Midterm Examination

CS 730 - Spring 2011

Wednesday, March 9, 2011, 7:15pm-9:15pm

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **Give reasoning and justify all your answers.**

P and D : 5 —
argument : 5 —

1. (10 points) Prove the following theorem of the alternative (attributed to Fan): Either there exists x with $Ax = 0$, $c^T x > 0$, and $x \geq 0$, or there exists y with $A^T y \geq c$, but not both.

2. (10 points)

   (a) Find the dual of the following problem:

   4 —

   $$\min -\ln x + 4x \text{ subject to } x \geq 1.$$

   2 — (b) Give the domain of the dual objective from (a).

   4 — (c) Find the solutions of the primal and dual.

3. (20 points) Consider the following problem:

   $$\min_{x \in \mathbf{R}^2} \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - t)^4 \text{ subject to } \|x\|_1 \leq 1,$$

   where t is a real parameter and $\|x\|_1 = |x_1| + |x_2|$.

   4 — (a) Rewrite the problem as a constrained optimization problem (with four smooth constraints).

   6 — (b) For what values of t does the point $x^* = (1, 0)^T$ satisfy the KKT conditions for your formulation?

2 — (c) For which of the values of t from (b) are strict complementarity conditions satisfied for your formulation?

4 — (d) For the values of t from (b), are the second-order sufficient conditions satisfied? Explain.

4 — (e) Find the solution of the problem when $t = 1$. (Hint: Probably just one of the constraints in your formulation will be active.)

4. (20 points) Consider the following nonlinear program:

$$\min_x f(x) \text{ subject to } c(x) \geq 0,$$

where $f : \mathbf{R}^n \to \mathbf{R}$ and $c : \mathbf{R}^n \to \mathbf{R}^m$ are smooth functions, and its ℓ_1-penalized counterpart

$$\min_{(x,t)} f(x) + \mu e^T t \text{ subject to } c(x) + t \geq 0,\ t \geq 0,$$

where $\mu \geq 0$ is a penalty parameter and $e = (1, 1, \ldots, 1)^T$.

6 — (a) Write down the KKT conditions for both problems.

7 — (b) Suppose that x^* is a KKT point for the first problem with optimal multipliers λ^*. Under what condition on μ is the same pair (x^*, λ^*) together with $t^* = 0$ a KKT point for the penalized formulation? If this condition holds, what are the optimal multipliers for the constraints $c(x) + t \geq 0$ and $t \geq 0$?

7 — (c) Consider now the following quadratic-penalty formulation:

$$\min_{(x,t)} f(x) + \frac{1}{2}\mu t^T t \text{ subject to } c(x) + t \geq 0,\ t \geq 0,$$

where $\mu > 0$ is again a penalty parameter. Suppose that x^* is a KKT point for the original problem with optimal multipliers λ^*, and that $\lambda^* \neq 0$. Under what conditions on μ is the same (x^*, λ^*) together with $t^* = 0$ a KKT point for the quadratic-penalty formulation?

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① Label the statements.

I $\exists_x$ st. $Ax = 0,\ x \geq 0,\ c^T x > 0$

II $\exists_y$ st $A^T y \geq c$

Consider this dual pair of CP:

(P) min $-c^T x$ st $Ax = 0,\ x \geq 0$

(D) max 0 st $A^T \lambda \leq -c$

By making the identification $y = -\lambda$, we can write (D) equivalently as

(D') max 0 st $A^T y \geq c$.

I true $\Rightarrow$ (P) unbounded $\Rightarrow$ (D') infeasible $\Rightarrow$ II false

II true $\Rightarrow$ (D') feasible and optimal, with optimal objective 0

$\Rightarrow$ (P) optimal with optimal obj 0

$\Rightarrow \{ Ax = 0,\ x \geq 0 \Rightarrow c^T x \leq 0 \}$

$\Rightarrow$ I false

② ⓐ $\mathcal{L}(x,\lambda) = -\ln x + 4x - \lambda(x-1)$

$q(\lambda) = \inf_x -\ln x + 4x - \lambda(x-1).$

min achieved when $-\frac{1}{x} + 4 - \lambda = 0 \Rightarrow x = \frac{1}{4-\lambda}$

We need $x > 0$ (since $\ln x$ not defined otherwise), so $\lambda < 4$.

By substituting, obtain explicitly:

$$q(\lambda) = \ln(4-\lambda) + \frac{4}{4-\lambda} - \lambda\left(\frac{1}{4-\lambda} - 1\right)$$

$$= \ln(4-\lambda) + \frac{4-\lambda+\lambda(4-\lambda)}{4-\lambda}$$

$\Rightarrow g(\lambda) = \ln(4-\lambda) + (\lambda+1)$. $\qquad \lambda < 4$

for $\lambda \geq 4$, have $g(\lambda) = -\infty$

Dual: $\max\limits_{\lambda \geq 0} g(\lambda)$

ⓑ Domain $= \{\lambda \mid g(\lambda) > -\infty\} = (-\infty, 4)$

ⓒ Dual: $\max\limits_{\lambda \geq 0} g(\lambda)$ $\qquad g'(\lambda) = \frac{-1}{4-\lambda} + 1$

$g'(\lambda) = 0$ when $1 = \frac{1}{4-\lambda} \Rightarrow \lambda = 3$.

solution: $\lambda^* = 3$

Primal: $\min\limits_{x \geq 1} f(x) = -\ln x + 4x$

$f'(x) = \frac{-1}{x} + 4 > 0$ for $x \geq 1$.

solution $x^* = 1$

③ ⓐ $\min \; (x_1 - \frac{3}{2})^2 + (x_2 - t)^4$ s.t.

$$\begin{aligned} x_1 + x_2 + 1 &\geq 0 \\ x_1 - x_2 + 1 &\geq 0 \\ -x_1 + x_2 + 1 &\geq 0 \\ -x_1 - x_2 + 1 &\geq 0 \end{aligned}$$

ⓑ KKT:

$$\begin{aligned} 2(x_1 - \tfrac{3}{2}) - \lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 &= 0 \\ 4(x_2 - t)^3 - \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 &= 0 \end{aligned}$$

$0 \leq \lambda_1 \perp x_1 + x_2 + 1 \geq 0$

$0 \leq \lambda_2 \perp x_1 - x_2 + 1 \geq 0$

$0 \leq \lambda_3 \perp -x_1 + x_2 + 1 \geq 0$

$0 \leq \lambda_4 \perp -x_1 - x_2 + 1 \geq 0$

At $x^* = \binom{1}{0}$: KKT become

$$-1 - \lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 = 0$$

$$-4t^3 - \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 = 0$$

$$0 \le \lambda_1 \perp 2 \ge 0 \qquad \Rightarrow \lambda_1 = 0$$

$$0 \le \lambda_2 \perp 2 \ge 0 \qquad \Rightarrow \lambda_2 = 0$$

$$0 \le \lambda_3 \perp 0 \qquad \Rightarrow \lambda_3 \ge 0$$

$$0 \le \lambda_4 \perp 0 \qquad \Rightarrow \lambda_4 \ge 0$$

Reduce to:

$$\lambda_3 + \lambda_4 = 1$$

$$-\lambda_3 + \lambda_4 = 4t^3$$

$$\lambda_3 \ge 0 , \ \lambda_4 \ge 0$$

Manipulate:

$$2\lambda_4 = 1 + 4t^3 \quad \Rightarrow \quad \lambda_4 = \tfrac{1}{2} + 2t^3$$

$$\Rightarrow \quad \lambda_3 = 1 - \lambda_4 = \tfrac{1}{2} - 2t^3 .$$

So t yields a KKT solution when

$$\tfrac{1}{2} + 2t^3 \ge 0 \Rightarrow t^3 \ge \tfrac{-1}{4} \Rightarrow t \ge \tfrac{-1}{4^{1/3}}$$

AND $\tfrac{1}{2} - 2t^3 \ge 0 \Rightarrow t^3 \le \tfrac{1}{4} \Rightarrow t \le \tfrac{1}{4^{1/3}}$

Hence the interval is

$$t \in \left[\tfrac{-1}{4^{1/3}}, \tfrac{1}{4^{1/3}} \right].$$

Ⓒ Strict complementarity when $t \in \left(\tfrac{-1}{4^{1/3}}, \tfrac{1}{4^{1/3}} \right)$ (open interval) since then $\lambda_3 > 0$ and $\lambda_4 > 0$

ⓓ Since constraints are linear, they do not contribute to $\nabla^2_{xx} \mathcal{L}(x, \lambda)$, so we have

$$\nabla^2_{xx}\mathcal{L}(x,\lambda) = \nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 12(x_2-t)^2 \end{bmatrix}$$

so $$\nabla^2_{xx}\mathcal{L}(x^*,\lambda) = \begin{bmatrix} 2 & 0 \\ 0 & 12t^2 \end{bmatrix}$$

We have

$$\mathcal{F}(x^*) = \left\{ d \;\middle|\; \begin{matrix} -d_1+d_2 \geq 0 \\ -d_1-d_2 \geq 0 \end{matrix} \right\},$$

$$\mathcal{C}(x^*,\lambda) = \left\{ d \in \mathcal{F}(x^*) : \begin{matrix} d_1 = d_2 \text{ if } \lambda_3 > 0 \\ d_1 = -d_2 \text{ if } \lambda_4 > 0 \end{matrix} \right\}.$$

If $t \in \left(\frac{-1}{4^{1/3}}, \frac{1}{4^{1/3}}\right)$ we have $\lambda_3 > 0$, $\lambda_4 > 0$ so

$$\mathcal{C}(x^*,\lambda) = \{0\},$$

So second-order sufficient conditions are satisfied trivially.

If $t = \frac{-1}{4^{1/3}}$ or $t = \frac{1}{4^{1/3}}$, we have

$$\nabla^2_{xx}\mathcal{L}(x^*,\lambda) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{12}{4^{2/3}} \end{bmatrix}$$

which is positive definite, so again 2oS conditions are satisfied trivially.

(e) When $t=1$, we guess that the constraint

$$-x_1 - x_2 + 1 \geq 0$$

is the active one. Thus seek a solution of KKT conditions with $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \geq 0$:

$$2\left(x_1 - \tfrac{3}{2}\right) + \lambda_4 = 0$$
$$4(x_2-1)^3 + \lambda_4 = 0.$$
$$x_1 + x_2 + 1 \geq 0$$
$$x_1 - x_2 + 1 \geq 0$$
$$-x_1 + x_2 + 1 \geq 0$$
$$x_1 + x_2 = 1, \quad \lambda_4 \geq 0.$$

By reducing the three equalities to an equation in x_2 alone we get.

$$2x_1 - 3 = -\lambda_4 = 4(x_2-1)^3, \qquad x_1 = 1 - x_2$$
$$\Rightarrow \quad 2 - 2x_2 - 3 = 4(x_2-1)^3$$
$$\Rightarrow \quad -1 - 2x_2 = 4x_2^3 - 12x_2^2 + 12x_2 - 4$$
$$\Rightarrow \quad 4x_2^3 - 12x_2^2 + 14x_2 - 3 = 0$$

this has a root in $(0,1)$ since the LHS is -3 when $x_2 = 0$ and 3 when $x_2 = 1$

Hence $x_1 \in (0,1)$ also.

Moreover $\lambda_4 = -4(x_2-1)^3 > 0$.

So we have a KKT point.

Second-order sufficient conditions hold, since

$$\nabla^2_{xx} \mathcal{L}(x,\lambda) = \nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 12(x_2-1)^2 \end{bmatrix}$$

which is positive definite

④ (a) First problem. Denote $A(x) = \nabla c(x)$

$$\left.\begin{aligned} &\nabla f(x) - A(x)\lambda = 0 \\ &0 \le \lambda \perp c(x) \ge 0. \end{aligned}\right\} \begin{matrix}(1a)\\(1b)\end{matrix}$$

Second problem.

$$\nabla f(x) - A(x)\lambda = 0 \qquad (2a)$$

$$\mu e - \lambda - \tau = 0 \qquad (2b)$$

$$0 \le \lambda \perp c(x) + t \ge 0. \qquad (2c)$$

$$0 \le \tau \perp t \ge 0. \qquad (2d)$$

(b) - If (x^*, λ^*) is a KKT point for first problem, (2b) and $\tau \ge 0$ tell us that $\underline{\underline{\mu \ge \|\lambda\|_\infty}}$

When this condition holds, we can satisfy (2) by setting

$$\tau^* = \mu e - \lambda^* \ge 0$$

$$t^* = 0$$

and setting x^*, λ^* exactly as in (1).

(c) - KKT for quadratic penalty u

$$\nabla f(x) - A(x)\lambda = 0 \qquad (3a)$$

$$\mu t - \lambda - \tau \ge 0. \qquad (3b)$$

$$0 \le \lambda \perp c(x) + t \ge 0 \qquad (3c)$$

$$0 \le \tau \perp t \ge 0. \qquad (3d)$$

if $t^* = 0$, (3b) becomes $-\lambda - \tau \ge 0$, which, since $\lambda \ge 0$ and $\tau \ge 0$, can hold only if $\lambda^* = \tau^* = 0$. Since $\lambda^* \ne 0$, the KKT point for (1) can <u>never</u> satisfy (3), for any $\mu > 0$.