1. Given $S \subset \mathbb{R}^n$, define its convex hull $\text{co}(S)$ to be the set of all convex combinations of points in $S$.

   (a) Show that if $S$ is compact, then $\text{co}(S)$ is also compact. (Hint: Use Carathéodory’s theorem, which states that any convex combination of points in $\mathbb{R}^n$ can be expressed as a convex combination of $n + 1$ or fewer points in $\mathbb{R}^n$.)

   (b) If $\text{co}(S)$ is compact, must $S$ be compact? Explain.

2. Use linear programming duality to prove Farkas’s Lemma: For any matrix $A \in \mathbb{R}^{p \times n}$ and any vector $b \in \mathbb{R}^n$, exactly one of these two statements is true:

   There exists $x$ such that $Ax \leq 0$ and $b^T x > 0$;

   or

   There exists $y$ such that $A^T y = b$ and $y \geq 0$.

3. Consider the equality constrained optimization problem

   $$\min f(x) \quad \text{subject to } c_i(x) = 0, \quad i = 1, 2, \ldots, m,$$

   (1)

   where $x \in \mathbb{R}^n$ and the functions $f$ and $c_i, i = 1, 2, \ldots, m$ are smooth. Suppose that for some $x^* \in \mathbb{R}^n$ there is $\lambda^* \in \mathbb{R}^m$ such that the KKT
conditions are satisfied by \((x^*, \lambda^*)\) and that LICQ (the linear independence constraint qualification) and second-order sufficient conditions are satisfied there.

Consider now the following inequality constrained problem, which is equivalent to (1) (i.e. they have the same solutions):

\[
\min f(x) \text{ subject to } c_i(x) \leq 0, \ c_i(x) \geq 0, \ i = 1, 2, \ldots, m. \tag{2}
\]

(a) Are KKT conditions satisfied at the solution of (2)?
(b) Is LICQ satisfied at the solution of (2)?
(c) Is the Mangasarian-Fromovitz constraint qualification (MFCQ) satisfied at the solution of (2)?
(d) Are second-order sufficient conditions satisfied at the solution of (2)?

4. Consider the problem

\[
\min_{(x_1, x_2) \in \mathbb{R}^2} \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - x_1 x_2 - 2x_2 \text{ subject to } x \in \Omega,
\]

where

\[
\Omega = \{(x_1, x_2) \mid x_2 \leq 1 - |x_1|\}.
\]

(a) Write down \(N_\Omega(x^*)\) and \(T_\Omega(x^*)\), the normal and tangent cones to \(\Omega\) at the point \(x^* = (0, 1)\). (There is no need to derive these cones rigorously from the definition; just write down what they are.)
(b) Using your answer to (a), show that the geometric first-order necessary condition \(-\nabla f(x^*) \in N_\Omega(x^*)\) is satisfied at \(x^* = (0, 1)\).
(c) By writing the problem in the standard form

\[
\min f(x) \text{ subject to } c_i \geq 0, \ i \in \mathcal{I},
\]

for some appropriate functions \(c_i\), show that the KKT conditions are satisfied at \(x^* = (0, 1)\).
(d) Verify that LICQ is satisfied for the formulation in (c) at \(x^* = (0, 1)\) and write down the critical cone \(\mathcal{C}(x^*, \lambda^*)\) at this point.
(e) Using the formulation of (c), show that the second-order sufficient conditions are also satisfied at \(x^* = (0, 1)\).