New Directions 2016: Mathematical Optimization
August 2016
Worksheet 1

1. Compute the gradient and Hessian of the Rosenbrock function:

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

2. Prove the all isolated minimizers are strict. (One way to do this is to prove that "not isolated" implies "not strict".)

3. (a) Give an example of a matrix that is *not* positive definite despite having all positive entries.

   (b) If A is a positive definite matrix, must its diagonal elements all be positive? Explain.

4. Suppose that $f(x) = \frac{1}{2}x^T Ax$ where A is symmetric positive semidefinite. Show that f is convex, by proving that the following is true for all $x, y \in \mathbb{R}^n$ and all $\alpha \in [0, 1]$:

$$f(x + \alpha(y - x)) \leq (1 - \alpha)f(x) + \alpha f(y).$$

5. Suppose that $f(x) = x^T Ax$ where A is an $n \times n$ matrix that is *not necessarily symmetric*. What is the Hessian $\nabla^2 f(x)$?

6. Consider the sequence $\{x_k\}$ defined by

$$x_k = \begin{cases} (1/4)^{2^k}, & \text{for } k \text{ even} \\ (x_{k-1}/k), & \text{for } k \text{ odd.} \end{cases}$$

   Is this sequence Q-superlinearly convergent? Q-quadratically convergent? R-quadratically convergent?

7. Give an example of a sequence $\{x_k\}$ of positive real numbers, that decreases to zero Q-superlinearly but not Q-quadratically.

8. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function, and suppose that the set of minimizers S defined by $S := \{x^* : f(x^*) \leq f(x) \text{ for all } x \in \mathbb{R}^n\}$ is nonempty. Show that S is a *closed* and *convex* set.

9. Show the interior and closure of a convex set are convex.

10. A point $x \in \mathbb{R}^n$ is a *convex combination* of the points $\{x^1, x^2, \ldots, x^r\}$ in $\mathbb{R}^n$ if for some real numbers $\lambda_1, \lambda_2, \ldots \lambda_r$ that satisfy $\sum \lambda_i = 1$ and $\lambda_i \geq 0$, we have $x = \sum_{i=1}^r \lambda_i x^i$. Show that a set S in $\mathbb{R}^n$ is convex if and only if every convex combination of a finite number of points of S is in S.

11. A set C is a *cone* if for every $\lambda > 0$ and every $x \in C$, $\lambda x \in C$. Prove that a cone is convex if and only if it is closed under addition, that is, $x, y \in C$ implies that $x + y \in C$.