Midterm Examination

CS 730 - Spring 2010

Wednesday, March 17, 2010, 7:15pm-9:15pm

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **Give reasoning and justify all your answers.**

1. Given $S \subset \mathbf{R}^n$, define its convex hull $\mathrm{co}(S)$ to be the set of all convex combinations of points in S.

   (a) Show that is S is compact, then $\mathrm{co}(S)$ is also compact. (Hint: Use Carathéodory's theorem, which states that any convex combination of points in $\mathbf{R}^n$ can be expressed as a convex combination of $n+1$ or fewer points in $\mathbf{R}^n$.)

   (b) If $\mathrm{co}(S)$ is compact, must S be compact? Explain.

2. Use linear programming duality to prove Farkas's Lemma: For any matrix $A \in \mathbf{R}^{p \times n}$ and any vector $b \in \mathbf{R}^n$, exactly one of these two statements is true:

   $$\text{There exists } x \text{ such that } Ax \leq 0 \text{ and } b^T x > 0;$$

   or

   $$\text{There exists } y \text{ such that } A^T y = b \text{ and } y \geq 0.$$

3. Consider the equality constrained optimization problem

   $$\min f(x) \text{ subject to } c_i(x) = 0, \quad i = 1, 2, \ldots, m, \qquad (1)$$

   where $x \in \mathbf{R}^n$ and the functions f and c_i, $i = 1, 2, \ldots, m$ are smooth. Suppose that for some $x^* \in \mathbf{R}^n$ there is $\lambda^* \in \mathbf{R}^m$ such that the KKT conditions are satisfied by (x^*, λ^*) and that LICQ (the linear independence constraint qualification) and second-order sufficient conditions are satisfied there.

Consider now the following inequality constrained problem, which is equivalent to (1) (i.e. they have the same solutions):

$$\min f(x) \text{ subject to } c_i(x) \le 0, \quad c_i(x) \ge 0, \quad i = 1, 2, \ldots, m. \tag{2}$$

(a) Are KKT conditions satisfied at the solution of (2)?

(b) Is LICQ satisfied at the solution of (2)?

(c) Is the Mangasarian-Fromovitz constraint qualification (MFCQ) satisfied at the solution of (2)?

(d) Are second-order sufficient conditions satisfied at the solution of (2)?

4. Consider the problem

$$\min_{(x_1,x_2)\in\mathbf{R}^2} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1x_2 - 2x_2 \text{ subject to } x \in \Omega,$$

where

$$\Omega = \{(x_1, x_2) \,|\, x_2 \le 1 - |x_1|\}.$$

(a) Write down $N_\Omega(x^*)$ and $T_\Omega(x^*)$, the normal and tangent cones to Ω at the point $x^* = (0, 1)$. (There is no need to derive these cones rigorously from the definition; just write down what they are.)

(b) Using your answer to (a), show that the geometric first-order necessary condition $-\nabla f(x^*) \in N_\Omega(x^*)$ is satisfied at $x^* = (0, 1)$.

(c) By writing the problem in the standard form

$$\min f(x) \text{ subject to } c_i \ge 0, \quad i \in \mathcal{I},$$

for some appropriate functions c_i, show that the KKT conditions are satisfied at $x^* = (0, 1)$.

(d) Verify that LICQ is satisfied for the formulation in (c) at $x^* = (0, 1)$ and write down the critical cone $\mathcal{C}(x^*, \lambda^*)$ at this point.

(e) Using the formulation of (c), show that the second-order sufficient conditions are also satisfied at $x^* = (0, 1)$.