condition on μ will strict complementarity also be satisfied at the corresponding KKT point for the penalized formulation?

3. (15 points)

   Consider the following convex quadratic programming problem:

$$\min \frac{1}{2} x^T D x + c^T x \text{ subject to } x \geq 0, \ e^T x = 1,$$

   where D is an $n \times n$ positive definite diagonal matrix and $e = (1, 1, \ldots, 1)^T$.

   (a) Write down the Wolfe dual of this problem, and eliminate variables as necessary to express it in the following form:

$$\max_{\lambda,\mu} \frac{1}{2} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}^T P \begin{bmatrix} \lambda \\ \mu \end{bmatrix} + t^T \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \text{ subject to } \lambda \geq 0,$$

   where $\lambda \in \mathbf{R}^n$ is the vector of multipliers for the constraints $x \geq 0$ and μ is the (scalar) multiplier for $e^T x = 1$. Give explicit formulas for P and t.

(b) Is it always possible to reformulate the dual as a problem involving λ alone (that is, to eliminate μ)? Explain your answer.

4. (25 points)

(a) Solve the following semidefinite program in the symmetric 2×2 matrix X:

$$\min_{X \succeq 0} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X \text{ s.t. } \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = -1, \; \begin{bmatrix} 0 & 0.5 \\ 0.5 & -1 \end{bmatrix} \bullet X = -1.$$

(b) Write down the dual of the problem in (a), formulated in terms of two real variables (say y_1 and y_2). (You do not need to solve it.)

(c) For the barrier function $f : S\mathbf{R}^{n \times n} \to \mathbf{R}$ defined by $f(X) = -\ln \det X$, we know that the second derivative operator $f''(X)$ is defined by

$$f''(X)UV = (X^{-1}UX^{-1}) \bullet V = \text{trace}(X^{-1}UX^{-1}V),$$

where U and V are any two matrices in $S\mathbf{R}^{n \times n}$. Show that

$$f''(X)UV = f''(X)VU.$$