In the questions below, we suppose that $\psi$ is a convex (possibly non-smooth) function.

1. Show that $\partial \psi(x)$ is a closed convex set.

2. Note from Theorem 23.4 of Rockafellar’s *Convex Analysis* (1970) that we have the following expression for directional derivative of $\psi$:

$$D(\psi(x); p) = \sup_{s \in \partial \psi(x)} s^T p.$$ 

Suppose that $x$ is not a minimizer of $\psi$ (that is, $0 \notin \partial \psi(x)$) and let $\bar{d}$ be the vector of minimum norm in $\partial \psi(x)$ (that is, $\bar{d} = \arg \min_{d \in \partial \psi(x)} \|d\|^2$). Show that $\psi(x - t\bar{d}) < \psi(x)$ for all $t$ positive and sufficiently small.

3. Divide the number 8 into two nonnegative parts $x$ and $y$ so as to maximize $xy(x - y)$.

4. Consider the problem

$$\max \quad x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

subject to $\sum_{i=1}^{n} x_i = 1$, $x_i \geq 0$, $i = 1, \ldots, n$,

where $a_i$ are given positive scalars. Find a global maximum and show that it is unique.