Midterm Examination

CS 730 - Spring 2010

Wednesday, March 17, 2010, 7:15pm-9:15pm

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. Give reasoning and justify all your answers.

1. Given $S \subset \mathbb{R}^n$, define its convex hull $\text{co}(S)$ to be the set of all convex combinations of points in $S$.

   (a) Show that is $S$ is compact, then $\text{co}(S)$ is also compact. (Hint: Use Carathéodory’s theorem, which states that any convex combination of points in $\mathbb{R}^n$ can be expressed as a convex combination of $n + 1$ or fewer points in $\mathbb{R}^n$.)

   (b) If $\text{co}(S)$ is compact, must $S$ be compact? Explain.

2. Use linear programming duality to prove Farkas’s Lemma: For any matrix $A \in \mathbb{R}^{p \times n}$ and any vector $b \in \mathbb{R}^n$, exactly one of these two statements is true:

   There exists $x$ such that $Ax \leq 0$ and $b^T x > 0$;

   or

   There exists $y$ such that $A^T y = b$ and $y \geq 0$.

3. Consider the equality constrained optimization problem

   \[
   \min f(x) \text{ subject to } c_i(x) = 0, \quad i = 1, 2, \ldots, m, \tag{1}
   \]

   where $x \in \mathbb{R}^n$ and the functions $f$ and $c_i$, $i = 1, 2, \ldots, m$ are smooth. Suppose that for some $x^* \in \mathbb{R}^n$ there is $\lambda^* \in \mathbb{R}^m$ such that the KKT
conditions are satisfied by \((x^*, \lambda^*)\) and that LICQ (the linear independence constraint qualification) and second-order sufficient conditions are satisfied there.

Consider now the following inequality constrained problem, which is equivalent to (1) (i.e. they have the same solutions):

\[
\min f(x) \text{ subject to } c_i(x) \leq 0, \ c_i(x) \geq 0, \ i = 1, 2, \ldots, m. \quad (2)
\]

5. (a) Are KKT conditions satisfied at the solution of (2)?

5. (b) Is LICQ satisfied at the solution of (2)?

5. (c) Is the Mangasarian-Fromovitz constraint qualification (MFCQ) satisfied at the solution of (2)?

5. (d) Are second-order sufficient conditions satisfied at the solution of (2)?

4. Consider the problem

\[
\min_{(x_1, x_2) \in \mathbb{R}^2} \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - x_1 x_2 - 2x_2 \text{ subject to } x \in \Omega,
\]

where

\[
\Omega = \{(x_1, x_2) \mid x_2 \leq 1 - |x_1|\}.
\]

4. (a) Write down \(N_\Omega(x^*)\) and \(T_\Omega(x^*)\), the normal and tangent cones to \(\Omega\) at the point \(x^* = (0, 1)\). (There is no need to derive these cones rigorously from the definition; just write down what they are.)

3. (b) Using your answer to (a), show that the geometric first-order necessary condition \(-\nabla f(x^*) \in N_\Omega(x^*)\) is satisfied at \(x^* = (0, 1)\).

5. (c) By writing the problem in the standard form

\[
\min f(x) \text{ subject to } c_i \geq 0, \ i \in \mathcal{I},
\]

for some appropriate functions \(c_i\), show that the KKT conditions are satisfied at \(x^* = (0, 1)\).

4. (d) Verify that LICQ is satisfied for the formulation in (c) at \(x^* = (0, 1)\) and write down the critical cone \(C(x^*, \lambda^*)\) at this point.

4. (e) Using the formulation of (c), show that the second-order sufficient conditions are also satisfied at \(x^* = (0, 1)\).
1. Let \( x_k \) be a sequence of points in \( S \).

By Carathéodory's theorem we can write

\[ x_k = \frac{1}{n} \sum_{i=1}^{n} a_{ki} u_i, \]

for some vectors \( u_i \in S \) and scalars \( a_{ki} \) with

\[ a_{ki} \geq 0, \quad \sum_{i=1}^{n} a_{ki} = 1. \]

Now by compactness of \( S \), we can choose a subsequence \( S_i \) s.t.

\[ \lim_{k \to i} u_{k,i} = u_i \quad \text{for some } u_i \in S. \]

Now choosing a further subsequence \( S_{j_i} \subset S_i \), we have by compactness that

\[ \lim_{k \to j_i} u_{k,j_i} = u_{j_i}. \]

(we still have \( \lim_{k \to j_i} u_{k,i} = u_i \)).

Continuing in this fashion, we can find a subsequence \( S_{j_{i,n}} \) such that

\[ \lim_{k \to j_{i,n}} u_{k,i} = u_i, \quad u_{j_{i,n}} \in S. \]

Note that for each \( k \), we have that

\[ (a_{k,1}, a_{k,2}, \ldots, a_{k,n+1}) \in U_{n+1} \subset \mathbb{R}^{n+1}. \]
where \( S_{mn} = \{ x \in \mathbb{R}^n \mid x \geq 0, \sum x_i = 1 \} \) is closed and bounded, hence compact.

By taking a further subsequence \( S_{m_k} \subseteq S_{mn} \), we have

\[
\lim_{k \to \infty} (x_{1k}, x_{2k}, \ldots, x_{nk}) = (a_1, a_2, \ldots, a_n)
\]

we thus have

\[
\lim_{k \to \infty} x_i = \frac{1}{n} \sum_{i=1}^{n} a_i \in \text{co} S
\]

so that \( \{ x_{n,k} \} \) has an accumulation point in \( \text{co} S \).

Hence \( \text{co} S \) is compact.

(1) \( \mathbb{N} \). Consider the set in \( \mathbb{R}^2 \):

\[
S = \{ 0 \} \cup \left( \frac{1}{4}, \frac{3}{4} \right) \cup \{ 1 \}
\]

Here \( \text{co} S = [0,1] \) is compact, but \( S \) is not compact.

(2) Define the following fixed-end pair:

(P) \( \lim \sup 0^+ y \text{ st. } A^T y = b, y \geq 0 \)

(D) \( \max b^T x \text{ st. } A x \leq 0 \)

and let the two alternatives be:

I. \( \exists x \text{ st. } A x \leq 0, b^T x > 0 \)

II. \( \exists y \text{ st. } A^T y = b, y \geq 0 \).
I is TRUE $\Rightarrow$ (D) is unbounded $\Rightarrow$ (P) is infeasible $\Rightarrow$ II is FALSE

II is TRUE $\Rightarrow$ (P) has a solution $y$ with optimal objective $0$ $\Rightarrow$ (P) has a solution with optimal objective $0$ $\Rightarrow b^T x \leq 0$. for all $x$. with $Ax \leq 0$ $\Rightarrow$ I is FALSE

3: Yes. Writing $y^{0}$ as

\[
\min f(y) \text{ s.t. } c(y) \leq 0, -c(y) \geq 0 \quad (1)
\]

the KKT conditions are that there exist $\alpha_i \geq 0$, $\beta_i \geq 0$, $\lambda_i \geq 0$, $\gamma_i \geq 0$.

\[

\nabla f(x^*) - \sum_{i=1}^{n} \alpha_i \nabla c_i(x^*) + \sum_{i=1}^{m} \beta_i \nabla l_i(x^*) = 0
\]

\[
0 \leq \alpha_i \perp c_i(x^*) = 0
\]

\[
0 \leq \beta_i \perp -c_i(x^*) = 0
\]

Setting $\alpha^* = \max (x^*_i, 0)$ $\beta^* = \max (-x^*_i, 0)$ we have

\[
\nabla f(x^*) - \sum_{i=1}^{n} \alpha_i \nabla c_i(x^*) + \sum_{i=1}^{m} \beta_i \nabla c_i(x^*)
\]

\[
= \nabla f(x^*) - \sum_{i=1}^{n} (\alpha_i - \beta_i) \nabla c_i(x^*)
\]

\[
= \nabla f(x^*) - \sum_{i=1}^{n} x^*_i \nabla l_i(x^*)
\]

\[
= 0
\]
Hence the first KKT condition holds. Since \( x \geq 20, \beta \geq 20 \), and \( \gamma \geq 0 \), the other conditions obviously hold.

(2) **No.** Achieve constraints for (2) are
\[
\mathbf{v}_c(\mathbf{x}) \quad \text{and} \quad -\mathbf{v}_c(\mathbf{x}), \quad \mathbf{v}_c(\mathbf{x}) = -\mathbf{v}_c(\mathbf{x})
\]
which is clearly not a linearly independent set (except in the trivial case \( m = 0 \)).

(3) **No.** MFCQ for (2) requires existence of a vector \( d \) such that
\[
\mathbf{v}_c(\mathbf{x})^T d < 0, \quad i = 1, \ldots, m
\]
\[-\mathbf{v}_c(\mathbf{x})^T d < 0, \quad i = 1, \ldots, m
\]
but these two conditions cannot hold simultaneously (except again in the trivial case \( m = 0 \)).

(4) **Yes.** 2st. condition for the original problem hold, so we have
\[
\mathbf{v}^T \left[ \mathbf{v}_c(\mathbf{x})^T - \frac{\alpha}{m} \mathbf{v}_c(\mathbf{x})^T \right] \mathbf{v} > 0
\]
for all \( \mathbf{v} \neq 0 \) with \( \mathbf{v}_c(\mathbf{x})^T \mathbf{v} = 0 \).

Turning to (2) and setting \( \alpha_i, \beta_i, \gamma_i \) as in part (2), we have the critical case
\[
E(x, \alpha, \beta) = \left\{ \begin{array}{ll}
\mathbf{v}_c(\mathbf{x})^T \mathbf{v} > 0 & i = 1, \ldots, m \\
\mathbf{v}_c(\mathbf{x})^T \mathbf{v} = 0 & \text{if } \alpha_i > 0 \\
-\mathbf{v}_c(\mathbf{x})^T \mathbf{v} > 0 & \text{if } \beta_i = 0 \\
-\mathbf{v}_c(\mathbf{x})^T \mathbf{v} = 0 & \text{if } \beta_i > 0
\end{array} \right.
\]
\[
= \left\{ \begin{array}{ll}
\mathbf{v}_c(\mathbf{x})^T \mathbf{v} = 0 & i = 1, \ldots, m
\end{array} \right.
\]
Hence for \( u \in C(x, y, k) \) with \( x > 0 \) we have

\[
\begin{align*}
&\quad u^T \left[ \varphi^2(x) - \sum x_i \varphi(x_i) + \sum x_i^2 \varphi(x_i) \right] u \\
&= u^T \left[ \varphi^2(x) - \sum y_i \varphi(y_i) \right] u \\
&> 0
\end{align*}
\]

\( x \) is unbounded.

\[
\begin{align*}
N_{\varphi}(x) &= \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_2 > 15u_1 \right\} \\
T_{\varphi}(x) &= \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_2 \leq -15u_1 \right\}
\end{align*}
\]

\( f(x, y) = \frac{1}{4} x_1^2 + \frac{1}{4} x_2^2 - x_1 x_2 - 2x_1 \)

\[
\begin{align*}
&f(x, y) = \begin{bmatrix} x_1 - x_2 \\ y_2 - x_1 - 2 \end{bmatrix} \\
\text{so } \quad &\begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow -2f(x) = (1) \in N_{\varphi}(x).
\end{align*}
\]

\( \text{Write as } \min f(x) \text{ s.t. } x_2 \leq 1 - x_1, \quad x_2 \leq -1 + x_1, \quad x_1 - x_2 - 1 \geq 0 \)

\[
\begin{align*}
&= \min f(x) \text{ s.t. } -x_1 + x_2 + 1 \geq 0, \quad x_1 - x_2 - 1 \geq 0
\end{align*}
\]
So \( \nabla_c (x^k) = (\frac{-1}{1}) \), \( \nabla_c (y^k) = (\frac{1}{-1}) \).

So \( f(x^k) - \lambda_1 \nabla_c (x^k) - \lambda_2 \nabla_c (y^k) \)

\[ = (\frac{-1}{1}) - \lambda_1 (\frac{-1}{-1}) - \lambda_2 (\frac{1}{1}) \]

By setting \( \lambda_1 = 1, \lambda_2 = 0 \), the above equation holds. All other KKT conditions also hold because

\[ 0 \leq 1 = \lambda_1 \leq c_1 (x^k) = 0 \geq 0 \]
\[ 0 \leq 0 = \lambda_2 \leq c_2 (y^k) = 0 \geq 0. \]

\( \nabla_c (x^k) = (\frac{-1}{1}) \), \( \nabla_c (x^k) = (\frac{1}{-1}) \)

are clearly the indefiniteness since

\[ \alpha_1 \nabla_c (x^k) + \alpha_2 \nabla_c (y^k) \geq 0 \]

\( \Rightarrow -\alpha_1 + \alpha_2 = 0 \) and \( -\alpha_1 - \alpha_2 \geq 0 \)

\( \Rightarrow \alpha_1 = \alpha_2 = 0 \).

\( \mathcal{L} (x^k, y^k) : \begin{cases} (\frac{1}{1}) \quad \nabla_c (x^k)^T \bar{y} = 0 \quad \text{(with } \lambda_1 = 0) \\ (\frac{-1}{1}) \quad \nabla_c (y^k)^T \bar{y} \geq 0 \quad \text{(with } \lambda_2 = 0) \end{cases} \)

\( \begin{cases} (\frac{1}{1}) \quad -\bar{y}, -\bar{y} = 0 \quad \Rightarrow \end{cases} \)

\( \begin{cases} (\frac{1}{1}) \quad \alpha \geq -\alpha \\ \bar{y} \geq 0 \end{cases} \)

\( \begin{cases} (\frac{1}{1}) \quad \alpha \geq -\alpha \\ \bar{y} \geq 0 \end{cases} \)
\[ \Delta^2 f \left( x^2 \right) - \lambda, \nabla^2 g \left( y^2 \right) - \lambda, \nabla^2 h \left( z^2 \right) \]

\[
\begin{pmatrix}
1 & -1 \\
-1 & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha \\
-\alpha
\end{pmatrix} + 
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha \\
-\alpha
\end{pmatrix} = 
\begin{pmatrix}
\alpha \\
-\alpha
\end{pmatrix} + 
\begin{pmatrix}
2\alpha \\
-2\alpha
\end{pmatrix}
\]

\[= 4\alpha^2 \quad > 0 \]

provided \( \alpha \neq 0 \).