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An Accelerated Newton Method for Equations with Semismooth Jacobians and Nonlinear Complementarity Problems

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Abstract We discuss local convergence of Newton's method to a singular solution x^* of the nonlinear equations $F(x) = 0$, for $F : \mathbb{R}^n \to \mathbb{R}^n$. It is shown that an existing proof of Griewank, concerning linear convergence to a singular solution x^* from a starlike domain around x^* for F twice Lipschitz continuously differentiable and x^* satisfying a particular regularity condition, can be adapted to the case in which F' is only strongly semismooth at the solution. Further, under this regularity assumption, Newton's method can be accelerated to produce fast linear convergence to a singular solution by over-relaxing every second Newton step. These results are applied to a nonlinear-equations reformulation of the nonlinear complementarity problem (NCP) whose derivative is strongly semismooth when the function f arising in the NCP is sufficiently smooth. Conditions on f are derived that ensure that the appropriate regularity conditions are satisfied for the nonlinear-equations reformulation of the NCP at x^*.

Keywords Nonlinear Equations · Semismooth Functions · Newton's Method · Nonlinear Complementarity Problems

Mathematics Subject Classification (2000) 65H10 · 90C33

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1 Introduction

Consider a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, and let $x^* \in \mathbb{R}^n$ be a solution to $F(x) = 0$. We consider the local convergence of Newton's method when the solution x^* is *singular* (that is, $\ker F'(x^*) \neq \{0\}$) and when F is once but possibly not twice differentiable. We also consider an accelerated variant of Newton's method that achieves a fast linear convergence rate under these conditions. Our technique can be applied to a nonlinear-equations reformulation of nonlinear complementarity problems (NCP) defined by

$$\text{(1)} \qquad \text{NCP}(f)\text{:}\ \ 0 \leq f(x), \ \ x \geq 0, \ \ x^T f(x) = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. At degenerate solutions of the NCP (solutions x^* at which $x_i^* = f_i(x^*) = 0$ for some i), this nonlinear-equations reformulation is not twice differentiable at x^*, and the weaker smoothness assumptions considered in this paper are required. Our results show that (i) the simple approach of applying Newton's method to the nonlinear-equations reformulation of the NCP converges inside a starlike domain centered at x^*, albeit at a linear rate if the solution is singular; (ii) a simple technique can be applied to accelerate the convergence in this case to achieve a faster linear rate. The simplicity of our approach contrasts with other nonlinear-equations-based approaches to solving (1), which are either nonsmooth (and hence require nonsmooth Newton techniques whose implementations are more complex; see for example Josephy [14] and the discussion in Facchinei and Pang [6, p. 663-674]) or else require classification of the indices $i = 1, 2, \ldots, n$ into those for which $x_i^* = 0$, those for which $f_i(x^*) = 0$, or both.

Around 1980, several authors, including Reddien [20], Decker and Kelley [3], and Griewank [8], proved linear convergence for Newton's method to a singular solution x^* of F from special regions near x^*, provided that F is twice Lipschitz continuously differentiable and a certain 2-regularity condition holds at x^*. (The "2" emphasizes the role of the second derivative of F in this regularity condition.)

In the first part of this work, we show that Griewank's analysis, which gives linear convergence from a partial neighborhood of x^* known as a *starlike domain*, can be extended to the case in which F' is strongly semismooth at x^*; see Section 4. In Section 5, we consider a standard acceleration scheme for Newton's method, which "overrelaxes" every second Newton step. By assuming that F' is at least strongly semismooth at x^* and that a 2-regularity condition holds, we show that this technique yields arbitrarily fast linear convergence from a partial neighborhood of x^*.

In the second part of this work, beginning in Section 6, we consider a particular nonlinear-equations reformulation of the NCP and interpret the regularity conditions for this reformulation as conditions on the NCP. We show that they reduce to previously known NCP regularity conditions in certain special cases. We conclude in Section 7 by presenting computational results for some simple NCPs, including a number of degenerate examples.

We start with certain preliminaries and definitions of notation and terminology (Section 2), followed by a discussion of prior relevant work (Section 3).

2 Definitions and Properties

For $G : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$, we follow convention in writing the derivative G' as a map from $\Omega \to \mathbb{R}^p \times \mathbb{R}^n$ when $p > 1$ and a map from Ω to $\mathbb{R}^n$ (equivalently $\mathbb{R}^n \times \mathbb{R}$) when $p = 1$.

The Euclidean norm is denoted by $\| \cdot \|$, and the unit sphere is $\mathcal{S} = \{t \mid \|t\| = 1\}$.

For any subspace X of $\mathbb{R}^n$, $\dim X$ denotes the dimension of X. The kernel of a linear operator M is denoted $\ker M$, the image or range of the operator is denoted $\operatorname{range} M$. $\operatorname{rank} M$ denotes the rank of the matrix M, which is the dimension of $\operatorname{range} M$.

A *starlike domain* with respect to $x^* \in \mathbb{R}^n$ is an open set $\mathcal{A}$ with the property that $x \in \mathcal{A} \Rightarrow \lambda x + (1-\lambda)x^* \in \mathcal{A}$ for all $\lambda \in (0,1)$. A vector $t \in \mathcal{S}$ is an *excluded direction* for $\mathcal{A}$ if $x^* + \lambda t \notin \mathcal{A}$ for all $\lambda > 0$.

2.1 Smoothness Conditions

We now list various definitions relating to the smoothness of a function.

Definition 1 Directionally differentiable. Let $G : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$, with Ω open, $x \in \Omega$, and $d \in \mathbb{R}^n$. If the limit

$$\lim_{t \downarrow 0} \frac{G(x+td) - G(x)}{t} \tag{2}$$

exists in $\mathbb{R}^p$, we say that G has a *directional derivative* at x along d and we denote this limit by $G'(x;d)$. If $G'(x;d)$ exists for every d in a neighborhood of the origin, we say that G is *directionally differentiable* at x.

Definition 2 B-differentiable. ([6, Definition 3.1.2]) $G : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$, with Ω open, is *B(ouligand)-differentiable* at $x \in \Omega$ if G is Lipschitz continuous in a neighborhood of x and directionally differentiable at x.

Definition 3 Strongly semismooth. ([6, Definition 7.4.2]) Let $G : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^p$, with Ω open, be locally Lipschitz continuous on Ω. We say that G is *strongly semismooth* at $\bar{x} \in \Omega$ if G is directionally differentiable near $\bar{x}$ and

$$\limsup\nolimits_{\bar{x} \neq x \to \bar{x}} \frac{\|G'(x; x-\bar{x}) - G'(\bar{x}; x-\bar{x})\|}{\|x-\bar{x}\|^2} < \infty.$$

Further, if G is strongly semismooth at every $\bar{x} \in \Omega$, we say that G is strongly semismooth on Ω.

If G is (strongly) semismooth at $\bar{x}$, then it is B-differentiable at $\bar{x}$. Further, if G is B-differentiable at $\bar{x}$, then $G'(\bar{x}; \cdot)$ is Lipschitz continuous [19]. Hence, for $F' : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ strongly semismooth at x^*, there is some L_{x^*} such that

$$\|(F')'(x^*; h_1) - (F')'(x^*; h_2)\| \leq L_{x^*} \|h_1 - h_2\|. \tag{3}$$

Provided F' is strongly semismooth at x^* and $\|x - x^*\|$ is sufficiently small, we have the following crucial estimate from equation (7.4.5) of [6].

$$F'(x) = F'(x^*) + (F')'(x^*; x - x^*) + O(\|x - x^*\|^2). \tag{4}$$

(We use $p = n^2$ in order to apply Definition 3 to F'.)

2.2 2-regularity

For $F : \mathbf{R}^n \to \mathbf{R}^n$, suppose x^* is a singular solution of $F(x) = 0$ and F' is strongly semismooth at x^*. We define $N := \ker F'(x^*)$. Let $N_\perp$ denote the complement of N, such that $N \oplus N_\perp = \mathbf{R}^n$, and let $N_* := \ker F'(x^*)^T$ with complement $N_{*\perp}$. We denote by P_N, $P_{N_\perp}$, P_{N_*}, and $P_{N_{*\perp}}$ the orthogonal projections onto N, $N_\perp$, N_*, and $N_{*\perp}$ respectively, while $(\cdot)|_N$ denotes the restriction map to N. Let $m := \dim N > 0$.

We say that F satisfies *2-regularity* for some $d \in \mathbf{R}^n$ at a solution x^* if

$$(P_{N_*}F')'(x^*; d)|_N \text{ is nonsingular.} \tag{5}$$

The 2-regularity conditions of Reddien [20], Decker and Kelley [3], and Griewank [8] require (5) to hold for certain $d \in N$. In fact, this property first appeared in the literature as $(P_{N_*}F''(x^*)d)|_N$; the form in (5) was introduced by Izmailov and Solodov in [11]. By applying P_{N_*} to F' before taking the directional derivative, the theory of 2-regularity may be applied to problems for which $P_{N_*}F'$ is directionally differentiable but F' is not [13].

Decker and Kelley [3] and Reddien [20] use the following definition of 2-regularity, which we call $2^\forall$-regularity.

Definition 4 $2^\forall$-regularity. $2^\forall$*-regularity* holds for F at x^* if (5) holds for every $d \in N \setminus \{0\}$.

$2^\forall$-regularity implies (geometric) isolation of the solution x^* [20,3]. For F twice differentiable at x^*, Decker and Kelley note that the $2^\forall$-regularity condition implies that the dimension of N is at most 2 [4].

Next, we define a weaker 2-regularity that can hold regardless of the dimension of N or whether x^* is isolated.

Definition 5 2^{ae}-regularity. 2^{ae}*-regularity* holds for F at x^* if (5) holds for almost every $d \in N$.

Weaker still is the condition we call 2^1-regularity.

Definition 6 2^1-regularity. 2^1*-regularity* holds for F at x^* if (5) holds for some $d \in N$.

For the case in which F is twice Lipschitz continuously differentiable, Griewank shows that 2^1-regularity and 2^{ae}-regularity are actually equivalent [8, p. 110]. This property fails to hold under the weaker smoothness conditions of this work. For example, the solution $x^* = [0, 1]^T$ of the smooth nonlinear equations reformulation (7) of the nonlinear complementarity problem affknot1 (defined in Appendix A) is 2^1-regular but not 2^{ae}-regular.

Izmailov and Solodov introduce a regularity condition and prove that it implies that x^* is an isolated solution [11, Theorem 5(a)]. The following form of this condition, which we call 2^T-regularity, is specific to our case $F : \mathbf{R}^n \to \mathbf{R}^n$ and is due to Daryina, Izmailov, and Solodov [1, Def. 2.1]. Consider the set

$$T_2 := \{d \in N \mid (P_{N_*} F')'(x^*; d)d = 0\}. \tag{6}$$

Definition 7 $\mathbf{2^T}$-regularity [1, Def. 2.1]. 2^T*-regularity* holds for F at x^* if $T_2 = \{0\}$.

As can be seen from Table 1 in Section 7, neither 2^T-regularity nor 2^{ae}-regularity implies the other. If dim $N = 1$, then 2^T-regularity is equivalent to $2^\forall$-regularity (which is trivially equivalent to 2^{ae}-regularity in this case).

3 Prior Work

In this section, we summarize briefly the prior work most relevant to this paper.

2-regularity Conditions. 2-regularity has been applied to a variety of uses including error bounds, implicit function theorems, and optimality conditions [11,13]. We focus on the use of 2-regularity conditions to prove convergence of Newton-like methods to singular solutions. As explained in Subsection 2.2, such conditions concern the behavior of the directional derivative of F' on the null spaces N and N_* of $F'(x^*)$ and $F'(x^*)^T$, respectively.

The 2^1-regularity condition (Definition 6) was used in [21] by Reddien and in [10] by Griewank and Osborne. The proofs therein show convergence of Newton's method (at a linear rate of 1/2) only for starting points x_0 such that $x_0 - x^*$ lies approximately along the particular direction d for which the nonsingularity condition (5) holds.

The more stringent $2^\forall$-regularity condition (Definition 4) was used by Decker and Kelley [3] to prove linear convergence of Newton's method from starting points in a particular truncated cone around N. The convergence analysis given for $2^\forall$-regularity [20,3,2] is much simpler than the analysis presented in Griewank [8], and in the current paper, under weaker smoothness assumptions.

Griewank [8] proves convergence of Newton's method from all starting points in a starlike domain with respect to x^*. If 2^1-regularity holds at x^*, then the starlike domain is nonempty. As mentioned in Subsection 2.2, 2^1-regularity is equivalent to 2^{ae}-regularity when F is twice Lipschitz continuously differentiable at x^*. In this case, 2^{ae}-regularity implies that the starlike domain is "dense" near x^* in the sense that the set of excluded directions has measure zero—a much more general set than the cones around N analyzed prior to that time.

Acceleration Techniques. When iterates $\{x_k\}$ generated by a Newton-like method converge to a singular solution, the error $x_k - x^*$ lies predominantly in the null space N of $F'(x^*)$. Acceleration schemes typically attempt to stay within a cone around N while lengthening ("overrelaxing") some or all of the Newton steps.

We discuss several of the techniques proposed in the early 1980s. All require starting points whose error lies in a cone around N, and all assume three times differentiability of F. Decker and Kelley [4] prove superlinear convergence for an acceleration scheme in which every second Newton step is essentially doubled in length along the subspace N. Their technique requires $2^{\forall}$-regularity at x^*, an estimate of N, and a nonsingularity condition over N on the third derivative of F at x^*. Decker, Keller, and Kelley [2] prove superlinear convergence when every third step is overrelaxed, provided that 2^1-regularity holds at x^* and the third derivative of F at x^* satisfies a nonsingularity condition on N. Kelley and Suresh [16] prove superlinear convergence of an accelerated scheme under less stringent assumptions. If 2^1-regularity holds at x^* and the third derivative of F at x^* is bounded over the truncated cone about N, then overrelaxing every other step by a factor approaching 2 results in superlinear convergence.

By contrast, the acceleration technique given in Section 5 of our paper does not require the starting point x_0 to be in a cone about N, and requires only strong semismoothness of F' at x^*. On the other hand, we obtain only fast linear convergence. We believe, however, that our analysis can be extended to use a scheme like that of Kelley and Suresh [16], increasing the overrelaxation factor to achieve superlinear convergence.

Smooth Nonlinear-Equations Reformulation of the NCP. In the latter part of this paper, we discuss a nonlinear-equations reformulation of the NCP Ψ based on the function $\psi_s(a,b) := 2ab - (\min(0, a+b))^2$, which has the property that $\psi_s(a,b) = 0$ if and only if $a \geq 0$, $b \geq 0$, and $ab = 0$. The function $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ is defined by

$$\Psi_i(x) := 2x_i f_i(x) - (\min(0, x_i + f_i(x)))^2, \quad i = 1, 2, \ldots, n. \tag{7}$$

This reformulation is apparently due to Evtushenko and Purtov [5] and was studied further by Kanzow [15]. The first derivative Ψ' is strongly semismooth at a solution x^* if f' is strongly semismooth at x^*. At a solution x^* for which $x_i^* = f_i(x^*) = 0$ for some i, x^* is a singular root of Ψ and Ψ fails to be twice differentiable.

Recently, Izmailov and Solodov [11–13] and Daryina, Izmailov, and Solodov [1] have investigated the properties of the mapping Ψ and designed algorithms around it. (Some of their investigations, like ours, have taken place in the more general setting of a mapping F for which F' has semismoothness properties.) In particular, Izmailov and Solodov [11,13] show that an error bound for NCPs holds whenever 2^T-regularity holds. Using this error bound to classify the indices $i = 1, 2, \ldots, n$, Daryina, Izmailov, and Solodov [1] present an active-set Gauss-Newton-type method for NCPs. They prove superlinear convergence to singular points which satisfy 2^T-regularity as well as another condition known as weak regularity, which requires full rank of a certain submatrix of $f'(x^*)$. These conditions are weaker than those required for superlinear convergence of known nonsmooth-nonlinear-equations reformulations of NCP. In [12], Izmailov and Solodov augment the reformulation $\Psi(x) = 0$ by adding a nonsmooth function containing second-order information. They apply the generalized Newton's method to the resulting function and prove superlinear convergence under 2^T-regularity and another condition called quasi-regularity. The quasi-regularity condition resembles the 2-regularity condition for the NCP; their relationship is discussed in Subsection 6.3 below.

In contrast to the algorithms of [1] and [12], the approach we present in this work has fast linear convergence rather than superlinear convergence. Our regularity conditions are comparable and may be weaker in some cases. (For example, the problem munson4 in Appendix A satisfies both 2^T-regularity and 2^{ae}-regularity but not weak regularity.) We believe that our algorithm has the advantage of simplicity. Near the solution, it modifies Newton's method only by incorporating a simple check to detect linear convergence and possibly overrelaxing every second step. There is no need to classify the constraints, add "bordering" terms, or switch to a different step computation strategy in the final iterations.

4 Convergence of the Newton Step to a Singularity

Griewank [8] extended the work of others [20, 3] to prove local convergence of Newton's method from a starlike domain $\mathcal{R}$ of a singular solution x^* of $F(x) = 0$. Specialized to the case of $k = 1$ (Griewank's notation), he assumes that $F''(x)$ is Lipschitz continuous near x^* and that x^* is a 2^1-regular solution. Griewank's convergence analysis shows that the first Newton step takes the initial point x_0 from the original starlike domain $\mathcal{R}$ into a simpler starlike domain $\mathcal{W}_s$, a wedge around a certain vector s in the null space N. The domain $\mathcal{W}_s$ is similar to the domains of convergence found in earlier works (Reddien [20], Decker and Kelley [3]). Linear convergence is then proved inside $\mathcal{W}_s$.

For F twice continuously differentiable, the convergence domain $\mathcal{R}$ is much larger than $\mathcal{W}_s$. In fact, the set of directions excluded from $\mathcal{R} - x^*$ has zero measure. As a result, the error in the initial iterate with respect to x^* need not lie near the null space N [8].

In this section, we weaken the smoothness assumption of Griewank in replacing the second derivative of F in (5) by a directional derivative of F'. Our assumptions follow:

Assumption 1 *For $F : \mathbf{R}^n \to \mathbf{R}^n$, x^* is a singular, 2^1-regular solution of $F(x) = 0$ and F' is strongly semismooth at x^*.*

We show that Griewank's convergence results hold under this assumption.

Theorem 1 *Suppose Assumption 1 holds. There exists a starlike domain $\mathcal{R}$ about x^* such that, if Newton's method for $F(x)$ is initialized at any $x_0 \in \mathcal{R}$, the iterates converge linearly to x^* with rate $1/2$. If the problem is converted* *to standard form (8) and* $x_0 = \rho_0 t_0$, *where* $\rho_0 = \|x_0\| > 0$ *and* $t_0 \in \mathcal{S}$, *then the iterates converge inside a cone with axis* $g(t_0)/\|g(t_0)\|$, *for* g *defined in (30).*

Only a few modifications to Griewank's proof [8] are necessary. We use the properties (3) and (4) to show that F is smooth enough for the main steps in the proof to hold. Finally, we make an insignificant change to a constant required by the proof due to a loss of symmetry in $\mathcal{R}$. (Symmetry is lost in moving from derivatives to directional derivatives because directional derivatives are positively but not negatively homogeneous.) The proof in [8] also considers regularities larger than 2, for which higher derivatives are required. We restrict our discussion to 2-regularity because we are interested in the application to a nonlinear-equations reformulation of NCP, for which such higher derivatives are unavailable.

In the remainder of this section, we develop some necessary preliminaries, discuss domains of invertibility of the Jacobian and convergence of the Newton iterates, analyze the form of a Newton step, and finally sketch the proof of Theorem 1. The proof is presented in full in the extended technical report [18], where its points of departure from Griewank's proof are highlighted.

4.1 Preliminaries

For simplicity of notation, we start by standardizing the problem. The Newton iteration is invariant with respect to nonsingular linear transformations of F and nonsingular affine transformations of the variables x. As a result, we can assume that

$$x^* = 0, \;\; F'(x^*) = F'(0) = (I - P_{N_*}), \text{ and } N_* = \mathbf{R}^m \times \{0\}^{n-m}. \tag{8}$$

(We revoke assumption (8) in our discussion of the NCP in Sections 6 and 7.)

For $x \in \mathbf{R}^n \setminus \{0\}$, we write $x = x^* + \rho t = \rho t$, where $\rho = \|x\|$ is the 2-norm distance to the solution and $t = x/\rho$ is a direction in the unit sphere $\mathcal{S}$. From the third assumption in (8), we have

$$P_{N_*} = \begin{bmatrix} I_{m\times m} & 0_{m\times n-m} \\ 0_{n-m\times m} & 0_{n-m\times n-m} \end{bmatrix},$$

where I represents the identity matrix and 0 the zero matrix, with subscripts indicating their dimensions. By substituting in the second assumption of (8), we obtain

$$F'(0) = \begin{bmatrix} 0_{m\times m} & 0_{m\times n-m} \\ 0_{n-m\times m} & I_{n-m\times n-m} \end{bmatrix}. \tag{9}$$

Since $F'(0)$ is symmetric, the null space N is identical to N_*.

Using (8), we partition $F'(x)$ as follows:

$$F'(x) = \begin{bmatrix} P_{N_*}F'(x)|_N & P_{N_*}F'(x)|_{N_\perp} \\ P_{N_{*\perp}}F'(x)|_N & P_{N_{*\perp}}F'(x)|_{N_\perp} \end{bmatrix} =: \begin{bmatrix} B(x) & C(x) \\ D(x) & E(x) \end{bmatrix}.$$

In conformity with the partitioning in (9), the submatrices B, C, D, and E have dimensions $m \times m, m \times n-m, n-m \times m$, and $n-m \times n-m$, respectively. Using $x^* = 0$, we define

$$\bar{B}(x) = \bar{B}(x - x^*) = (P_{N_*}F')'(x^*; x - x^*)|_N = (P_{N_*}F')'(0; x)|_N, \tag{10a}$$

$$\bar{C}(x) = \bar{C}(x - x^*) = (P_{N_*}F')'(x^*; x - x^*)|_{N_\perp} = (P_{N_*}F')'(0; x)|_{N_\perp}. \tag{10b}$$

From $x = \rho t$, the expansion (4) with $x^* = 0$ yields

$$\begin{aligned} B(x) &= \bar{B}(x) + O(\rho^2) = \rho\bar{B}(t) + O(\rho^2), \\ C(x) &= \bar{C}(x) + O(\rho^2) = \rho\bar{C}(t) + O(\rho^2), \\ D(x) &= O(\rho), \quad \text{and} \quad E(x) = I + O(\rho). \end{aligned} \tag{11}$$

Note that the constants that bound the $O(\cdot)$ terms in these expressions can be chosen independently of t, by compactness of $\mathcal{S}$. This is the first difference between our analysis and Griewank's analysis; we use (4) to arrive at (11), while he uses Taylor's theorem.

For some $r_b > 0$, E is invertible for all $\rho < r_b$ and all $t \in \mathcal{S}$, with $E^{-1}(x) = I + O(\rho)$. Invertibility of $F'(x)$ is equivalent to invertibility of the Schur complement of $E(x)$ in $F'(x)$, which we denote by $G(x)$ and define by

$$G(x) := B(x) - C(x)E(x)^{-1}D(x).$$

This claim follows from the determinant formula

$$\det(F'(x)) = \det(G(x))\det(E(x)).$$

By reducing r_b if necessary to apply (11), we have

$$G(x) = B(x) + O(\rho^2) = \rho\bar{B}(t) + O(\rho^2). \tag{12}$$

Hence,

$$\det(F'(x)) = \rho^m \det\bar{B}(t) + O(\rho^{m+1}).$$

As in the proof of [8, Lemma 3.1 (iii)], we note that all but the smallest m singular values of $F'(x)$ are close to 1 in a neighborhood of x^*. Letting $\nu(s)$ denote the smallest singular value of $F'(s)$, we have by the expression above that

$$\nu(\rho t) = O((\det F'(\rho t))^{1/m}) = \begin{cases} O(\rho), & \text{if } \bar{B}(t) \text{ is nonsingular,} \\ o(\rho), & \text{if } \bar{B}(t) \text{ is singular.} \end{cases} \tag{13}$$

For later use, we define γ to be the smallest positive constant such that

$$\|G(x) - \rho\bar{B}(t)\| \le \gamma\rho^2, \quad \text{for all } x = \rho t, \text{ all } t \in \mathcal{S}, \text{ and all } \rho < r_b.$$

Following Griewank [8], we define the function $\sigma(t)$ to be the minimum of 1 and the L_2 operator norm of the smallest singular value of $\bar{B}(t)$, that is,

$$\sigma(t) := \begin{cases} 0 & \text{if } \bar{B}(t) \text{ is singular} \\ \min(1, \|\bar{B}^{-1}(t)\|^{-1}) & \text{otherwise.} \end{cases} \tag{14}$$

It is a fact from linear algebra that the individual singular values of a matrix vary continuously with respect to perturbations of the matrix [7, Theorem 8.6.4]. By (3), $\bar{B}(t)$ is Lipschitz continuous in t, so that $\sigma(t)$ is continuous in t. This is the second difference between our analysis and Griewank's analysis: We require (3) to prove continuity of $\sigma(t)$, while he uses the fact that $\bar{B}(t)$ is linear in t which holds under his stronger smoothness assumptions.

Let

$$\Pi_0(d) := \det \bar{B}(d), \qquad \text{for } d \in \mathbb{R}^n. \tag{15}$$

In contrast to the smooth case considered by Griewank, $\Pi_0(d)$ is not a homogeneous polynomial in d, but rather a positively homogeneous, piecewise-smooth function. Hence, 2^1-regularity does not necessarily imply 2^{ae}-regularity. Since the determinant is the product of singular values, we can use the same reasoning as for $\sigma(t)$ to deduce that $\Pi_0(t)$ is continuous in t for $t \in \mathcal{S}$.

4.2 Domains of Invertibility and Convergence

In this section we define the domains $\mathcal{W}_s$ and $\mathcal{R}$. The definitions of $\mathcal{W}_s$ and $\mathcal{R}$ depend on several functions that we first introduce. If we define $\min(\emptyset) = \pi$, the angle

$$\phi(s) := \frac{1}{4} \min\{\cos^{-1}(t^T s) \,|\, t \in \mathcal{S} \cap {\Pi_0}^{-1}(0)\}, \;\; \text{for } s \in N \cap \mathcal{S} \tag{16}$$

is a well defined, nonnegative continuous function, bounded above by $\frac{\pi}{4}$. For the smooth case considered by Griewank, if $t \in {\Pi_0}^{-1}(0)$, then $-t \in {\Pi_0}^{-1}(0)$ and the maximum angle if ${\Pi_0}^{-1}(0) \neq \emptyset$ is $\frac{\pi}{2}$. This assertion is no longer true in our case; the corresponding maximum angle is π. Hence, we have defined $\min(\emptyset) = \pi$ (instead of Griewank's definition $\min(\emptyset) = \frac{\pi}{2}$) and the coefficient of $\phi(s)$ is $\frac{1}{4}$ instead of $\frac{1}{2}$. This is the third and final difference between our analysis and Griewank's analysis. Now, $\phi^{-1}(0) = N \cap \mathcal{S} \cap {\Pi_0}^{-1}(0)$ because the set $\{s \in \mathcal{S} \,|\, \Pi_0(s) \neq 0\}$ is open in $\mathcal{S}$ since $\Pi_0(\cdot)$ is continuous on $\mathcal{S}$, by (3).

In [8, Lemma 3.1], Griewank defines the auxiliary starlike domain of invertibility $\bar{\mathcal{R}}$,

$$\bar{\mathcal{R}} := \{x = \rho t \,|\, t \in \mathcal{S},\, 0 < \rho < \bar{r}(t)\}, \tag{17}$$

where

$$\bar{r}(t) := \min\left\{ r_b, \frac{1}{2}\gamma^{-1}\sigma(t) \right\}. \tag{18}$$

Note that the excluded directions, $t \in \mathcal{S}$ for which $\sigma(t) = 0$, are the directions along which the smallest singular value of the determinant of $F'(\rho t)$ is $o(\rho)$ by (13) and (14).

As in [8, Lemma 5.1], we define

$$\hat{r}(s) := \min\{\bar{r}(t) \,|\, t \in \mathcal{S},\, \cos^{-1}(t^T s) \leq \phi(s)\}, \;\; \text{for } s \in N \cap \mathcal{S} \tag{19}$$

and

$$\hat{\sigma}(s) := \min\{\sigma(t) \,|\, t \in \mathcal{S},\ \cos^{-1}(t^T s) \leq \phi(s)\},\ \text{ for } s \in N \cap \mathcal{S}. \tag{20}$$

These minima exist and both are nonnegative and continuous on $\mathcal{S} \cap N$ with $\hat{\sigma}^{-1}(0) = \hat{r}^{-1}(0) = \phi^{-1}(0)$. Note that since $\sigma(t) \leq 1$ by definition, we have $\hat{\sigma}(s) \leq 1$ for $s \in N \cap \mathcal{S}$.

Let c be the positive constant defined by

$$c := \max\{\|\bar{C}(t)\| + \sigma(t) \mid t \in \mathcal{S}\}. \tag{21}$$

In the following, we use the abbreviation

$$q(s) := \frac{1}{4} \sin \phi(s) \leq \frac{1}{4},\ \text{ for } s \in N \cap \mathcal{S}. \tag{22}$$

We define the angle $\hat{\phi}(s)$, for which $0 \leq \hat{\phi}(s) \leq \pi/2$, by the equality

$$\sin \hat{\phi}(s) := \min \left\{ \frac{q(s)}{c/\hat{\sigma}(s) + 1 - q(s)}, \frac{2\delta \hat{r}(s)}{(1 - q(s))\hat{\sigma}^2(s)} \right\},\ \text{ for } s \in N \cap \mathcal{S}, \tag{23}$$

where δ is a problem-dependent, positive number to be specified in (39).

We now define

$$\hat{\rho}(s) := \frac{(1 - q(s))\hat{\sigma}^2(s)}{2\delta} \sin \hat{\phi}(s),\ \text{ for } s \in N \cap \mathcal{S}. \tag{24}$$

Both $\hat{\phi}$ and $\hat{\rho}$ are nonnegative and continuous on $N \cap \mathcal{S}$ with

$$\hat{\phi}^{-1}(0) = \hat{\rho}^{-1}(0) = \phi^{-1}(0) = \Pi_0^{-1}(0) \cap N \cap \mathcal{S}. \tag{25}$$

Now we can define the starlike domain $\mathcal{W}_s$,

$$\mathcal{W}_s := \{x = \rho t \,|\, t \in \mathcal{S},\ \cos^{-1}(t^T s) < \hat{\phi}(s), 0 < \rho < \hat{\rho}(s)\}, \tag{26}$$

and the starlike domain $\mathcal{I}_s$,

$$\mathcal{I}_s := \{x = \rho t \mid t \in \mathcal{S}, \cos^{-1}(t^T s) < \phi(s), 0 < \rho < \hat{\rho}(s)\}. \tag{27}$$

By the first inequality in (23), $\sin \hat{\phi}(s) \leq \sin \phi(s)$. Since both $\hat{\phi}(s)$ and $\phi(s)$ are acute angles, we have $\hat{\phi}(s) \leq \phi(s)$ and thus $\mathcal{W}_s \subseteq \mathcal{I}_s$. For $s \in \mathcal{S} \cap N$, $\mathcal{W}_s = \emptyset$ if and only if $\Pi_0(s) = 0$. The second implicit inequality in the definition of $\sin \hat{\phi}(s)$, ensures that $\hat{\rho}(s)$ satisfies

$$\hat{\rho}(s) \leq \hat{r}(s) \leq \bar{r}(t) \leq r_b,\ \text{ for all } t \in \mathcal{S} \text{ with } \cos^{-1} t^T s \leq \phi(s). \tag{28}$$

It follows that

$$\mathcal{I}_s \subset \bar{\mathcal{R}},\ \text{ for all } \ s \in \mathcal{S} \cap N \setminus \Pi_0^{-1}(0). \tag{29}$$

(The justification given in [8] that $\hat{r}(s) \leq \bar{r}(s)$ is insufficient.)

Consider the positively homogeneous vector function $g : (\mathbb{R}^n \setminus \Pi_0^{-1}(0)) \to N \subseteq \mathbb{R}^n$,

$$g(x) = \rho g(t) = \begin{bmatrix} I & \bar{B}^{-1}(t)\bar{C}(t) \\ 0 & 0 \end{bmatrix} x. \tag{30}$$

It is shown in (40) that the Newton iteration from a point x near x^* is, to first order, the map $x^* + \frac{1}{2}g(x)$, provided $g(x)$ is defined at x.

The starlike domain of convergence $\mathcal{R}$, which lies inside the domain of invertibility $\bar{\mathcal{R}}$, is defined as follows (where $x = \rho t$ as usual):

$$\mathcal{R} := \{x = \rho t \,|\, t \in \mathcal{S},\, 0 < \rho < r(t)\}, \tag{31}$$

where

$$r(t) := \min \left\{ \bar{r}(t), \frac{\sigma^2(t)\hat{\rho}(s(t))}{2\delta r_b + c\sigma(t) + \sigma^2(t)}, \frac{\|g(t)\|\sigma^2(t)\sin\hat{\phi}(s(t))}{2\delta} \right\}, \tag{32}$$

where we define

$$s(t) := \frac{g(t)}{\|g(t)\|} \in N \cap \mathcal{S},$$

and δ is the constant to be defined in (39). (The factor of 2, or $k+1$ for the general case, is missing from the denominator of the second term in the definition of $r(t)$ in [8] but should have been included, as it is necessary for the proof of convergence.)

We conclude this subsection by characterizing the excluded directions of $\mathcal{R}$, that is, $t \in \mathcal{S}$ for which $r(t) = 0$. By the definition of $r(t)$ (32), these are directions for which at least one of $\bar{r}(t)$, $\sigma(t)$, $\|g(t)\|$, $\hat{\rho}(s(t))$, or $\sin\hat{\phi}(s(t))$ is zero. Let us inspect each of these possibilities. By definition, $\bar{r}(t)$ (18) is zero if and only if $\sigma(t)$ is zero. If $\sigma(t)$ is nonzero, that is, $t \notin \Pi_0^{-1}(0)$ then $g(t)$ is well defined. If additionally $\|g(t)\| \neq 0$, then $s(t)$ is well defined. Since $s(t) \in N \cap \mathcal{S}$, by (25) $\hat{\rho}(s(t))$ or $\sin\hat{\phi}(s(t))$ is zero if and only if $s(t) \in \Pi_0^{-1}(0)$. To summarize, $r(t)$ is zero for $t \in S$ if and only if one of the following conditions is true:

$$t \in \Pi_0^{-1}(0), \quad g(t) = 0, \quad \text{or} \quad g(t)/\|g(t)\| \in \Pi_0^{-1}(0). \tag{33}$$

The first condition fails if t is 2-regular (5). Likewise, the third condition fails if $g(t)/\|g(t)\|$ is 2-regular. Let us consider the second condition. For $d \in \mathbb{R}^n \setminus \Pi_0^{-1}(0)$, by the definition of g (30) we have

$$g(d) = 0 \Leftrightarrow \bar{B}(d)d_N + \bar{C}(d)d_{N_\perp} = 0,$$

where d_N is the orthogonal projection of d onto N and $d_{N_\perp}$ is the orthogonal projection of d onto $N_\perp$. By the definitions (10a) and (10b), we have

$$g(d) = 0 \Leftrightarrow (P_{N_*}F')'(x^*; d)d = 0, \quad \text{for } d \in \mathbb{R}^n \setminus \Pi_0^{-1}(0). \tag{34}$$

The right-hand side of this condition is identical to the condition defining the set T_2 (6), though the domain of d differs. Due to the limited smoothness of F, it is possible for either Π_0, g, or $\Pi_0(g)$ to map a set of positive measure in $\mathbb{R}^n$ to 0. This is despite the facts that Π_0 g, and $\Pi_0(g)$ may be nonzero elsewhere, Π_0 and g are continuous and positively homogeneous, and g is the identity on its range N.

4.3 The Form of a Newton Step from $\mathbf{x} \in \bar{\mathcal{R}}$

The content of this subsection is taken directly from [8] (with k set to 1); we include it here for readability of this section and for further reference in Section 5.

We consider the form of the Newton step from a point $x = \rho t$ in the domain of invertibility $\bar{\mathcal{R}}$ defined in (17) to the point $\bar{x}$.

$$\bar{x} = x - F'(x)^{-1}F(x). \tag{35}$$

For $x \in \bar{\mathcal{R}}$, we have $\sigma(t) > 0$. In the remainder of this discussion, we drop the argument t from $\sigma(t)$ and dependence of various matrix quantities on x. Using positivity of σ and (11), it can be checked that the following expressions from [8] are also true here.

$$F'(x)^{-1} = \begin{bmatrix} G^{-1} & -G^{-1}CE^{-1} \\ -E^{-1}DG^{-1} & E^{-1} + E^{-1}DG^{-1}CE^{-1} \end{bmatrix},$$

(see [8, (12)]),where

$$G^{-1}(x) = \rho^{-1}\bar{B}^{-1}(t) + \sigma^{-2}O(\rho^0) = \sigma^{-2}O(\rho^{-1}), \tag{36}$$

(see [8, (13)]). As in the proof of [8, Lemma 4.1] with $k = 1$, we have

$$F(x) = \begin{bmatrix} \frac{1}{2}G + O(\rho^2) & \frac{1}{2}C + O(\rho^2) \\ \frac{1}{2}D + O(\rho^2) & E + O(\rho) \end{bmatrix} x.$$

Using (11) to aggregate the order terms, as in Griewank [8], we have

$$F'(x)^{-1}F(x) = \begin{bmatrix} \frac{1}{2}I + \|G^{-1}\|O(\rho^2) & -\frac{1}{2}G^{-1}C + \|G^{-1}\|O(\rho^2) \\ O(\rho) + \|G^{-1}\|O(\rho^3) & I + O(\rho) + \|G^{-1}\|\rho^2 \end{bmatrix} x.$$

Due to (11), (36), and the positivity of σ, we have

$$G^{-1}(x)C(x) = \bar{B}^{-1}(t)\bar{C}(t) + \sigma^{-2}O(\rho).$$

Hence,

$$F'(x)^{-1}F(x) = \tag{37}$$
$$\begin{bmatrix} \frac{1}{2}I + \|G^{-1}\|O(\rho^2) & -\frac{1}{2}\bar{B}^{-1}(t)\bar{C}(t) + \sigma^{-2}O(\rho) + \|G^{-1}\|O(\rho^2) \\ O(\rho) + \|G^{-1}\|O(\rho^3), & I + O(\rho) + \|G^{-1}\|\rho^2 \end{bmatrix} x.$$

Since $\|G^{-1}\| = \sigma^{-2}O(\rho^{-1})$, we can write

$$F'(x)^{-1}F(x) = \begin{bmatrix} \frac{1}{2}I & -\frac{1}{2}\bar{B}^{-1}(t)\bar{C}(t) \\ 0 & I \end{bmatrix} x - e(x), \tag{38}$$

where the remainder vector $e(x)$ can be bounded as follows:

$$\|e(x)\| \le \delta \frac{\|x\|^2}{\sigma(x/\|x\|)^2} = \delta \frac{\rho^2}{\sigma^2}, \tag{39}$$

where the constant δ is positive and finite; in fact, it is a product of finite powers of the constants in the $O(\cdot)$ terms in (11) which, as we have already noted, are finite. The definition of $r(t)$ (32) uses this value of δ.

Using (38), we have

$$\bar{x} = x - F'(x)^{-1}F(x) = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}\bar{B}^{-1}(t)\bar{C}(t) \\ 0 & 0 \end{bmatrix} x + e(x) = \frac{1}{2}g(x) + e(x), \tag{40}$$

where $g(x)$ is defined in (30). In other words, if $x_k = \rho_k t_k$ for $t_k \in \mathcal{S}$ is sufficiently close to x^* and $\sigma(t_k)$ is bounded below by a positive number, then the Newton iterate x_{k+1} satisfies

$$x_{k+1} = \frac{1}{2}g(x_k) + O(\|x_k\|^2).$$

The proof provides a single positive lower bound for $\sigma(t_k)$ for all subsequent Newton iterates $\{x_k\}$. Hence, $\frac{1}{2}g(x_k)$ is a first order approximation to the Newton step from x_k.

4.4 Outline of the Proof of Theorem 1

Consider the Newton iterates $\{x_j = \rho_j t_j\}_{j \geq 0}$ with $t_j \in \mathcal{S}$. For $s \in \mathcal{S}$, let $\psi_j(s)$ denote the angle between t_j and s, that is,

$$\psi_j(s) = \cos^{-1} t_j^T s. \tag{41}$$

Let $s_j = g(x_j)/\|g(x_j)\|$. The first phase of the proof is to show from (40) and the definition of $\mathcal{R}$ that if $x_0 = \rho_0 t_0 \in \mathcal{R}$, then

$$\psi_1(s_0) < \hat{\phi}(s_0) \;\text{ and }\; \rho_1 < \hat{\rho}(s_0),$$

so that $x_1 \in \mathcal{W}_{s_0}$.

The second phase of the proof analyzes convergence from inside the domain $\mathcal{W}_{s_0}$. Letting θ_j denote the angle between x_j and the null space N, it is shown that the sequence of Newton iterates $\{x_j = \rho_j t_j\}_{j \geq 1}$ starting from any point $x_1 \in \mathcal{W}_{s_0}$ maintains the properties

$$\rho_j < \hat{\rho}(s_0), \quad \theta_j < \hat{\phi}(s_0), \quad \psi_j(s_0) < \phi(s_0). \tag{42}$$

By the first and third properties, the iterates remain in $\mathcal{I}_{s_0}$ (27). Further, because of (20), the third property implies that

$$\sigma(t_j) \geq \hat{\sigma}(s_0) > 0, \tag{43}$$

a fact that is used often in the proof. Finally, it can be shown that ρ_j and θ_j go to zero as j goes to infinity and

$$\lim_{j \to \infty} \frac{\rho_{j+1}}{\rho_j} = \frac{1}{2}.$$

These facts are formally stated in Lemma 5.1 of [8]:

Lemma 1 *Suppose Assumption 1 and the standardizations (8) are satisfied. Then for any* $s \in N \cap \mathcal{S} \setminus {\mathit{\Pi}_0}^{-1}(0)$ *the Newton iteration converges linearly with common ratio* $1/2$ *from all points in the nonempty starlike domain* $\mathcal{W}_s$*. Further, the iterates remain in the starlike domain* $\mathcal{I}_s$*.*

Theorem 1 is obtained by combining this result with the analysis of the first step from x_0 to x_1, discussed above.

5 Acceleration of Newton's Method

Overrelaxation is known to improve the rate of convergence of Newton's method converging to a singular solution [9]. The overrelaxed iterate is

$$x_{j+1} = x_j - \alpha F'(x_j)^{-1} F(x_j), \tag{44}$$

where α is some fixed parameter in the range $[1, 2)$. (Of course, $\alpha = 1$ corresponds to the usual Newton step.)

If every step is overrelaxed, it can be shown that the condition $\alpha < \frac{4}{3}$ must be satisfied to ensure convergence and, as a result, the rate of linear convergence is no faster than $\frac{1}{3}$.

In this section, we focus on a technique in which overrelaxation occurs only on every second step; that is, standard Newton steps are interspersed with steps of the form (44) for some fixed $\alpha \in [1, 2)$. Broadly speaking, each pure Newton step refocuses the error along the null space N. Kelley and Suresh prove superlinear convergence for this method when α is systematically increased to 2 as the iterates converge. However, their proof requires the third derivative of F evaluated at x^* to satisfy a boundedness condition and assumes a starting point x_0 that lies near a 2^1-regular direction of x^* in N.

We state our main result and motivate its proof. The lengthy proof appears in full in [18, Section 5].

We assume that 2^1-regularity holds at x^*, F' is strongly semismooth at x^*, and that $x_0 \in \mathcal{R}_\alpha$, where $\mathcal{R}_\alpha$ is a starlike domain defined in (61) whose excluded directions are identical to those of $\mathcal{R}$ defined in Section 4 but whose rays are shorter. In fact, as α is increased to 2, the rays of the starlike domain $\mathcal{R}_\alpha$ shrink in length to zero.

Theorem 2 *Suppose Assumption 1 holds and let* $\alpha \in [1, 2)$*. There exists a starlike domain* $\mathcal{R}_\alpha \subseteq \mathcal{R}$ *about* x^* *such that if* $x_0 \in \mathcal{R}_\alpha$ *and for* $j = 0, 1, 2, \ldots$ *we have*

$$x_{2j+1} = x_{2j} - F'(x_{2j})^{-1} F(x_{2j}) \quad \textit{and} \tag{45}$$

$$x_{2j+2} = x_{2j+1} - \alpha F'(x_{2j+1})^{-1} F(x_{2j+1}), \tag{46}$$

then the iterates $\{x_i\}$ *for* $i = 0, 1, 2, \ldots$ *converge linearly to* x^* *and*

$$\lim_{j \to \infty} \frac{\|x_{2j+2} - x^*\|}{\|x_{2j} - x^*\|} = \frac{1}{2}\left(1 - \frac{\alpha}{2}\right).$$

We first describe a key step of the proof. Since the problem is in standard form, we have from (40) that the Newton step (45) satisfies the following relationships for $x_{2k} \in \bar{\mathcal{R}}$:

$$x_{2k+1} = \frac{1}{2}\begin{bmatrix} I & \bar{B}(t_{2k})^{-1}\bar{C}(t_{2k}) \\ 0 & 0 \end{bmatrix} x_{2k} + e(x_{2k}) = \frac{1}{2} g(x_{2k}) + e(x_{2k}), \tag{47}$$

for all $k \geq 0$, where $g(\cdot)$ is defined in (30) and the remainder term $e(\cdot)$ is defined in (38). As in (39), we have

$$\|e(x_{2k})\| \leq \delta \frac{\rho_{2k}^2}{\sigma_{2k}^2}. \tag{48}$$

For the accelerated Newton step (46), using manipulations similar to those for (40), we have for $x_{2k+1} \in \bar{\mathcal{R}}$ that

$$x_{2k+2} = \begin{bmatrix} (1-\frac{\alpha}{2})I & \frac{\alpha}{2}\bar{B}(t_{2k+1})^{-1}\bar{C}(t_{2k+1}) \\ 0 & (1-\alpha)I \end{bmatrix} x_{2k+1} + \alpha e(x_{2k+1}), \tag{49}$$

for all $k \geq 0$.

By substituting (47) into (49), we obtain

$$\begin{aligned} x_{2k+2} = \frac{1}{2} & \begin{bmatrix} (1-\frac{\alpha}{2})I & \frac{\alpha}{2}\bar{B}(t_{2k+1})^{-1}\bar{C}(t_{2k+1}) \\ 0 & (1-\alpha)I \end{bmatrix} \begin{bmatrix} I & \bar{B}(t_{2k})^{-1}\bar{C}(t_{2k}) \\ 0 & 0 \end{bmatrix} x_{2k} \\ & + \tilde{e}_\alpha(x_{2k}, x_{2k+1}), \end{aligned} \tag{50}$$

where

$$\tilde{e}_\alpha(x_{2k}, x_{2k+1}) = \begin{bmatrix} (1-\frac{\alpha}{2})I & \frac{\alpha}{2}\bar{B}(t_{2k+1})^{-1}\bar{C}(t_{2k+1}) \\ 0 & (1-\alpha)I \end{bmatrix} e(x_{2k}) + \alpha e(x_{2k+1}). \tag{51}$$

Multiplying the matrices in (50), we have

$$\begin{aligned} x_{2k+2} &= \frac{1}{2}\left(1 - \frac{\alpha}{2}\right) \begin{bmatrix} I & \bar{B}(t_{2k})^{-1}\bar{C}(t_{2k}) \\ 0 & 0 \end{bmatrix} x_{2k} + \tilde{e}_\alpha(x_{2k}, x_{2k+1}) \\ &= \frac{1}{2}\left(1 - \frac{\alpha}{2}\right) g(x_{2k}) + \tilde{e}_\alpha(x_{2k}, x_{2k+1}), \qquad (52) \end{aligned}$$

To bound the remainder term, note that $|1 - \frac{\alpha}{2}| + |1 - \alpha| = \frac{\alpha}{2}$ for $\alpha \in [1, 2)$. Hence, we have from (51) that

$$\begin{aligned} \|\tilde{e}_\alpha(x_{2k}, x_{2k+1})\| &\leq \frac{\alpha}{2}\left(1 + \|\bar{B}(t_{2k+1})^{-1}\|\|\bar{C}(t_{2k+1})\|\right)\|e(x_{2k})\| + \alpha\|e(x_{2k+1})\| \\ &\leq \left(\frac{\sigma_{2k+1} + \|\bar{C}(t_{2k+1})\|}{\sigma_{2k+1}}\right)\delta\frac{\rho_{2k}^2}{\sigma_{2k}^2} + \alpha\delta\frac{\rho_{2k+1}^2}{\sigma_{2k+1}^2} \\ &\qquad \text{from } \alpha < 2, \text{ (14), and (39)} \\ &\leq c\delta\frac{\rho_{2k}^2}{\sigma_{2k+1}\sigma_{2k}^2} + \alpha\delta\frac{\rho_{2k+1}^2}{\sigma_{2k+1}^2} \\ &\qquad \text{from (21)} \\ &\leq \tilde{\delta}\frac{\rho_{2k}^2 + \rho_{2k+1}^2}{\mu_{2k}^3}, \qquad (53) \end{aligned}$$

where

$$\mu_{2k} := \min(\sigma_{2k}, \sigma_{2k+1}) \tag{54}$$

and $\tilde{\delta} := \delta \max(c, \alpha)$. By combining (52) with (53), we obtain

$$\left\| x_{2k+2} - \frac{1}{2}\left(1 - \frac{\alpha}{2}\right) g(x_{2k}) \right\| \le \tilde{\delta} \frac{\rho_{2k}^2 + \rho_{2k+1}^2}{\mu_{2k}^3}. \tag{55}$$

In other words, if $x_{2k} = \rho_{2k} t_{2k}$ with $t_{2k} \in \mathcal{S}$ and $x_{2k+1} = \rho_{2k+1} t_{2k+1}$ with $t_{2k+1} \in \mathcal{S}$ are sufficiently close to x^* and $\sigma(t_{2k})$ and $\sigma(t_{2k+1})$ are bounded below by a positive number, then the accelerated Newton iterate x_{2k+2} satisfies

$$x_{2k+2} = \frac{1}{2}(1 - \frac{\alpha}{2}) g(x_{2k}) + O(\|x_{2k}\|^2).$$

The proof provides a single positive lower bound for $\sigma(t_{2k})$ and $\sigma(t_{2k+1})$ for all subsequent iterates. Hence, $\frac{1}{2}(1 - \frac{\alpha}{2}) g(x_{2k})$ is a first order approximation to the double step achieved by applying a Newton step followed by an accelerated Newton step from x_{2k}.

Before discussing the remainder of the proof, we introduce the following new parameters:

$$q_\alpha(s) := \frac{1 - \alpha/2}{4} \sin \phi(s), \text{ for } s \in N \cap \mathcal{S}, \tag{56}$$

We define the angle $\tilde{\phi}_\alpha(s)$, for which $0 \le \tilde{\phi}_\alpha(s) \le \pi/2$, by the equality

$$\sin \tilde{\phi}_\alpha(s) := \min \left\{ \frac{q_\alpha(s)}{c/\hat{\sigma}(s) + 1 - q_\alpha(s)}, \frac{2\delta \hat{r}(s)}{(1 - q_\alpha(s))\hat{\sigma}^2(s)} \right\}, \text{ for } s \in N \cap \mathcal{S}. \tag{57}$$

We further define

$$\tilde{\rho}_\alpha(s) := \frac{(1 - \alpha/2 - q_\alpha(s))\hat{\sigma}^3(s)}{4\tilde{\delta}} \sin \tilde{\phi}_\alpha(s) \text{ for } s \in N \cap \mathcal{S}, \tag{58}$$

$$\mathcal{W}_{s,\alpha} := \{x = \rho t \,|\, t \in \mathcal{S},\, \cos^{-1}(t^T s) < \tilde{\phi}_\alpha(s), 0 < \rho < \tilde{\rho}_\alpha(s)\}, \tag{59}$$

and

$$\mathcal{I}_{s,\alpha} := \{x = \rho t \,|\, t \in \mathcal{S},\, \cos^{-1}(t^T s) < \phi(s), 0 < \rho < \tilde{\rho}_\alpha(s)\}. \tag{60}$$

It can be shown that $\mathcal{I}_{s_0,\alpha} \subseteq \bar{\mathcal{R}}$. The starlike domain of convergence is defined as follows:

$$\mathcal{R}_\alpha := \{x = \rho t \,|\, t \in \mathcal{S},\, 0 < \rho < r_\alpha(t)\}, \tag{61}$$

where

$$r_\alpha(t) := \min \left\{ \bar{r}(t), \frac{\sigma^2(t)\tilde{\rho}_\alpha(s(t))}{2\delta r_b + c\sigma(t) + \sigma^2(t)}, \frac{\|g(t)\|\sigma^2(t)(1 - \alpha/2)\sin \tilde{\phi}_\alpha(s(t))}{8\delta} \right\} \tag{62}$$

and $s(t) = g(t)/\|g(t)\| \in N \cap \mathcal{S}$.

As in Section 4, the angle between iterate $x_i = \rho_i t_i$ and the null space N is denoted by θ_i, while $\psi_i(s_0)$ denotes the angle between x_i and s_0 (41). The proof of Theorem 2 is by induction. The induction step consists of showing that if

$$\begin{aligned} &\rho_{2k+\iota} < \tilde{\rho}_\alpha(s_0), \quad \theta_{2k+\iota} < \tilde{\phi}_\alpha(s_0), \text{ and } \psi_{2k+\iota}(s_0) < \phi(s_0), \\ &\qquad\qquad \text{for } \iota \in \{1,2\}, \quad \text{all } k \text{ with } 0 \le k < j, \end{aligned} \tag{63}$$

then

$$\rho_{2j+\iota} < \tilde{\rho}_\alpha, \quad \theta_{2j+\iota} < \tilde{\phi}_\alpha(s_0), \text{ and } \psi_{2j+\iota}(s_0) < \phi(s_0) \quad \text{for } \iota \in \{1,2\}. \tag{64}$$

For all $i = 1, 2, \ldots$, the third property in (63) and (64)—$\psi_i(s_0) < \phi(s_0)$—implies the crucial fact that $\sigma(t_i) \ge \hat{\sigma}(s_0) > 0$; see (20) and (43). By the first and third properties, the iterates remain in $\mathcal{I}_{s_0,\alpha}$. Since $\mathcal{I}_{s_0,\alpha} \subseteq \bar{\mathcal{R}}$, the bounds of Subsection 4.3 together with (47) and (49) are valid for our iterates.

The anchor step of the induction argument consists of showing that for $x_0 \in \mathcal{R}_\alpha$, we have $x_1 \in \mathcal{W}_{s_0,\alpha}$ and $x_2 \in \mathcal{I}_{s_0,\alpha}$ with $\theta_2 < \tilde{\phi}_\alpha$. Indeed, these facts yield (63) for $j = 1$.

The convergence rate claimed in the theorem is a byproduct of the proof of the induction step.

6 Application to Nonlinear Complementarity Problems

The nonlinear complementarity problem for the function $f : \mathbb{R}^n \to \mathbb{R}^n$, denoted by NCP($f$), is as follows: Find an $x \in \mathbb{R}^n$ such that

$$0 \le f(x), \;\; x \ge 0, \;\; x^T f(x) = 0.$$

Let x^* be a solution of NCP(f). We assume throughout this section and the next that f' is well defined and strongly semismooth at x^*. We do not standardize the problem (as we did earlier in (8) to simplify the discussions of Sections 4 and 5), as the rescaling and shifting needed to enforce this assumption would complicate this section considerably.

In this section we discuss a nonlinear-equations reformulation for the NCP. We tailor the convergence theorems, Theorems 1 and 2, to this reformulation, interpret the 2-regularity condition for the NCP(f), and provide conditions under which the starlike domain of convergence is "directionally dense" at the solution.

6.1 NCP Notation, Definitions, and Properties

For any matrix $M \in \mathbb{R}^p \times \mathbb{R}^q$ and any sets $\mathcal{U} \subseteq \{1, 2, \ldots, p\}$ and $\mathcal{V} \subseteq \{1, 2, \ldots, q\}$, we write $M_{\mathcal{U},\mathcal{V}}$ to denote the submatrix of M whose rows lie in $\mathcal{U}$ and columns lie in $\mathcal{V}$. To The row submatrix corresponding to indices in the set $\mathcal{U}$ is denoted by $M_{\mathcal{U}}$. We denote the number of elements in any set $\mathcal{U}$

by $|\mathcal{U}|$. Let e_i denote the ith column of the identity matrix. In this section, we use the notation $\langle \cdot, \cdot \rangle$ to denote the inner product between two vectors. For any $x \in \mathbb{R}^n$, we use $\operatorname{diag} x$ to denote the $\mathbb{R}^{n \times n}$ diagonal matrix formed from the components of x.

We define the inactive, biactive, and active index sets, α, β, and γ respectively, at a solution x^* of NCP(f) as follows,

$$\begin{cases} i \in \alpha, \text{ if } x_i^* = 0,\ f(x^*)_i > 0, \\ i \in \beta, \text{ if } x_i^* = 0,\ f(x^*)_i = 0, \\ i \in \gamma, \text{ if } x_i^* > 0,\ f(x^*)_i = 0. \end{cases}$$

6.2 The Nonlinear-Equations Reformulation

We recall the nonlinear-equations reformulation Ψ (7) of the NCP (1), and consider the use of Newton's method for solving $\Psi(x) = 0$. In this section, we establish the structure of null space N and the form of 2-regularity (5) for the NCP function f with the reformulation Ψ, then tailor the local convergence results of Sections 4 and 5 to f with the reformulation Ψ.

Taking the derivative of Ψ, we have

$$\begin{aligned} \Psi_i'(x) = 2\{&(f_i(x) - \min(0, x_i + f_i(x)))e_i \\ &+ (x_i - \min(0, x_i + f_i(x)))f_i'(x)\}, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{65}$$

It can be seen that Ψ' is strongly semismooth when f' is strongly semismooth by applying the following two facts: From [6, Proposition 7.4.4], the composition of strongly semismooth functions is strongly semismooth, and from [6, Proposition 7.4.7], every piecewise-affine map is strongly semismooth.

At the solution x^*, Ψ_i' simplifies to

$$\Psi_i'(x^*) = 2\{f_i(x^*)e_i + x_i^* f_i'(x^*).\}$$

By inspection, we have

$$\begin{cases} \Psi_i'(x^*) = 2f_i(x^*)e_i, & i \in \alpha, \\ \Psi_i'(x^*) = 0, & i \in \beta, \\ \Psi_i'(x^*) = 2x_i^* f_i'(x^*), & i \in \gamma. \end{cases}$$

The null space of $\Psi'(x^*)$ (whose ith row is the transpose of Ψ_i') is

$$N \equiv \ker \Psi'(x^*) = \{\xi \in \mathbb{R}^n \mid f_\gamma'(x^*)\xi = 0,\ \xi_\alpha = 0\}, \tag{66}$$

so that

$$\dim N = \dim \ker f_{\gamma, \beta\cup\gamma}'(x^*).$$

In particular, if $\beta \neq \emptyset$, then $\dim N > 0$ and x^* is a singular solution of $\Psi(x) = 0$. The null space of $\Psi'(x^*)^T$ is

$$\begin{aligned} N_* = \{\xi \in \mathbb{R}^n \mid \xi_\alpha = -(\operatorname{diag} f_\alpha(x^*))^{-1}(f_{\gamma,\alpha}'(x^*))^T(\operatorname{diag} x_\gamma^*)\xi_\gamma, \\ f_{\gamma,\beta\cup\gamma}'(x^*)^T(\operatorname{diag} x_\gamma^*)\xi_\gamma = 0\}. \end{aligned} \tag{67}$$

If rank $f'_{\gamma,\beta\cup\gamma}(x^*) = |\gamma|$, then $N_* = \{\xi \in \mathbb{R}^n \mid \xi_\alpha = 0,\ \xi_\gamma = 0\}$.

The 2-regularity condition (5) for Ψ at x^* and $d \in \mathbb{R}^n$ is

$$(P_{N_*}\Psi')'(x^*;d)|_N \ \text{ is nonsingular.} \tag{68}$$

By direct calculation, we have

$$\frac{1}{2}(\Psi')'_i(x;d) = (\langle f'_i(x), d\rangle - \eta_i)e_i + (d_i - \eta_i)f'_i(x) + (x_i - \min(0, x_i + f_i(x)))(f'_i)'(x;d),$$

where $\eta_i := \min(0, x_i + f_i(x))'(x;d)$. We can calculate this directional derivative using the result [6, Proposition 3.1.6] for the composition of B-differentiable functions.

$$\eta_i = \begin{cases} \min(0, d_i + \langle f'_i(x), d\rangle), & \text{if } x_i + f_i(x) = 0, \\ 0, & \text{if } x_i + f_i(x) > 0, \\ d_i + \langle f'_i(x), d\rangle, & \text{if } x_i + f_i(x) < 0. \end{cases}$$

At a solution x^*, we have $\eta_i = 0$ for $i \in \alpha\cup\gamma$, and $\eta_i = \min(0, d_i + \langle f'_i(x^*), d\rangle)$ for $i \in \beta$. Hence, we have

$$\frac{1}{2}(\Psi'_i)'(x^*;d) = \begin{cases} \langle f'_i(x^*), d\rangle e_i + d_i f'_i(x^*), & i \in \alpha, \\ ((\langle f'_i(x^*), d\rangle - \min(0, d_i + \langle f'_i(x^*), d\rangle))e_i & \\ \quad + (d_i - \min(0, d_i + \langle f'_i(x^*), d\rangle))f'_i(x^*), & i \in \beta, \\ \langle f'_i(x^*), d\rangle e_i + d_i f'_i(x^*) + x^*_i (f'_i)'(x^*;d), & i \in \gamma \end{cases} \tag{69}$$

By noting that for any scalars s_1, s_2 we have

$$s_1 - \min(0, s_2) = s_1 + \max(0, -s_2) = \max(s_1, s_1 - s_2) = -\min(-s_1, s_2 - s_1),$$

we can rewrite (69) as follows

$$\frac{1}{2}(\Psi'_i)'(x^*;d) = \begin{cases} \langle f'_i(x^*), d\rangle e_i + d_i f'_i(x^*), & i \in \alpha, \\ \max(\langle f'_i(x^*), d\rangle, -d_i)e_i - \min(\langle f'_i(x^*), d\rangle, -d_i) f'_i(x^*), & i \in \beta, \\ \langle f'_i(x^*), d\rangle e_i + d_i f'_i(x^*) + x^*_i (f'_i)'(x^*;d), & i \in \gamma. \end{cases} \tag{70}$$

Using the notation

$$r = \operatorname{rank} f'_{\gamma,\beta\cup\gamma}(x^*),$$

we define an orthonormal matrix Z of dimension $|\gamma| \times r$ such that the columns of Z span range $f'_{\gamma,\beta\cup\gamma}(x^*)$, and another orthonormal matrix $Z_\perp$ of dimensions $|\gamma| \times (|\gamma| - r)$ such that the columns of $Z_\perp$ span $\ker f'_{\gamma,\beta\cup\gamma}(x^*)^T$. Note that $[Z \mid Z_\perp]$ is an orthogonal matrix of dimensions $|\gamma| \times |\gamma|$. We note that the matrices Z and $Z_\perp$ are not uniquely defined by the conditions above; there are infinitely many possible choices in general for orthonormal matrices that span the subspaces in question. However the properties discussed below are independent of the particular choices for these matrices.

In the remainder of this section, we often drop the argument x^* from f and f', for clarity.

Proposition 1 *2-regularity (68) holds for* $d \in \mathbf{R}^n$ *at a solution* x^* *of* $\Psi(x) = 0$ *if and only if the matrix*

$$\begin{bmatrix} \left[e_i^T\right]_{i\in\alpha} \\ [\max(\langle f_i', d\rangle, -d_i)e_i - \min(\langle f_i', d\rangle, -d_i) f_i']_{i\in\beta}^T \\ Z^T f_\gamma' \\ Z_\perp^T \left[(f_\gamma')'(x^*; d) + \left[\frac{1}{x_i^*}\langle f_i', d\rangle e_i^T\right]_{i\in\gamma} - \langle f_{\gamma,\alpha}', \left(\operatorname{diag} \frac{d_j}{f_j}\right)_{j\in\alpha} f_\alpha' \rangle \right] \end{bmatrix} \tag{71}$$

is nonsingular. Further, for $d \in N$, *2-regularity holds if and only if the simpler matrix*

$$\begin{bmatrix} \left[e_i^T\right]_{i\in\alpha} \\ [\max(\langle f_i', d\rangle, -d_i)e_i - \min(\langle f_i', d\rangle, -d_i) f_i']_{i\in\beta}^T \\ Z^T f_\gamma' \\ Z_\perp^T (f_\gamma')'(x^*; d) \end{bmatrix} \tag{72}$$

is nonsingular.

Proof Consider any $d \in \mathbf{R}^n$. The claim that $(P_{N_*}\Psi')'(x^*; d)|_N$ is nonsingular for some $d \in \mathbf{R}^n$ (68) is equivalent to

$$P_{N_*}(\Psi')'(x^*; d)v = 0 \text{ and } v \in N \;\Rightarrow\; v = 0.$$

For $v \in N$, we have from (66) and (70) that

$$\frac{1}{2}(\Psi_i')'(x^*; d)v = \begin{cases} d_i\langle f_i', v\rangle, & i \in \alpha \\ \max(\langle f_i', d\rangle, -d_i)v_i - \min(\langle f_i', d\rangle, -d_i)\langle f_i', v\rangle, & i \in \beta \\ \langle f_i', d\rangle v_i + x_i^*\langle (f_i')'(x^*; d), v\rangle, & i \in \gamma. \end{cases} \tag{73}$$

Since N_* is defined in (67) to have the form $\{\xi \in \mathbf{R}^n \mid A\xi = 0\}$ for some matrix A, we have that $P_{N_*}w = 0$ if and only if $w = A^T z$ for some z. That is,

$$\frac{1}{2}(\Psi')'(x^*; d)v = \begin{bmatrix} \operatorname{diag} f_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ (\operatorname{diag} x_\gamma^*) f_{\gamma,\alpha}' & (\operatorname{diag} x_\gamma^*) f_{\gamma,\beta}' & (\operatorname{diag} x_\gamma^*) f_{\gamma,\gamma}' \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_\beta \\ z_\gamma \end{bmatrix}, \tag{74}$$

for some $z \in \mathbf{R}^n$. Hence, by matching components from this expression and (73), we have that $P_{N_*}(\Psi')'(x^*; d)v = 0$ if for some $z \in \mathbf{R}^n$ we have

$$\begin{aligned} & d_i\langle f_i', v\rangle = z_i f_i, && i \in \alpha, \\ & 0 = \max(\langle f_i', d\rangle, -d_i)v_i - \min(\langle f_i', d\rangle, -d_i)\langle f_i', v\rangle, && i \in \beta, \\ & \langle f_i', d\rangle v_i + x_i^*\langle (f_i')'(x^*; d), v\rangle = x_i^*\left[f_{i,\alpha}' z_\alpha + f_{i,\beta}' z_\beta + f_{i,\gamma}' z_\gamma\right], && i \in \gamma. \end{aligned}$$

Rearranging the first equation above yields an expression for z_α, which can be substituted into the third equation to give the following equivalent expressions.

$$0 = \max(\langle f_i', d\rangle, -d_i)v_i - \min(\langle f_i', d\rangle, -d_i)\langle f_i', v\rangle, \qquad i \in \beta, \tag{75a}$$

$$\begin{aligned}&\langle f_i', d\rangle v_i + x_i^* \langle (f_i')'(x^*; d), v\rangle - x_i^* \left\langle f_{i,\alpha}', \operatorname{diag}\, (d_j/f_j)_{j\in\alpha} \langle f_\alpha', v\rangle \right\rangle \\ &\qquad = x_i^* \left[f_{i,\beta}' z_\beta + f_{i,\gamma}' z_\gamma \right], \qquad i \in \gamma.\end{aligned} \tag{75b}$$

Using the definition of the orthonormal matrix Z, we can rewrite (75b) as follows:

$$\left[(1/x_i^*)\langle f_i', d\rangle v_i + \langle (f_i')'(x^*; d), v\rangle - \left\langle f_{i,\alpha}', \operatorname{diag}\, (d_j/f_j)_{j\in\alpha} \langle f_\alpha', v\rangle \right\rangle \right]_{i\in\gamma} = Zt,$$

for some $t \in \mathbb{R}^r$, so that

$$Z_\perp^T \left[\frac{1}{x_i^*} \langle f_i', d\rangle e_i^T + (f_i')'(x^*; d) - \langle f_{i,\alpha}', \operatorname{diag} \left(\frac{d_j}{f_j} \right)_{j\in\alpha} f_\alpha' \rangle \right]_{i\in\gamma} v = 0. \tag{76}$$

Since $v \in N$, we have from (66) that

$$v_\alpha = 0, \tag{77a}$$

$$f_{\gamma,\alpha}' v_\alpha + f_{\gamma,\beta}' v_\beta + f_{\gamma,\gamma}' v_\gamma = 0. \tag{77b}$$

The second condition (77b) is equivalent to

$$\begin{bmatrix} Z^T \\ Z_\perp^T \end{bmatrix} \left[f_{\gamma,\alpha}' \; f_{\gamma,\beta}' \; f_{\gamma,\gamma}' \right] v = 0. \tag{78}$$

Because

$$Z_\perp^T \left[f_{\gamma,\alpha}' \; f_{\gamma,\beta}' \; f_{\gamma,\gamma}' \right] v = \left[Z_\perp^T f_{\gamma,\alpha}' \; 0 \; 0 \right] v = Z_\perp^T f_{\gamma,\alpha}' v_\alpha$$

and $v_\alpha = 0$, the second block row in the system (78) does not add any information and can be dropped. Hence, we can write (77) equivalently as

$$v_\alpha = 0, \qquad Z^T \left[f_{\gamma,\alpha}' \; f_{\gamma,\beta}' \; f_{\gamma,\gamma}' \right] v = 0. \tag{79}$$

By gathering the conditions equivalent to $v \in N$ and $P_{N_*}(\Psi')'(x^*; d)v = 0$, namely, (75a), (76), and (79), we have

$$\begin{bmatrix} \left[e_i^T \right]_{i\in\alpha} \\ \left[\max(\langle f_i', d\rangle, -d_i)e_i - \min(\langle f_i', d\rangle, -d_i) f_i' \right]_{i\in\beta}^T \\ Z^T f_\gamma' \\ Z_\perp^T \left[(f_\gamma')'(x^*; d) + \left[(1/x_i^*)\langle f_i', d\rangle e_i^T \right]_{i\in\gamma} - \langle f_{\gamma,\alpha}', \operatorname{diag}\, (d_j/f_j)_{j\in\alpha} f_\alpha' \rangle \right] \end{bmatrix} v = 0,$$

from which we deduce that $v = 0$ whenever the coefficient matrix in this expression is nonsingular. Hence x^* is 2-regular with respect to $d \in \mathbb{R}^n$ if the matrix (71) is nonsingular. For $d \in N$, we have by the definition of N (66) that $\langle f_i', d\rangle = 0$ for $i \in \gamma$ and $d_\alpha = 0$. Upon applying these simplifications to the above matrix, we have precisely the matrix (72).

Recall that Ψ (7) is 2^1-regular (6) at x^* if $(P_{N_*}\Psi')'(x^*;d)|_N$ is nonsingular for some d in N, that is, if the matrix (72) is nonsingular for some $d \in N$.

The following theorem specializes Theorems 1 and 2 for applying Newton's method to the nonlinear-equations reformulation $\Psi(x)$ of NCP(f).

Theorem 3 *Consider the solution x^* of NCP(f) for $f : \mathbb{R}^n \to \mathbb{R}^n$ with f' strongly semismooth at x^*. Suppose that x^* is a singular solution in the sense that $N = \ker f'_{\gamma,\beta\cup\gamma}(x^*)$ is nontrivial. Suppose also that the matrix (72) is nonsingular for some $d \in N$. Then there exists a starlike domain $\mathcal{R}$ about x^*, such that, if Newton's method for the nonlinear-equations reformulation $\Psi(x)$ is initialized at any $x_0 \in \mathcal{R}$, the iterates converge linearly to x^* with rate $1/2$. Furthermore, if Newton's method is accelerated according to (45) and (46) for some $\alpha \in [1,2)$, then there exists a starlike domain $\mathcal{R}_\alpha \subseteq \mathcal{R}$ about x^*, such that if $x_0 \in \mathcal{R}_\alpha$ then the accelerated iterates $\{x_i\}$ for $i = 0,1,2,\ldots,$ converge linearly to x^* and*

$$\lim_{j\to\infty} \frac{\|x_{2j+2} - x^*\|}{\|x_{2j} - x^*\|} = \frac{1}{2}\left(1 - \frac{\alpha}{2}\right).$$

6.3 2-regularity Conditions for Special Cases of the NCP

In this section we show that the regularity conditions (71) and (72) simplify to more familiar regularity conditions in special cases of the NCP.

Nondegenerate NCP. Consider the case of nondegenerate NCP. We obtain a simpler regularity condition, related to 2-regularity for nonlinear equations, that ensures that 2-regularity holds for some $d \in N$, and hence that the conditions of Theorem 3 are satisfied.

Theorem 4 *Suppose that $\beta = \emptyset$. Then the NCP satisfies 2-regularity for $d \in N$ at the solution x^* if and only if*

$$P_{N^f_{*\gamma}}(f'_{\gamma,\gamma})'(x^*;d)|_{N^f_\gamma} \tag{80}$$

is nonsingular for $d \in N$, where

$$N^f_\gamma = \{\xi_\gamma \in \mathbb{R}^{|\gamma|} \mid f'_{\gamma,\gamma}\xi_\gamma = 0\}, \qquad N^f_{*\gamma} = \{\xi_\gamma \in \mathbb{R}^{|\gamma|} \mid (f'_{\gamma,\gamma})^T\xi_\gamma = 0\}.$$

Proof Let the orthonormal matrices $Z_\perp$ and Z be as in (72), and define two additional orthonormal matrices $\bar{Z}$ and $\bar{Z}_\perp$ such that the columns of $\bar{Z}_\perp$ span $\ker f'_{\gamma,\gamma}$ (and hence the space N^f_γ) and the columns of $\bar{Z}$ span range$\,(f'_{\gamma,\gamma})^T$. We have that $\bar{Z} \in \mathbb{R}^{|\gamma|\times r}$ and that $\bar{Z}_\perp \in \mathbb{R}^{|\gamma|\times(|\gamma|-r)}$ and, by the fundamental theorem of linear algebra, that $\left[\bar{Z} \,|\, \bar{Z}_\perp\right]$ is orthogonal. Specializing 2-regularity for $d \in N$ (72) to the case of $\beta = \emptyset$, we have that 2-regularity is equivalent to nonsingularity of the following matrix for some $d \in N$:

$$\begin{bmatrix} \left[e_i^T\right]_{i\in\alpha} \\ Z^T\left[f'_{\gamma,\alpha}(x^*)\; f'_{\gamma,\gamma}(x^*)\right] \\ Z_\perp^T(f'_\gamma)'(x^*;d) \end{bmatrix} \begin{bmatrix} I_\alpha & 0 \\ 0 & \left[\bar{Z}\; \bar{Z}_\perp\right] \end{bmatrix},$$

where I_α is the identity matrix of dimension $|\alpha|$. By forming the matrix product, we find that it is block lower triangular. Therefore, nonsingularity of the matrix product is equivalent to nonsingularity of the three (square) diagonal blocks, which are

$$I_\alpha, \qquad Z^T f'_{\gamma,\gamma}(x^*)\bar{Z}, \qquad Z_\perp^T (f'_{\gamma,\gamma})'(x^*;d)\bar{Z}_\perp,$$

which have dimensions $|\alpha|$, r, and $|\gamma| - r$, respectively. It is easy to see that $Z^T f'_{\gamma,\gamma}(x^*)\bar{Z}$ is nonsingular by the definition of Z and $\bar{Z}$. Since the columns of $Z_\perp$, as defined earlier, must span the subspace $N^f_{*\gamma}$, and since the columns of $\bar{Z}_\perp$ span the subspace N^f_γ, nonsingularity of $Z_\perp^T (f'_{\gamma,\gamma})'(x^*;d)\bar{Z}_\perp$ is equivalent to condition (80).

Nonlinear Equations. We now consider the case in which $\alpha = \beta = \emptyset$, so that the NCP reduces essentially to a system of nonlinear equations $f(x) = 0$ whose solution is at $x = x^*$. In the nondegenerate case in which $f'_{\gamma,\gamma}(x^*) \equiv f'(x^*)$ has full rank n, we have from definition (66) that $N = \{0\}$, so that x^* is a nonsingular solution and Theorem 3 does not apply.

Consider now the case in which $\alpha = \beta = \emptyset$ but $f'(x^*)$ has rank less than n—essentially the case of singular nonlinear equations. By specializing the discussion of nondegenerate NCP, we have from the definitions in Theorem 4 that

$$N^f = \ker f'(x^*), \qquad N^f_* = \ker f'(x^*)^T,$$

where we have dropped the subscript γ. Hence, 2-regularity is satisfied for some $d \in N$ if

$$P_{N^f_*}(f')'(x^*;d)|_{N^f} \qquad \text{is nonsingular for some } d \in N.$$

This is the 2^1-regularity condition for nonlinear equations (6).

NCP with a Modified Weak Regularity Condition. We now consider another special case in which we remove the condition $\beta = \emptyset$ and assume that the matrix $f'_{\gamma,\beta\cup\gamma}(x^*)$ has full rank. This assumption is similar to the weak regularity condition of Daryina et al. [1], which is a full-rank assumption on $f'_{\beta\cup\gamma,\gamma}(x^*)$. (The two assumptions are identical when $\beta = \emptyset$ or f' is symmetric, as is the case when f is the gradient of a scalar function.)

Theorem 5 *If for $d \in \mathbb{R}^n$ the set of n vectors in $\mathbb{R}^n$*

$$\begin{aligned}\{e_i\}_{i\in\alpha} \cup \{f'_i(x^*)\}_{i\in\gamma} \cup \{\langle f'_i(x^*), d\rangle e_i + d_i f'_i(x^*)\}_{i\in\beta_1} \cup \\ \{\langle f'_i(x^*), d\rangle f'_i(x^*) + d_i e_i\}_{i\in\beta_2},\end{aligned} \tag{81}$$

where $\beta_1 := \beta_1(d)$ and $\beta_2 := \beta_2(d)$, with

$$\beta_1(d) := \{i \in \beta \,|\, \langle f'_i(x^*), d\rangle > -d_i\}, \tag{82a}$$

$$\beta_2(d) := \{i \in \beta \,|\, \langle f'_i(x^*), d\rangle \le -d_i\}, \tag{82b}$$

is linearly independent, then 2-regularity (71) is satisfied by the NCP at x^* for $d \in \mathbb{R}^n$. Conversely, if $f'_{\gamma,\beta\cup\gamma}(x^*)$ has full rank and 2-regularity holds for $d \in \mathbb{R}^n$ at x^*, then the set of vectors (81) is linearly independent.

Proof Observe that if $f'_{\gamma,\beta\cup\gamma}(x^*)$ has full rank, we can set $Z = I$ and $Z_\perp$ null, so the matrix in (71) reduces to

$$\begin{bmatrix} \left[e_i^T\right]_{i\in\alpha} \\ \left[\max(\langle f_i'(x^*), d\rangle, -d_i)e_i - \min(\langle f_i'(x^*), d\rangle, -d_i) f_i'(x^*)\right]^T_{i\in\beta} \\ f'_\gamma(x^*) \end{bmatrix}.$$

By partitioning the index set β according to (82), we see that nonsingularity of this matrix is equivalent to linear independence of the vectors (81). This proves the converse implication, since it assumes that $f'_{\gamma,\beta\cup\gamma}(x^*)$. The first implication follows by noting that linear independence of the set (81) implies that $f'_{\gamma,\beta\cup\gamma}(x^*)$ has full rank.

The quasi-regularity condition of Izmailov and Solodov [12, Definition 4.1] requires the set (81) to be linearly independent for some $d \in \mathbf{R}^n$, for each possible partition $\beta = \beta_1 \cup \beta_2$, with β_1 and β_2 independent of d.

6.4 "Directional Denseness" of the Starlike Domain.

In this subsection, we give sufficient conditions for the starlike domain of convergence $\mathcal{R}$ (31) (or $\mathcal{R}_\alpha$ (61)), to be "directionally dense" at the solution x^*.

Definition 8 A starlike domain $\mathcal{R}$ about $x^* \in \mathbf{R}^n$ is *directionally dense* at x^* if for almost every $t \in \mathcal{S}$,

$$\text{there exists a positive number } C_t \text{ such that} \\ \text{if } \rho < C_t \text{ then } x = x^* + \rho t \in \mathcal{R}. \tag{83}$$

A direction t satisfies (83) if and only if t is not an *excluded direction*, as defined in Section 2.

We recall the characterization of the excluded directions of $\mathcal{R}$ from (33): A direction $t \in \mathcal{S}$ is excluded if and only if one of the following conditions is true:

$$t \in \Pi_0^{-1}(0), \tag{84a}$$

$$g(t) = 0, \text{ or} \tag{84b}$$

$$g(t)/\|g(t)\| \in \Pi_0^{-1}(0). \tag{84c}$$

In the following, we tailor the definitions of Π_0 and $g(t)$ to our application. (We do not use the standardizing assumptions (8) in the following definitions.) Consider the first and third conditions (84a) and (84c). We recall the definition of $\Pi_0 : \mathbf{R}^n \to \mathbf{R}$,

$$\Pi_0(d) := \det(P_{N_*}\Psi')'(x^*; d)|_N, \qquad \text{for } d \in \mathbf{R}^n. \tag{85}$$

Observe that the condition $d \notin \Pi_0^{-1}(0)$ is equivalent to the 2-regularity condition (68) for $d \in \mathbf{R}^n$, which is itself equivalent to nonsingularity of the matrix (71) by Proposition 1. Further, if $d \in N$, the condition $d \notin \Pi_0^{-1}(0)$ is equivalent to nonsingularity of the simpler matrix (72).

Now consider the second condition (84b). For $x \in \mathbb{R}^n$ with $\Pi_0(x-x^*) \neq 0$ and $\|x - x^*\|$ sufficiently small, recall that the Newton iterate from x is $x^* + \frac{1}{2}g(x-x^*) + O(\|x-x^*\|^2)$, where $g : (\mathbb{R}^n \setminus \Pi_0^{-1}(0)) \to N \subseteq \mathbb{R}^n$ is the positively homogeneous vector defined by

$$(86) \qquad \begin{aligned} g(x-x^*) = \rho g(t) = {} & P_N(x-x^*) \\ & + ((P_{N_*}F')'(x^*;t)|_N)^{-1}(P_{N_*}F')'(x^*;t)|_{N_\perp} P_{N_\perp}(x-x^*), \end{aligned}$$

for $x = x^* + \rho t$, $\rho = \|x - x^*\|$, and $t \in \mathcal{S}$. As in (34), we have

$$(87) \qquad g(d) = 0 \Leftrightarrow (P_{N_*}\Psi')'(x^*;d)d = 0, \qquad \text{for } d \in \mathbb{R}^n \setminus \Pi_0^{-1}(0).$$

From (70) and dividing the set β into $\beta_1(d)$ and $\beta_2(d)$ (82) for $d \in \mathbb{R}^n$, we have

$$(88) \qquad \frac{1}{2}(\Psi_i')'(x^*;d)d = \begin{cases} 2d_i\langle f_i'(x^*), d\rangle, & i \in \alpha, \\ 2d_i\langle f_i'(x^*), d\rangle, & i \in \beta_1(d), \\ -d_i^2 - \langle f_i'(x^*), d\rangle^2, & i \in \beta_2(d), \\ 2d_i\langle f_i'(x^*), d\rangle + x_i^*\langle (f_i')'(x^*;d), d\rangle, & i \in \gamma. \end{cases}$$

In order to express $(P_{N_*}\Psi')'(x^*;d)d = 0$ in terms of f, recall from the proof of Proposition 1 that $P_{N_*}w = 0$ if and only if $w = A^T z$ for some z, where $A^T z$ is the right-hand side of (74). That is, $(P_{N_*}\Psi')'(x^*;d)d = 0$ for $d \in \mathbb{R}^n$ if and only if

$$(89a) \qquad 2d_i\langle f_i'(x^*), d\rangle = f_i z_i, \qquad i \in \alpha,$$

$$(89b) \qquad 2d_i\langle f_i'(x^*), d\rangle = 0, \qquad i \in \beta_1(d),$$

$$(89c) \qquad d_i^2 + \langle f_i'(x^*), d\rangle^2 = 0, \qquad i \in \beta_2(d),$$

$$(89d) \qquad 2d_i\langle f_i'(x^*), d\rangle + x_i^*\langle (f_i')'(x^*;d), d\rangle = x_i^*\langle f_i', z\rangle, \qquad i \in \gamma.$$

If $\beta \neq \emptyset$ and $f_\beta' \not\equiv 0$, then (89b) and (89c) fail almost surely. This is because, for any $d \in \mathbb{R}^n$, d_i is almost surely nonzero for $i = 1, 2, \cdots, n$, and, if $f_\beta' \not\equiv 0$, then $\langle f_i', d\rangle$ is almost surely nonzero for $i \in \beta$. In this case, we have $(P_{N_*}\Psi')'(x^*;d)d \neq 0$ almost surely for $d \in \mathbb{R}^n$. If $\beta = \emptyset$, the conditions (89) can be simplified as follows. Solve (89a) for z_α. Substituting z_α into (89d), we find that $(P_{N_*}\Psi')'(x^*;d)d = 0$ for $d \in \mathbb{R}^n$ if and only if there is some $z_\gamma \in \mathbb{R}^{|\gamma|}$ that solves

$$(90) \quad 2\left(\operatorname{diag}\frac{d_\gamma}{x_\gamma^*}\right)\langle f_\gamma'(x^*), d\rangle + \langle (f_\gamma')'(x^*;d), d\rangle - f_{\gamma,\alpha}'(x^*)z_\alpha = f_{\gamma,\gamma}'(x^*)z_\gamma.$$

Since, by assumption, the (left) null space N_* is nontrivial, $\ker(f_{\gamma,\gamma}'(x^*))^T$ (67) is nontrivial. Hence, the complementary space range $(f_{\gamma,\gamma}'(x^*))$ must be a strict subspace of $\mathbb{R}^{|\gamma|}$. Thus, equation (90) is solvable only if the left-hand side, which is an element of $\mathbb{R}^{|\gamma|}$, lies in the subspace spanned by range $(f_{\gamma,\gamma}'(x^*))$ as is required by the right-hand side. Although counterexamples can be

Table 1 Convergence rate of Newton's Method on Ψ for the Simple NCP test problems, showing regularity properties. (● = property satisfied, blank = property not satisfied, — = property not applicable.)

Problem, s.p.	n	dim N	cgce rate	$\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\lvert\gamma\rvert$	full rank		regularity		
							$f'_{\gamma,\gamma}$	$f'_{\gamma,\beta\cup\gamma}$	b	2^T	2^{ae}
quarp, 1	1	0	suplin	1	0	0	—	—	●	—	—
aff1	2	0	suplin	1	0	1	●	●	●	—	—
DIS61, 2	2	0	suplin	1	0	1	●	●	●	—	—
quarquad, 1	2	1	1/2	0	1	1	●	●	●	●	●
affknot1	2	1	1/2	0	1	1		●			
affknot2	2	1	1/2	0	1	1	●	●	●	●	●
quadknot	2	2	1/2	0	1	1					●
munson4	2	2	1/2	0	0	2				●	●
DIS61, 1	2	2	1/2	0	1	1				●	●
DIS64	2	2	1/2	0	2	0	—	—	●	●	●
ne-hard	3	2	1/2	0	2	1	●	●			●
doubleknot	4	2	1/2	0	2	2	●	●	●	●	
quad1	2	1	2/3	0	1	1	●	●			
quarquad, 2	2	1	3/4	1	0	1					
quarp, 2	1	1	3/4	0	0	1					
quarn	1	1	3/4	0	0	1					

constructed, it seems likely that this containment will typically fail for almost all directions $d \in \mathbb{R}^n$. Under this assumption, $(P_{N_*}\Psi')'(x^*;d)d \neq 0$ almost surely for $d \in \mathbb{R}^n$. (By positive homogeneity, this is equivalent to $(P_{N_*}\Psi')'(x^*;t)t \neq 0$ almost surely for $t \in \mathcal{S}$.)

In summary, the starlike domain of convergence $\mathcal{R}$ is *directionally dense* (8) at the solution x^* if each of the three conditions of (84) fails almost surely for $t \in \mathcal{S}$. As discussed above, if 2-regularity (nonsingularity of (71)) holds for almost every $t \in \mathcal{S}$, then the first condition (84a) fails almost surely. Under this assumption, the second condition (84b) is almost surely equivalent to $(P_{N_*}\Psi')'(x^*;t)t = 0$. In the preceding paragraph, we suggest that this condition fails almost surely for $t \in \mathcal{S}$ for most problems, especially problems with $\beta \neq \emptyset$. Finally, the third condition fails almost surely if 2-regularity of $g(t)/\|g(t)\|$ (nonsingularity of (72) with d replaced by $g(t)/\|g(t)\|$) holds for almost every $t \in \mathcal{S}$. (We use (72) instead of (71) here because the range of g is N.)

7 Numerical Results on Simple NCPs

We describe here some computational results obtained from a simple test set of NCPs of small dimension, defined in Appendix A. Properties of the problems are shown in Table 1. If the problem has more than one default starting point/solution pair, the pair's number is given following the problem name. The convergence rate shown is for Newton's method with unit step length. We also tabulate the sizes of the sets α, β, and γ, and the satisfaction of various rank and regularity properties at the solution in question.

We report the numbers of iterations required for local convergence of Newton's method and the Accelerated-Newton method of Section 5 using

Table 2 Performance of Accelerated Newton Method (with $\alpha = 1.9$) for the NCP test problems for which the convergence rate of pure Newton is linear with factor 1/2. We show iterations for the pure Newton method, iterations for Accelerated Newton Method, and the iterations required by the Accelerated Newton Method in the accelerated phase, after a convergence rate of 1/2 had been detected in the pure Newton method.

Problem, Starting Point	Newton Iters	Accel Newton Iters	Accel Phase Iters
quarquad,1	16	10	5
affknot1	20	10	7
affknot2	19	10	5
quadknot	18	8	5
munson4	19	12	4
DIS61, 1	19	12	5
DIS64	21	11	7
ne-hard	25	19	5
doubleknot	22	14	5

the subset of Simple NCP test problems with convergence rates for Newton's method of 1/2. This is the subset of problems with a nontrivial null space N for which 2^{ae}-regularity may hold. (In fact, affknot1 and doubleknot have convergence rates of $\frac{1}{2}$ for Newton's method but do not satisfy 2^{ae}-regularity. Despite the absence of 2^{ae}-regularity, the acceleration technique of Section 5 hastens the convergence.) We detect linear convergence at a rate of 1/2 by applying the following tests to successive Newton steps p_i:

$$\left| \frac{\|p_i\|}{\|p_{i-1}\|} - \frac{\|p_{i-1}\|}{\|p_{i-2}\|} \right| < \texttt{cCauchy} \quad \text{and} \quad \left| \frac{\|p_i\|}{\|p_{i-1}\|} - \frac{1}{2} \right| < \texttt{cLinear}$$

with $\texttt{cCauchy} = .005$ and $\texttt{cLinear} = .01$. If both tests are satisfied at iteration i, we scale the next step p_{i+1} (and every second step thereafter) by the acceleration factor $\alpha = 1.9$. Convergence is declared when $\|\Psi(x)\| \leq 10^{-11}$.

Table 2 shows the number of iterations required by the Newton and Accelerated Newton methods for the subset of problems discussed above. The final column shows the number of steps taken in the "accelerated phase," following detection of a linear convergence rate in the pure Newton method. Note that the accelerated phase was present for all problem instances and that the number of steps taken in this phase is similar for all problems. For $\alpha = 1.9$, the convergence rate in the accelerated phase predicted by Theorem 2 was observed for all problems.

A Simple NCP Test Set: Problem Descriptions

Below we list the Simple NCP test problems, their solutions, and the corresponding starting points used to initialize Newton's method. A solution is any x satisfying

$$0 \leq x \ \perp \ f(x) \geq 0,$$

and we denote such x by x^*. Table 3 lists the starting point x_0 that was used for each solution x^*.

1. quarp

$$f(x) = (1-x)^4.$$

2. aff1
$$f(x) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 - 1 \end{bmatrix}.$$
3. DIS61 ([1, Example 6.1])
$$f(x) = \begin{bmatrix} (x_1 - 1)^2 \\ x_1 + x_2 + x_2^2 - 1 \end{bmatrix}.$$
4. quarquad
$$f(x) = \begin{bmatrix} -(1 - x_1)^4 + x_2 \\ 1 - x_2^2 \end{bmatrix}.$$
5. affknot1
$$f(x) = \begin{bmatrix} x_2 - 1 \\ x_1 \end{bmatrix}.$$
6. affknot2
$$f(x) = \begin{bmatrix} x_2 - 1 \\ x_1 + x_2 - 1 \end{bmatrix}.$$
7. quadknot
$$f(x) = \begin{bmatrix} x_2 - 1 \\ x_1^2 \end{bmatrix}.$$
8. munson4 (from MCPLIB [17])
$$f(x) = \begin{bmatrix} -(x_2 - 1)^2 \\ -(x_1 - 1)^2 \end{bmatrix}.$$
9. DIS64 ([1, Example 6.4])
$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ -x_2 \end{bmatrix}.$$
10. ne-hard (from MCPLIB [17])
$$f(x) = \begin{bmatrix} \sin x_1 + x_1^2 \\ x_2^3 + x_1 x_3 \\ x_3^2 - 200 + x_1 x_2 \end{bmatrix}.$$
11. doubleknot
$$f(x) = \begin{bmatrix} 1 - x_1 + x_2 + x_3 \\ x_1 - 1 \\ x_4 - 1 \\ 1 + x_3 - x_4 \end{bmatrix}.$$
12. quad1
$$f(x) = \begin{bmatrix} x_1 - 1 \\ x_2^2 \end{bmatrix}.$$
13. quarn
$$f(x) = -(1 - x)^4$$

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Table 3 Starting Point/Solution Pairs

Problem, s.p.	x_0	x^*
quarp, 1	0.1	0
aff1	(0.1, 0.9)	(0,1)
DIS61, 2	(0.2, 0.85)	$(0, (\sqrt{5}-1)/2)$
quarquad, 1	(0.1, 0.9)	(0, 1)
affknot1	(0.9, 0.1)	(0, 1)
affknot2	(0.5, 0.5)	(0, 1)
quadknot	(0.5, 0.5)	(0, 1)
munson4	(0, 0)	(1, 1)
DIS61, 1	(1.5, -0.5)	(1, 0)
DIS64	(2, 4)	(0, 0)
ne-hard	(10, 1, 10)	$(0, 0, \sqrt{200})$
doubleknot	(0.5, 0.5, 0.5, 0.5)	(1, 0, 0, 1)
quad1	(0.9, 0.1)	(1, 0)
quarquad, 2	(0.9, 0.1)	(1, 0)
quarp, 2	0.9	1
quarn	0.9	1

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