New Directions 2016: Mathematical Optimization
August 2016
Worksheet 4

1. Consider the NLP

$$\begin{aligned}\min_{x\in\mathbb{R}^n}\ & f(x)\\ \text{s.t. } & c_E(x)=0\\ & c_I(x)\le 0\end{aligned}$$

where $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ and $c_I:\mathbb{R}^n\longrightarrow\mathbb{R}^{m_I}$ are convex, and where $c_E:\mathbb{R}^n\longrightarrow\mathbb{R}^{m_E}$ is affine.

Prove that any local minimizer of the NLP is a global minimizer.

2. Consider the equality-constrained QP

$$\begin{aligned}\min_{x\in\mathbb{R}^n}\ & \frac{1}{2}x^TQx+g^Tx\\ \text{s.t. } & Ax+b=0\end{aligned}$$

where $g\in\mathbb{R}^n$, $b\in\mathbb{R}^m$, $A\in\mathbb{R}^{m\times n}$ has full rank, and $Q\in\mathbb{R}^{n\times n}$ is symmetric. Let $Z\in\mathbb{R}^{n\times(n-m)}$ be a null-space matrix of A; i.e., the columns of Z form a basis of the null space of A.

Suppose Z^TQZ has a negative eigenvalue. Prove that then the QP is unbounded.

3. Consider the QP

$$\begin{aligned}\min_{x\in\mathbb{R}^4}\ & \frac{1}{2}x^TQx+g^Tx\\ \text{s.t. } & Ax+b=0\end{aligned}$$

with

$$Q=\begin{bmatrix}1 & 0 & 1 & 2\\ 0 & 0.25 & -1.5 & -2\\ 1 & -1.5 & 11 & 14\\ 2 & -2 & 14 & 22\end{bmatrix}\qquad g\begin{pmatrix}-3\\ 2\end{pmatrix}$$

$$A=\begin{bmatrix}1 & 0 & 1 & 2\\ 0 & -0.5 & 3 & 4\end{bmatrix}\qquad b=\begin{pmatrix}-3\\ 1\end{pmatrix}$$

Solve this problem, using the step decomposition method. Compute x^* and λ^*.

4. Recall the primal active-set QP method from the lecture, and let $\mathcal{W}_0 \subseteq \mathcal{I}$ be an initial working set. Furthermore, suppose that the constraint gradients $\{a_i\}_{i \in \mathcal{E} \cup \mathcal{W}_0}$ are linearly independent. Prove that in all later iterations, the vectors $\{a_i\}_{i \in \mathcal{E} \cup \mathcal{W}}$ remain linearly independent, as long as the step size α is positive in all iterations.

5. Recall the QP

$$
\begin{aligned}
\min\ & (x_1 - 1)^2 + (x_2 - 2.5)^2 \\
\text{s.t.}\ & -x_1 + 2x_2 - 2 \le 0 && (1) \\
& x_1 + 2x_2 - 6 \le 0 && (2) \\
& x_1 - 2x_2 - 2 \le 0 && (3) \\
& -x_1 \le 0 && (4) \\
& -x_2 \le 0 && (5)
\end{aligned}
$$

   from the lecture. Execute the active-set QP algorithm starting from $x_0 = (4, 1)$ and $\mathcal{W}_0 = \{2, 3\}$.

6. Write down one NLP and give two separate iterates, one at which the (QP$_k$) in the SQP method is feasible, and one at which (QP$_k$) is infeasible.

7. Suppose, $p_k = 0$ is the optimal solution of (QP$_k$), the QP solved in the SQP method at an iteration x_k for an NLP with equality and inequality constraints. Prove that then x_k is a point at which the KKT conditions for the NLP hold.

8. Consider the following NLP from the lecture:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^2}\ & x_1 + (x_2 - 1)^2 \\
\text{s.t.}\ & x \ge 0.
\end{aligned}
$$

   Compute the central path, i.e., the set of the minimizers $x^*(\mu)$ of (BP$_\mu$) for all $\mu > 0$, together with their dual variables $\lambda^*(\mu)$ and $z^*(\mu)$.

9. Prove that (in the notation introduced in the Nonlinear Optimization Theory slides) we have $T_\Omega(\bar{x}) \subset \mathcal{F}(\bar{x})$, without the need for any constraint qualifications. (You need to use the definition of limiting feasible directions, in conjunction with Taylor's theorem.)

10. Suppose that the Mangasarian-Fromowitz constraint qualification is satisfied at $\bar{x}$. Show that there are no coefficients μ_i for $i \in \mathcal{A}(\bar{x})$ with μ_i *not all zero*, and $\mu_i \geq 0$ for all $i \in \mathcal{A}(\bar{x}) \cap \mathcal{I}$, such that

$$\sum_{i \in \mathcal{A}(\bar{x})} \mu_i \nabla c_i(\bar{x}) = 0.$$