(a) \(-\frac{\nabla F(x)}{\epsilon} + \frac{1}{\epsilon}(x - y) \in N_x(y)\)

(b) Suppose there exist two points \(x^1, x^2\) at which the minimizer is achieved. We have

\[
-\frac{\nabla F(x^1)}{\epsilon} + \frac{1}{\epsilon}(x^1 - y) \in N_{x^1}(y)
\]

(1.0)

\[
-\frac{\nabla F(x^2)}{\epsilon} + \frac{1}{\epsilon}(x^2 - y) \in N_{x^2}(y)
\]

(1.6)

By definition of \(N\), we have from (1.0) and

\[
\langle -\frac{\nabla F(x^1)}{\epsilon} + \frac{1}{\epsilon}(x^1 - y), x - x^1 \rangle \leq 0 \quad \forall x \in \mathbb{R}
\]

thus with \(x \in x^1\) we have

\[
\langle -\nabla F(x), x^1 - x \rangle - \frac{1}{\epsilon} \langle x^1 - y, x^1 - x \rangle \leq 0
\]

Similarly from (1.6) we have

\[
\langle -\nabla F(x^2), x^1 - x^2 \rangle - \frac{1}{\epsilon} \langle x^2 - y, x^1 - x^2 \rangle \leq 0
\]

By adding these two expressions, we obtain

\[
\langle -\nabla F(x^2) + \nabla F(x^1), x^1 - x^2 \rangle + \frac{1}{\epsilon} \langle x^1 - x^2, x^2 - x^1 \rangle \leq 0.
\]

and so

\[
\langle -\nabla F(x), x^2 - x^1 \rangle \leq -\frac{1}{\epsilon} \|x^2 - x^1\|^2 < 0
\]

(2)

\[
\|\nabla F(x)\| < \epsilon^2
\]
By convexity of $F$, we have

$$F(x') \geq F(x) + \langle \nabla F(x), x' - x \rangle$$

and by adding these two expressions, we obtain

$$0 \geq \langle \nabla F(x') - \nabla F(x), x' - x \rangle$$

which contradicts (2).

(c) $\overline{O}_c(y) = \min_{x \in X} F(x) + \frac{1}{2c} \|x - y\|^2$

\[ \leq F(y) \] (since $y \in X$ is feasible for the subproblem.)

(d) $\overline{O}_c(x) = \min_{x \in X} F(x) + \frac{1}{2c} \|x - x'\|^2$

\[ \geq \min_{x \in X} F(x) + \min_{x \in X} \frac{1}{2c} \|x - x'\|^2 \]

\[ = F(x^*) \]

The result follows by combining this inequality with (c).

(b) Use a similar argument to (d):

$\overline{O}_c(y) = \min_{x \in X} F(x) + \frac{1}{2c} \|x - y\|^2$

\[ \geq \min_{x \in X} F(y) + \min_{x \in X} \frac{1}{2c} \|x - y\|^2 \]

\[ = F(x^*) \]
(5) It follows from (3) that all $x^* \in X^*$ are minimal of $\delta_0$.

To show the converse, suppose that $x^*$ is not a minimizer of $\delta_0$. We have $\delta(x^*) > \delta(x^*) = F(x^*)$ for all $x \in X$. From (3) we have

$$F(y) \geq F(x^*) > \delta(x^*) = F(x^*)$$

and so $x^* \in X^*$.

(2) Consider the situation with $x = x^*$ and $x = y$. We have

$$\begin{bmatrix}
\delta(x^*) & \gamma(x^*) \\
\gamma(x^*) & 1
\end{bmatrix}
\begin{bmatrix}
\Delta x' \\
\Delta x^*
\end{bmatrix}
\leq 0. \tag{3}$$

Suppose for $C'$ that there are two solutions $x^*$, namely $(x^*, x^*) : 0 = (\Delta x^*, \Delta x^*)$ with $\Delta x^* \neq \Delta x^*$. Then

$$\begin{bmatrix}
\delta(x^*) & \gamma(x^*) \\
\gamma(x^*) & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x' - \Delta x^* \\
\Delta x^* - \Delta x^*
\end{bmatrix}
\leq 0. \tag{3}$$

We have

$$\delta(x^*) (\Delta x^* - \Delta x^*) = 0$$

and thus $\Delta x^* - \Delta x^* \in E(x^*, x^*)$ (by Exercise (a) and (b)). Hence by 2.5,

$$(\Delta x^* - \Delta x^*)^T \frac{\partial^2 F(x^*)}{\partial x^* x^*} (\Delta x^* - \Delta x^*) > 0 \quad \mathcal{G}.$$
By multiplying the first block row in (4) by \\
\((\Delta x^1 - \Delta x^2)^T\), we obtain \\
\((\Delta x^1 - \Delta x^2)^T \Delta x^2 \{(x^1)^T \{\Delta x^1 - \Delta x^2\} = 0 \)

which contradicts (5).

(b) No, because \(\Delta x^2 \) need not have full column rank, we may have \(\Delta x^1 \neq \Delta x^2 \) in (6).

\[ \frac{1}{2} \mathbf{w}^T \mathbf{x} \leq x_i^2 - 1 = 0, \quad i = 1, \ldots, n \]

Let \( \mathbf{x} \) be a matrix \( \mathbf{x} \) representing \( \mathbf{x} X \). Thus \\
\( X_j \) represents \( x_j \).

We can write (6) as \\
\[ \min \quad \frac{1}{2} \mathbf{w}^T \mathbf{x} \quad \text{s.t.} \quad A_i \mathbf{x} = 1 \quad i = 1, \ldots, n \]

where \( A_i \) is all zero except for a 1 in the \( (i) \) position.

(c) \( f''(x) + H = (X'^T X') + H \)
\[ = \text{trace} \left( X'^T H X' \right) \]

Since \( X'^T \) is symmetric (not stated, \( X'^T \) exists)
\[ = \text{trace} \left( \frac{1}{2} X'^T (x'^T + \sum X'^T x'^T + X'^T + X'^T H) \right) \]
\[ = \text{trace} \left( \frac{1}{2} X'^T (x'^T H + x'^T H + x'^T H) \right) \]

\[ = \frac{1}{2} \text{trace}(x'^T H x'^T) \]

\[ = \| X'^T H X' \|_F^2 \]
so clearly $f''(y)H \geq 0$.

If $f''(x)H = 0$, we have $x^2 + x^2 - 1 = 0$.

which implies $H = 0$, since $x, x^2, y, z$ are all solvable.

(4) \[ \min -x^2 x_2 \text{ s.t. } 1 - x^2 - x^2 \geq 0 \]

(5) \[ \min -x_1 x_3 + 2x_1 \lambda = 0 \]

(6) \[ -x_2^2 + 2x_2 \lambda = 0 \]

\[ 0 \leq 1 - x_1^2 - x_1 \lambda \geq 0 \]

(7) \[ x_1 \geq \delta, \quad x_2 = 0 \quad \text{or} \quad x_2 = \lambda \]

(8) \[ x_1 = 0 \quad \text{has two minima (7b), so} \quad x_2 = 0 \quad \text{or} \quad \lambda = 0 \]

(9) \[ x_2 = \min \{ x_2 \} \text{ is solved only if } \lambda = 0 \]

Thus \[ x_2 (0), \lambda = 0 \] is a KKT point.

(10) \[ x_2 > 0 \text{ the point is solved by any} \]

(11) \[ (0), \quad x_2 (0), \quad \lambda = 0 \]

(12) \[ \lambda = 0 \quad \text{the point is solved by any} \]

Thus \[ (x_2), \quad y_2 \in [0, 1] \text{ and } \lambda = 0 \]

(13) \[ x_2 = \lambda \quad \text{then (7b)} \Rightarrow x_2 = 2x_2 \lambda = 2x^2 \]

\[ 0 \leq 1 - 3x^2 \quad \text{or} \quad \lambda \geq 0 \]
\[ \lambda = 0 \quad \text{we get} \quad x = (0), \quad y = 0 \]

(one of the faces already identified)

If \( \lambda = \frac{1}{\sqrt{3}} \), then:

\[ x = \left( \frac{+\sqrt{2}}{\sqrt{3}} \right), \quad \lambda = \frac{\sqrt{3}}{2}. \]

clearly both are KKT points and by inspection, both are local minimizers.