A Practical Algorithm for Structure Embedding

Charlie Murphy

Overview

1. Structure Embedding

2. Use in Multi-threaded Verification

3. MatchEmbeds

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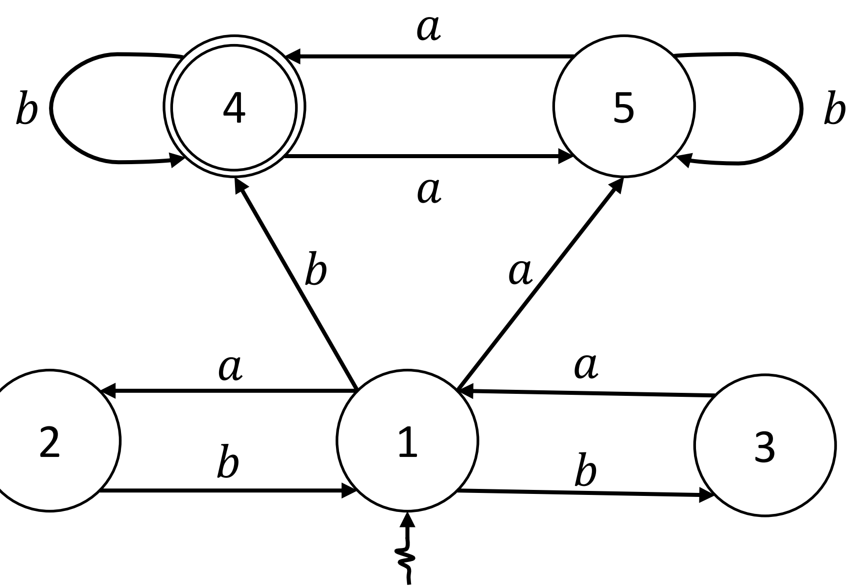
  - Graph $\equiv \langle V, edge \rangle$
  - NFA $\equiv \langle S, \{final, start\} \cup \{\Delta_a : a \in \Sigma\} \rangle$
  - Database $\equiv \langle Values, \{table_1, \dots, table_n\} \rangle$

Structures



$$\mathfrak{F} \stackrel{\text{def}}{=} \langle \{1,2,3,4,5\}, Start, Final, \Delta_a, \Delta_b \rangle$$

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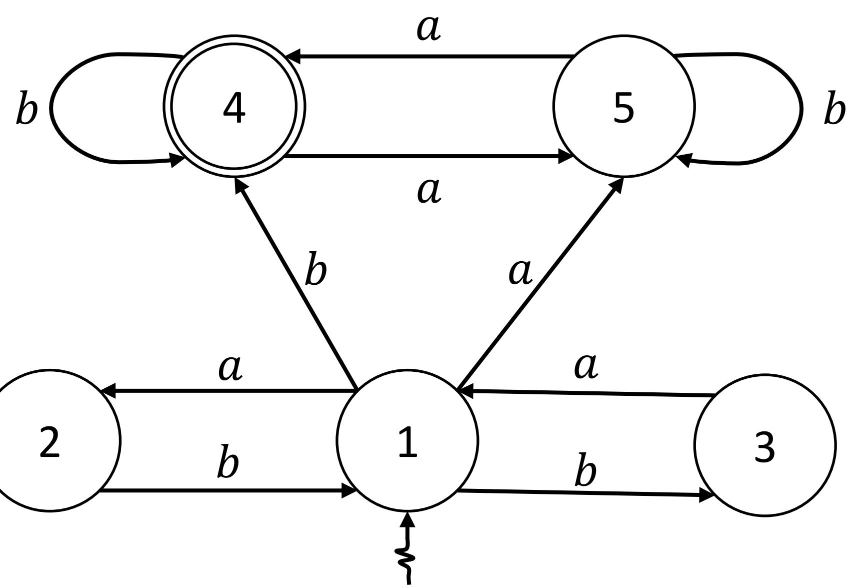

$$\mathfrak{F} \stackrel{\text{def}}{=} \langle \{1,2,3,4,5\}, Start, Final, \Delta_a, \Delta_b \rangle$$

where:

$Start \stackrel{\text{def}}{=} \{1\}$

$Final \stackrel{\text{def}}{=} \{4\}$

$\Delta_a \stackrel{\text{def}}{=} \{\langle 1,2 \rangle, \langle 1,5 \rangle, \langle 3,1 \rangle, \langle 4.5 \rangle, \langle 5,4 \rangle\}$

$\Delta_b \stackrel{\text{def}}{=} \{\langle 1,3 \rangle, \langle 1,4 \rangle, \langle 2,1 \rangle, \langle 4,4 \rangle, \langle 5,5 \rangle\}$

Structure Embedding

$$\mathfrak{A} \stackrel{\text{def}}{=} \langle \{1,2,3,4\}, p^{\mathfrak{A}}, q^{\mathfrak{A}}, r^{\mathfrak{A}} \rangle$$

$$p^{\mathfrak{A}} \stackrel{\text{def}}{=} \{1\}$$

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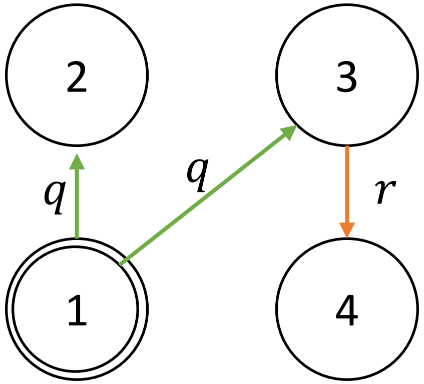

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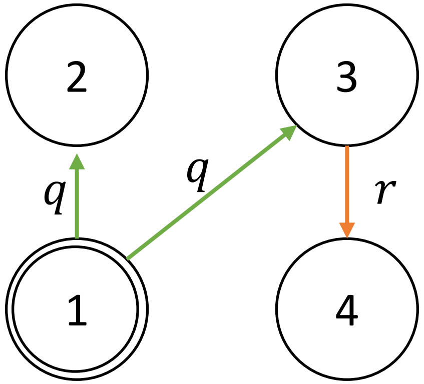

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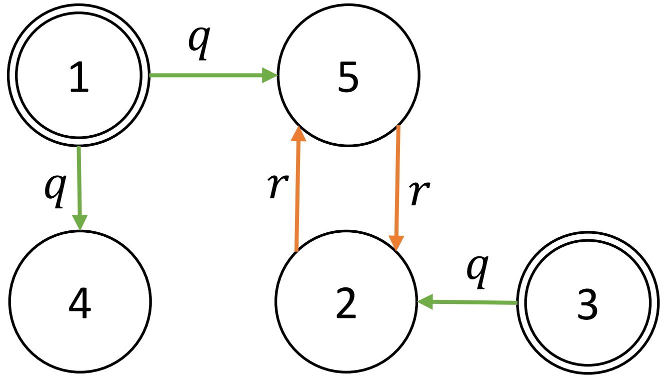

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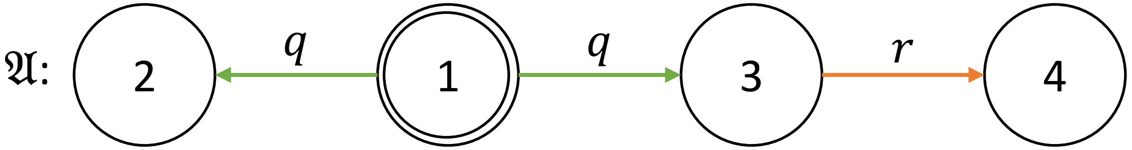

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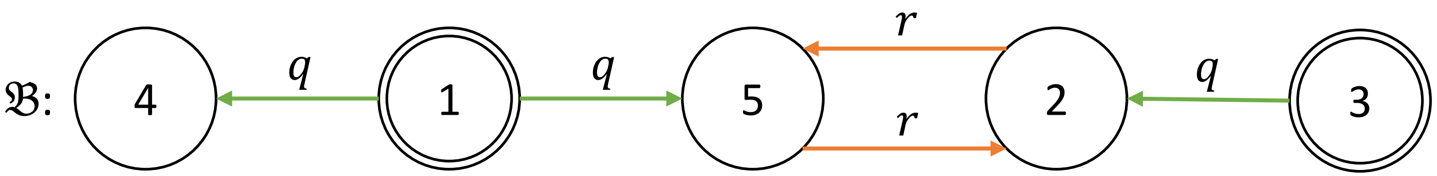

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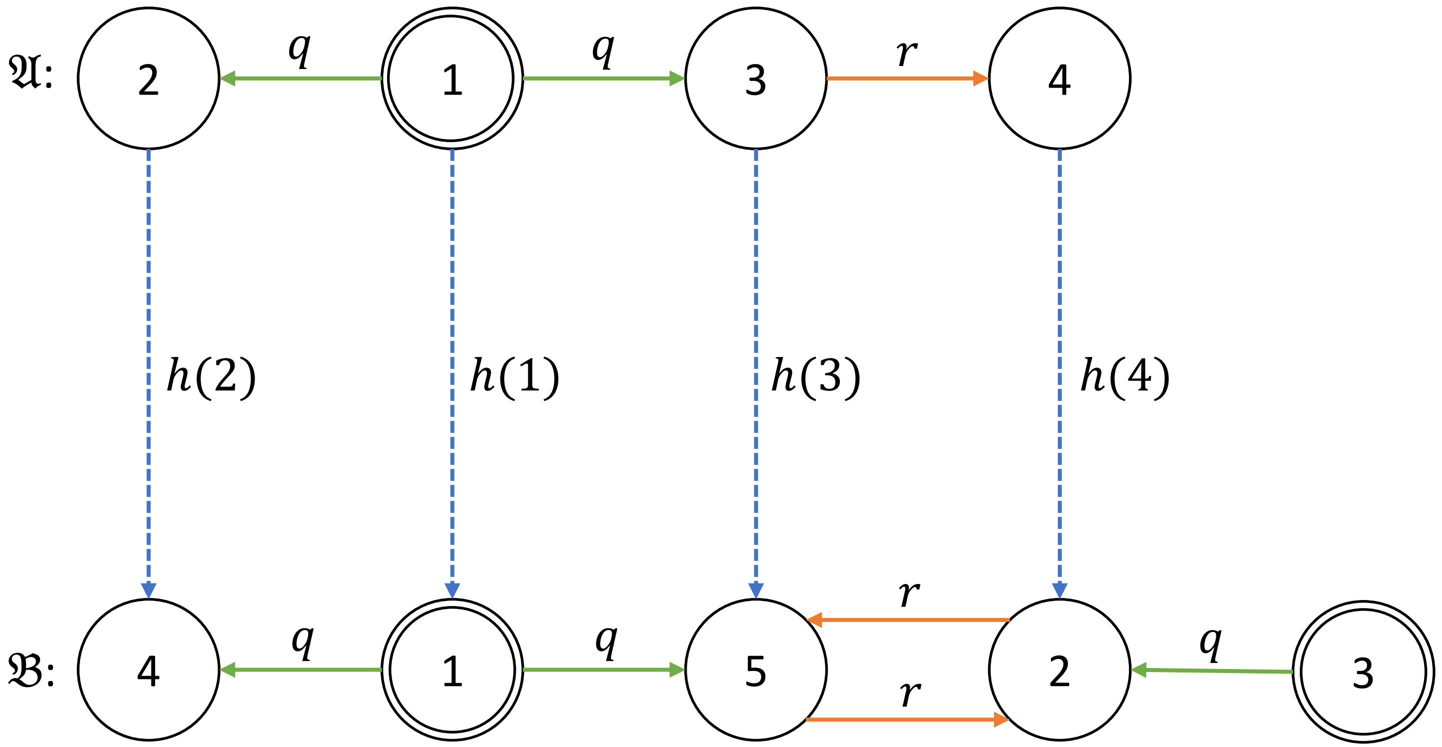

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  - Backtracking search
    - Polytime for monadic case
    - Practical for "real life" instances
    - Solves difficult instances quickly

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Multi-threaded Program Verification

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main_count():
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  for i = 1 to N:
    fork thread_count
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What about multi-threaded programs?

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$$l_4(1) \wedge l_6(2) \wedge l_7(3) \wedge S_{lt}(2) \wedge M_{lt}(2,3)$$

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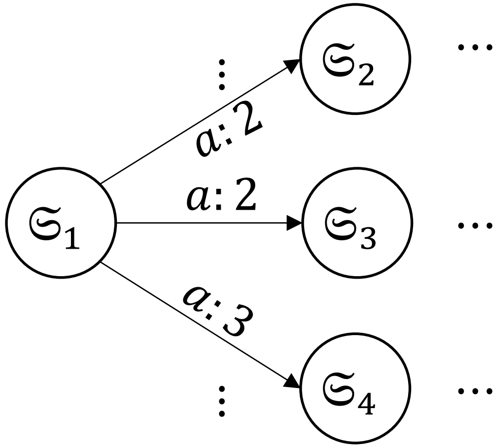
- Infinite state automata over infinite alphabet ($\Sigma \times \mathbb{N}$)



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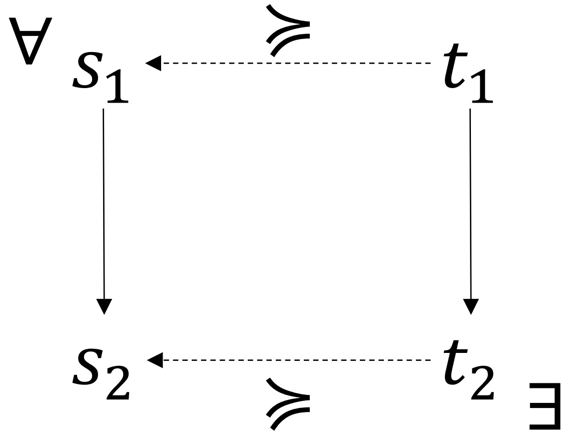
    - Universe of the current structure and 1 fresh element
  - Use **embeddings** to prune search space (Downward Compatibility)
    - Well structured transition system [Finkel and Schnoebelen. 2001]

Downward Compatibility

A wqo, $\preccurlyeq$, is downward compatible with transition system, $\langle S, \rightarrow \rangle$, if

$$\forall\, t_1 \preccurlyeq s_1 \text{ and transition } s_1 \rightarrow s_2 \text{ then } \exists t_2\ s.t.\ t_1 \rightarrow t_2 \text{ and } t_2 \preccurlyeq s_2$$



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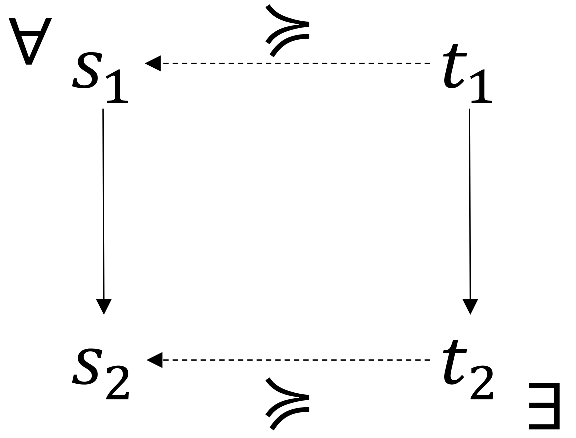

For PA and embedding if a path from s_1 accepts then a path from t_1 will accept.

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Match Embeds

Joint work with Zak Kincaid

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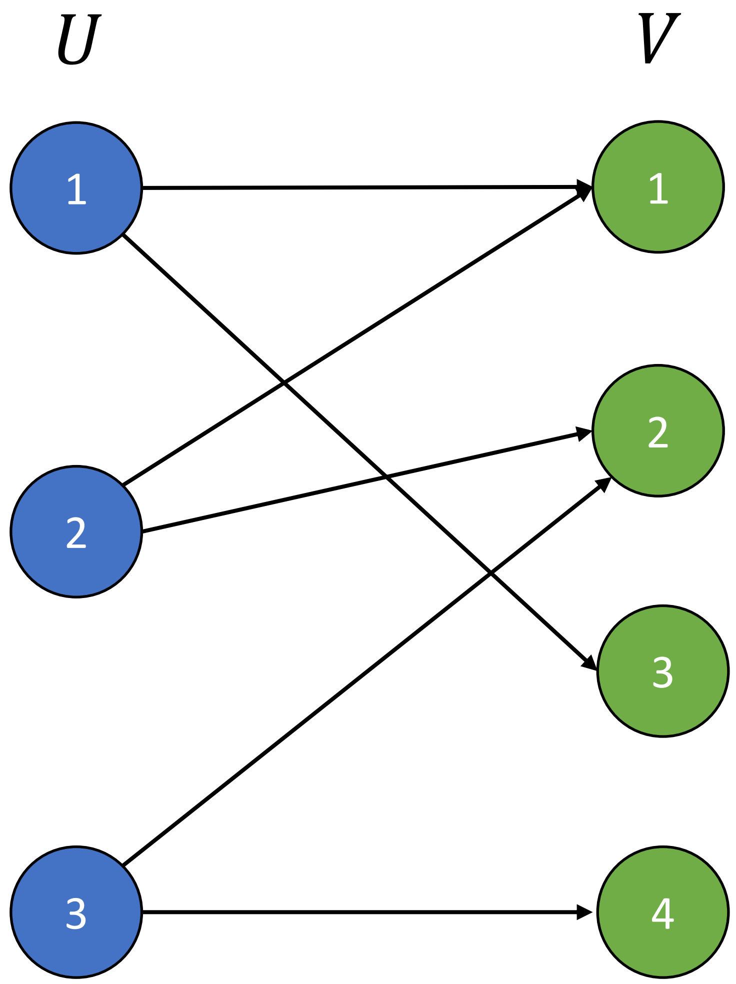
  - Reduction to bipartite graph matching
- Generalize bipartite graph matching strategy to general structures
  - Construct bipartite graph
  - Search matchings of graph for an embedding

Bipartite Graphs

- Bipartite Graphs, $G = \langle U, V, E \rangle$
  - U and V are disjoint
  - $E \subseteq U \times V$



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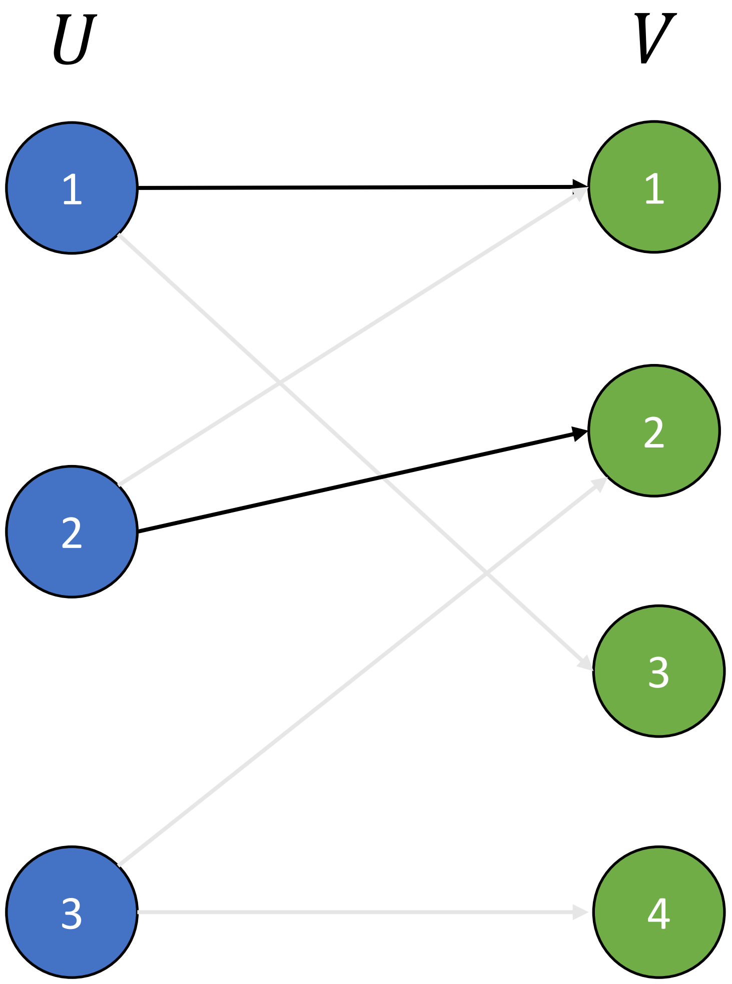

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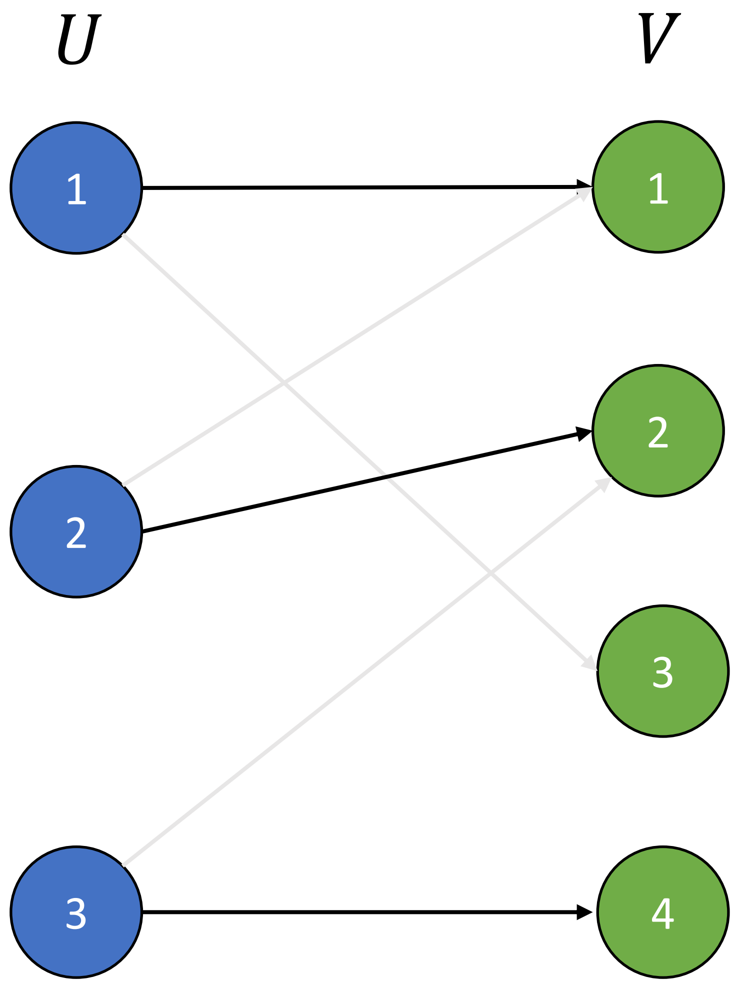
- Total Matching, M
  - M is a matching
  - M covers U ($|M| = |U|$)



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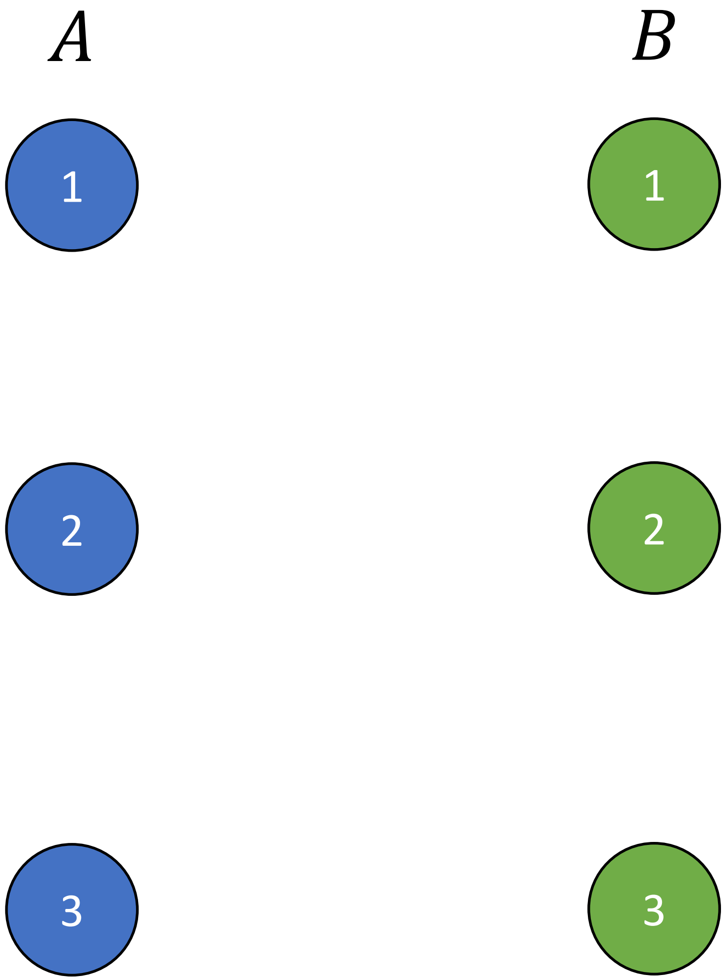

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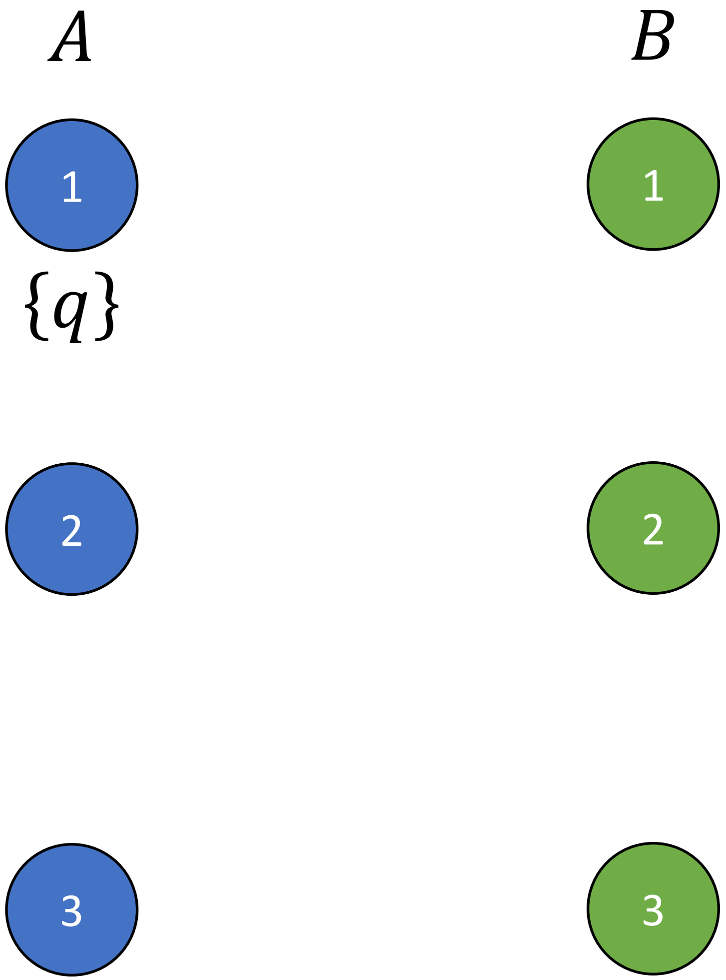

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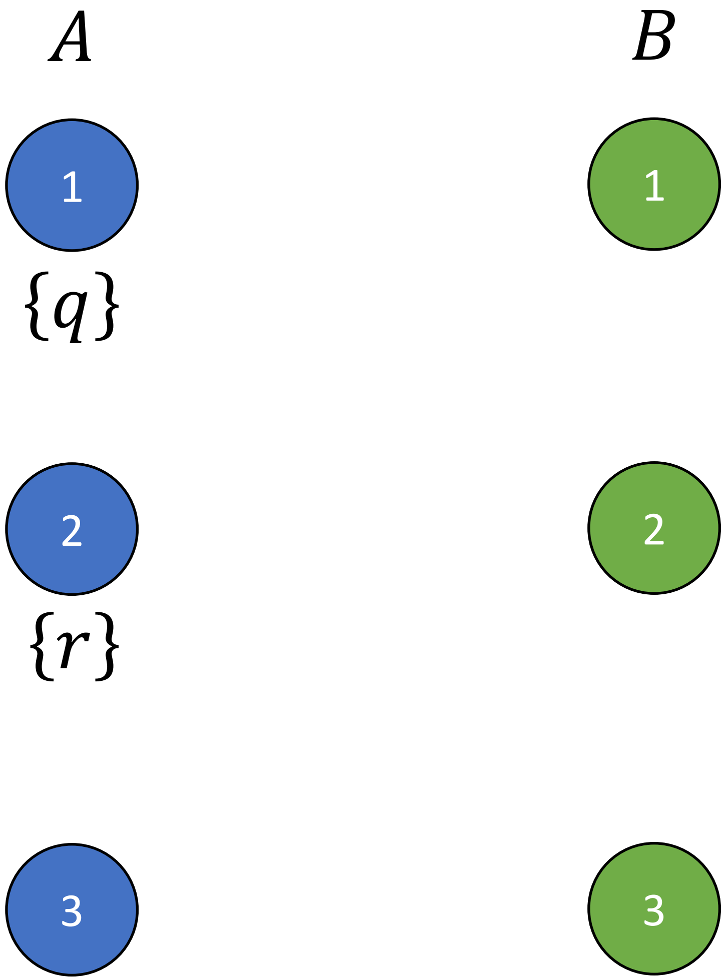

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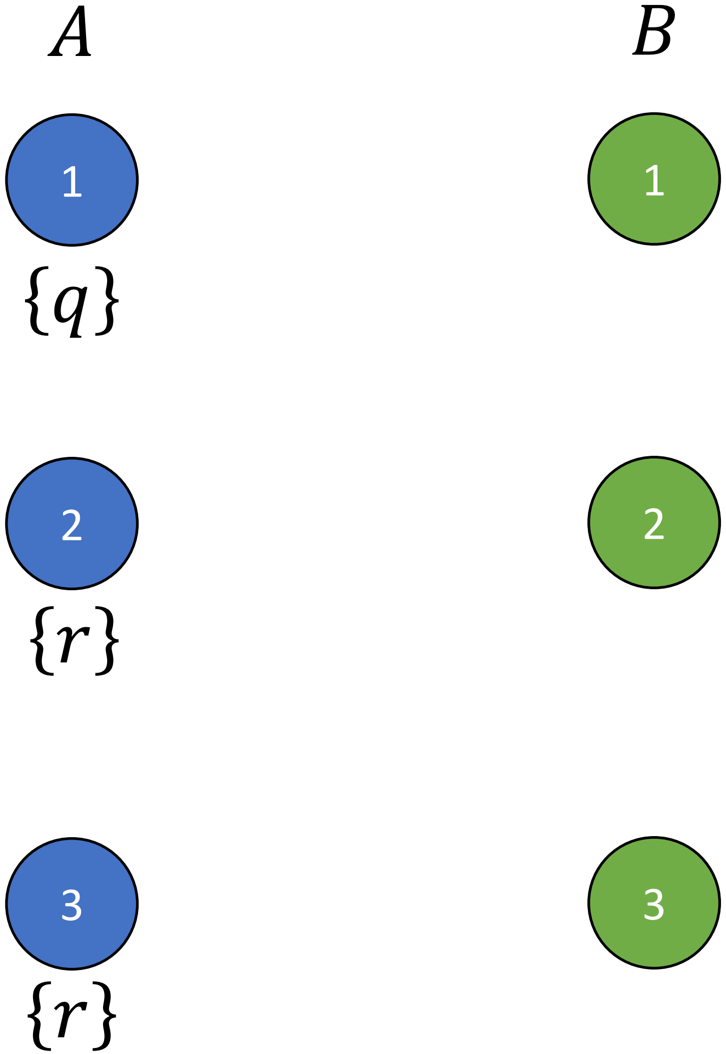

Monadic Case

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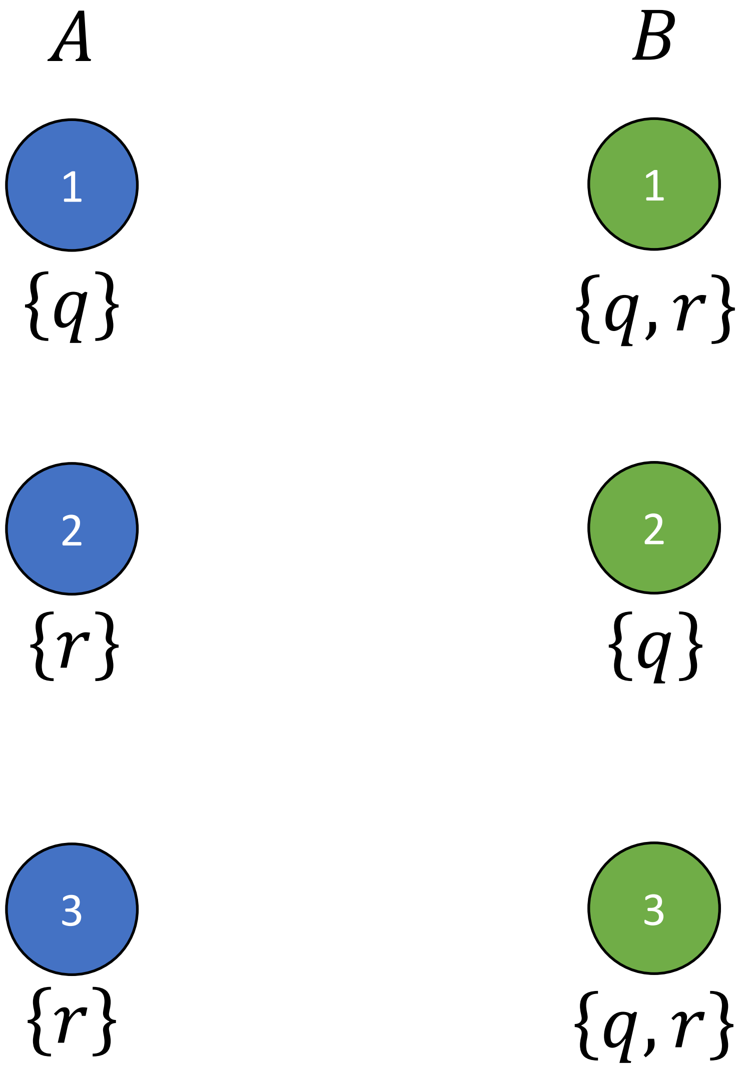

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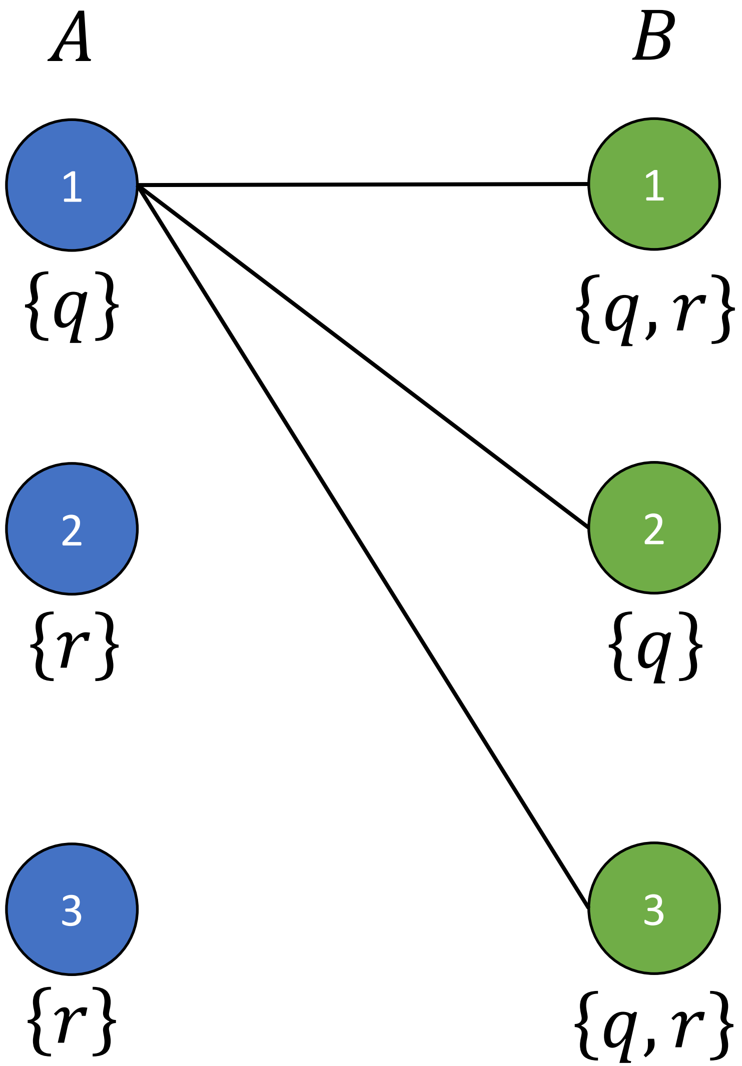

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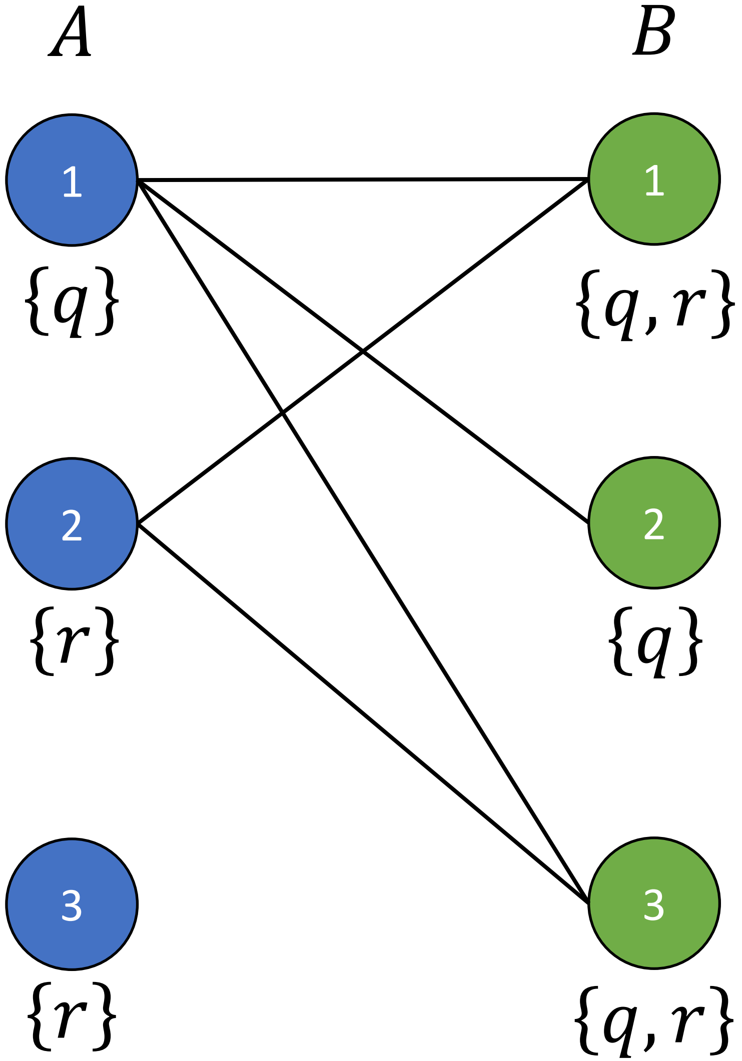

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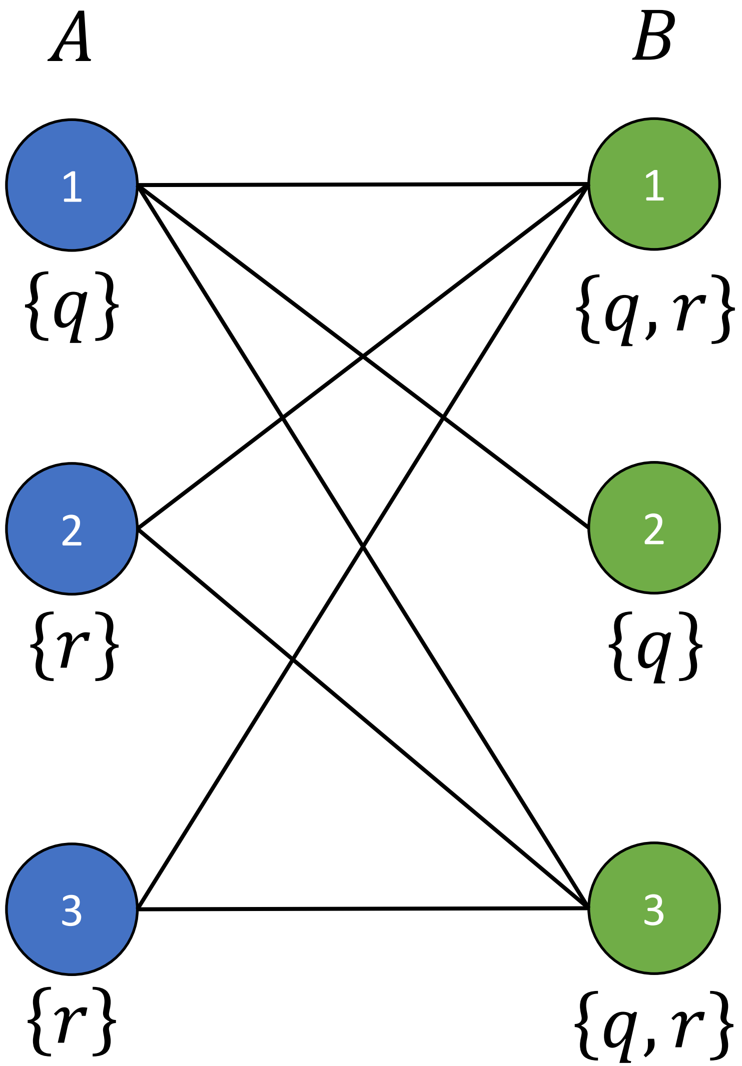

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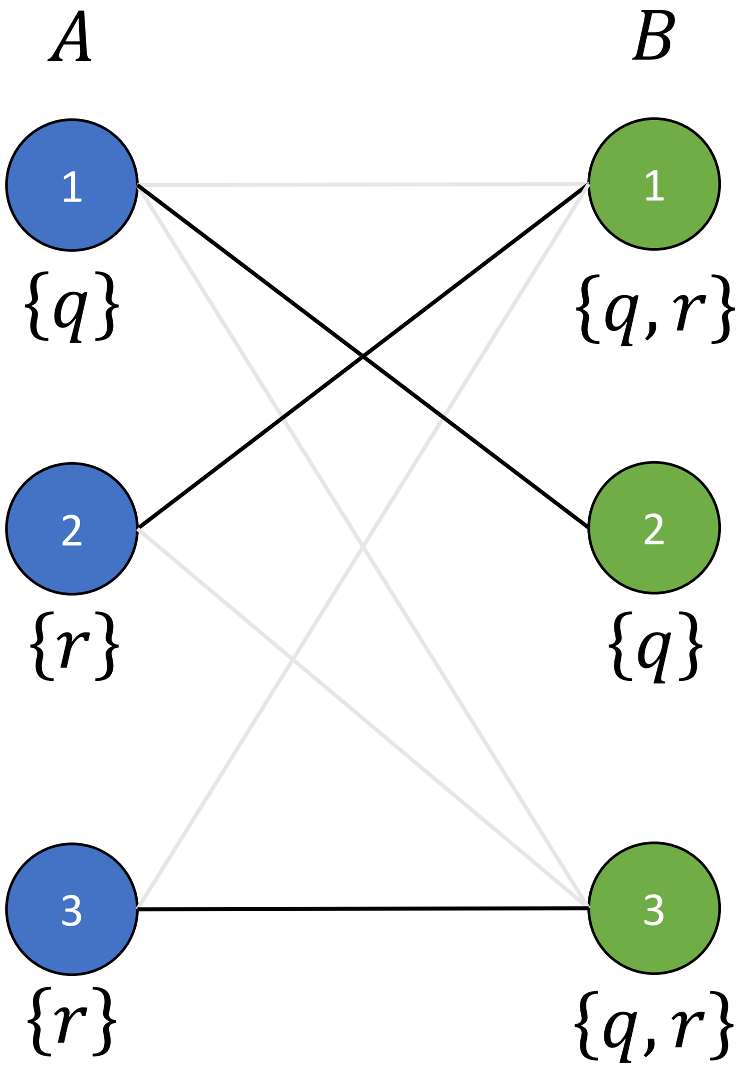

Maximum Matchings[1]

$$M_1 \overset{\text{def}}{=} \left\{ \begin{matrix} \langle 1,2 \rangle, \\ \langle 2,1 \rangle, \\ \langle 3,3 \rangle \end{matrix} \right\}$$

[Hopcroft and Karp. 1973][1]

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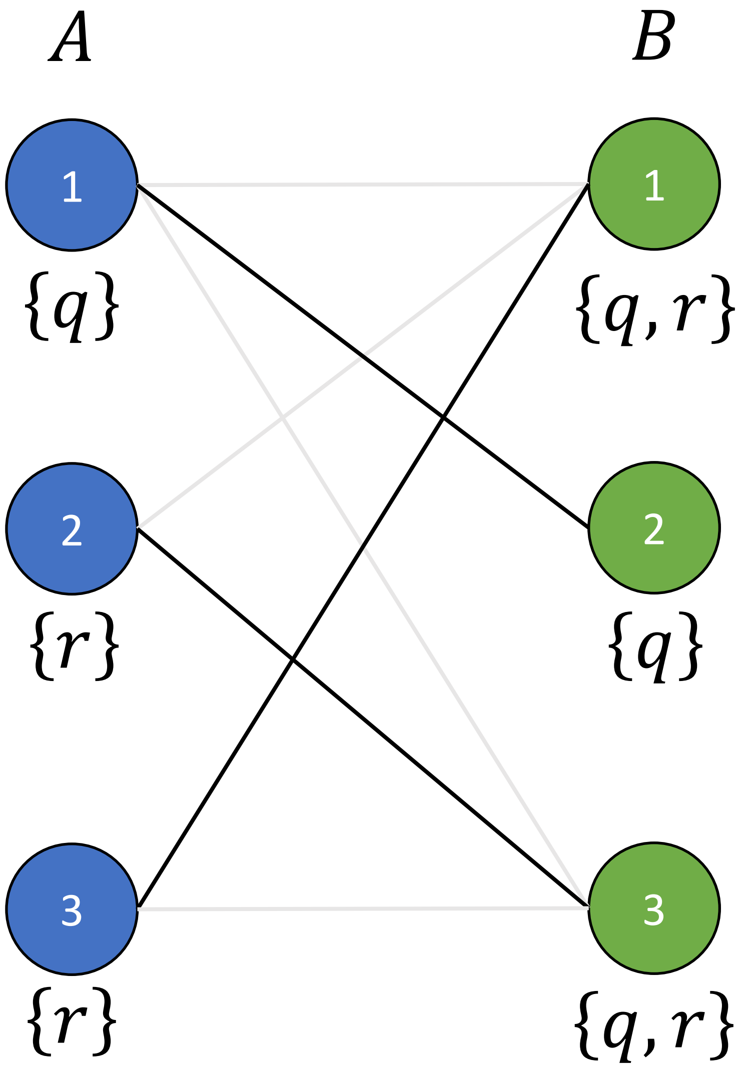

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- Structure embedding takes $O(|A||B|\sqrt{|A| + |B|})$ [Hopcroft and Karp. 1973]

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- Inspired by monadic reduction to bipartite graph matching

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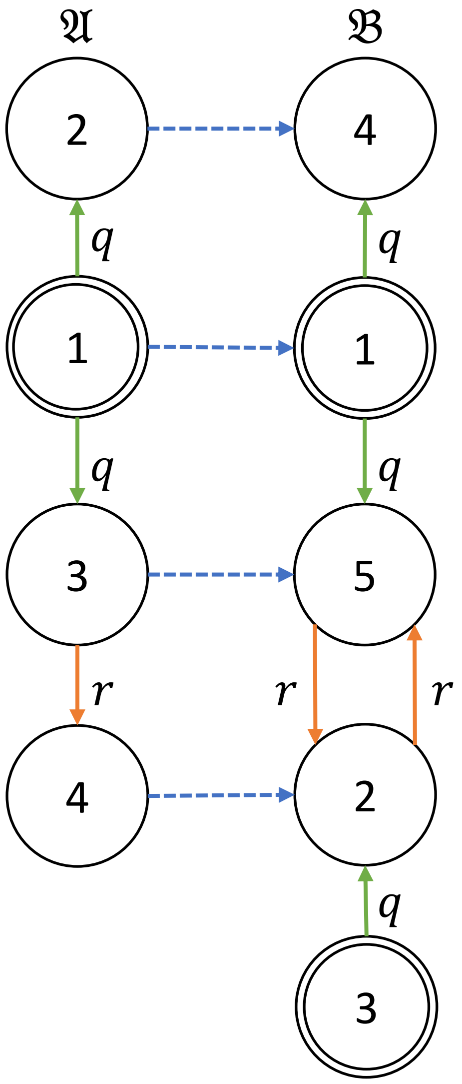
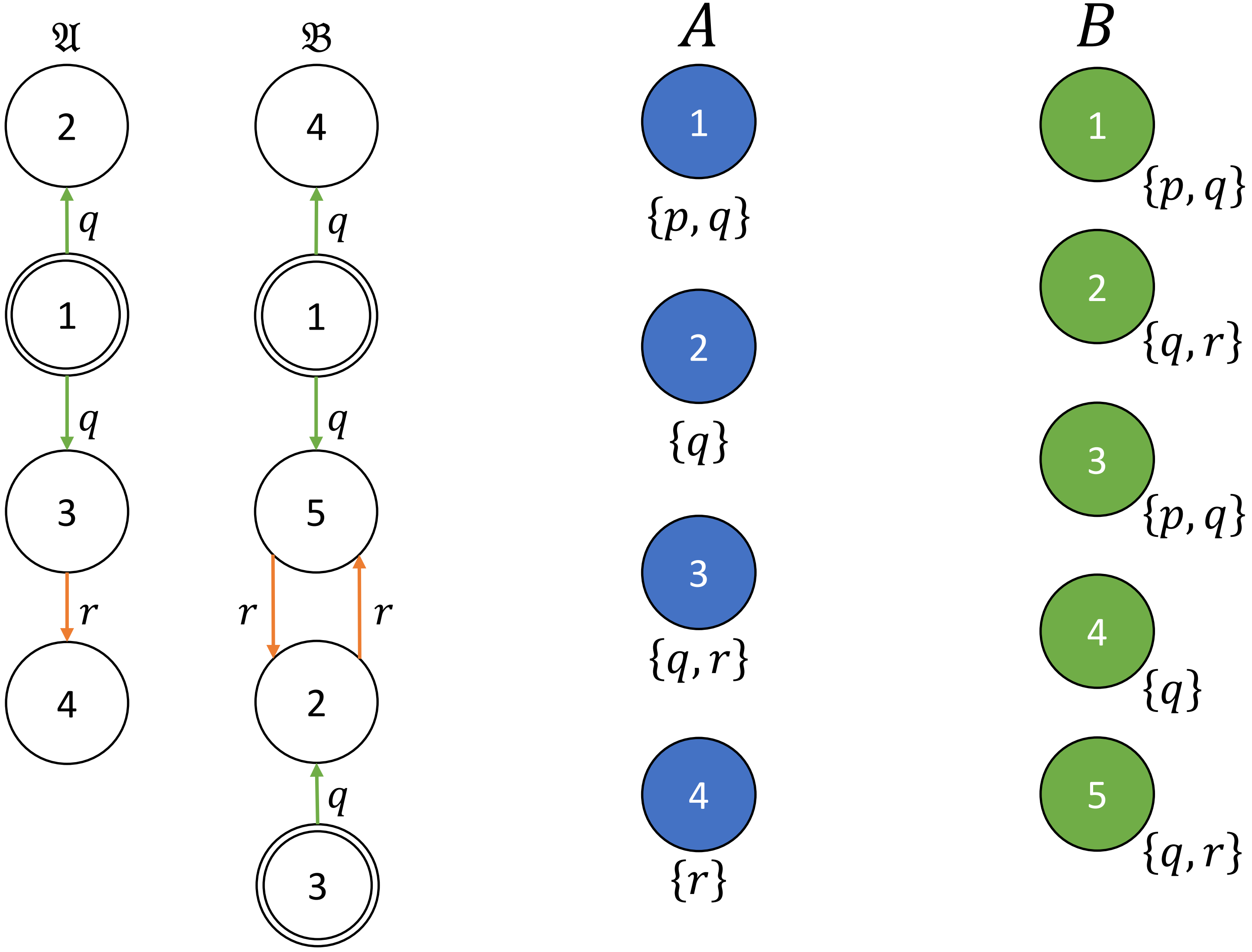
  4. Decide on edges in matching and recurse

General Case



General Case



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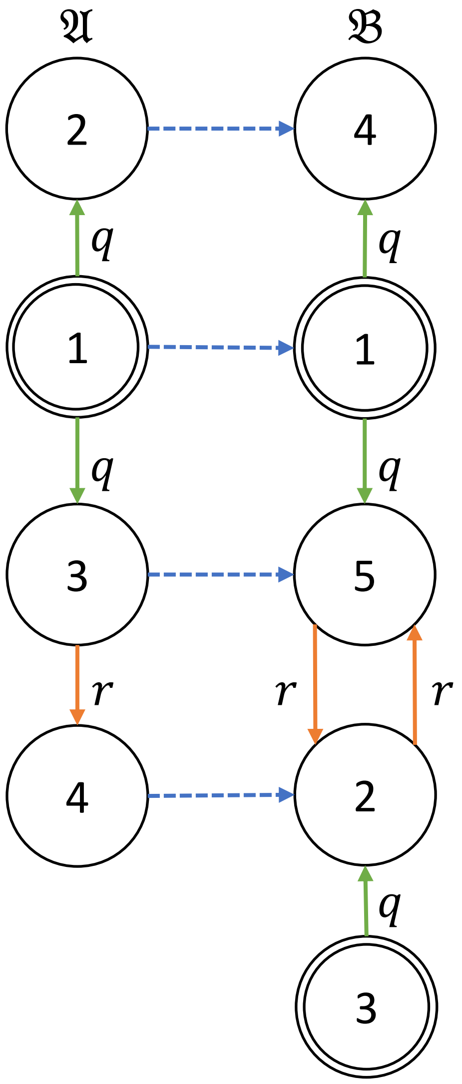

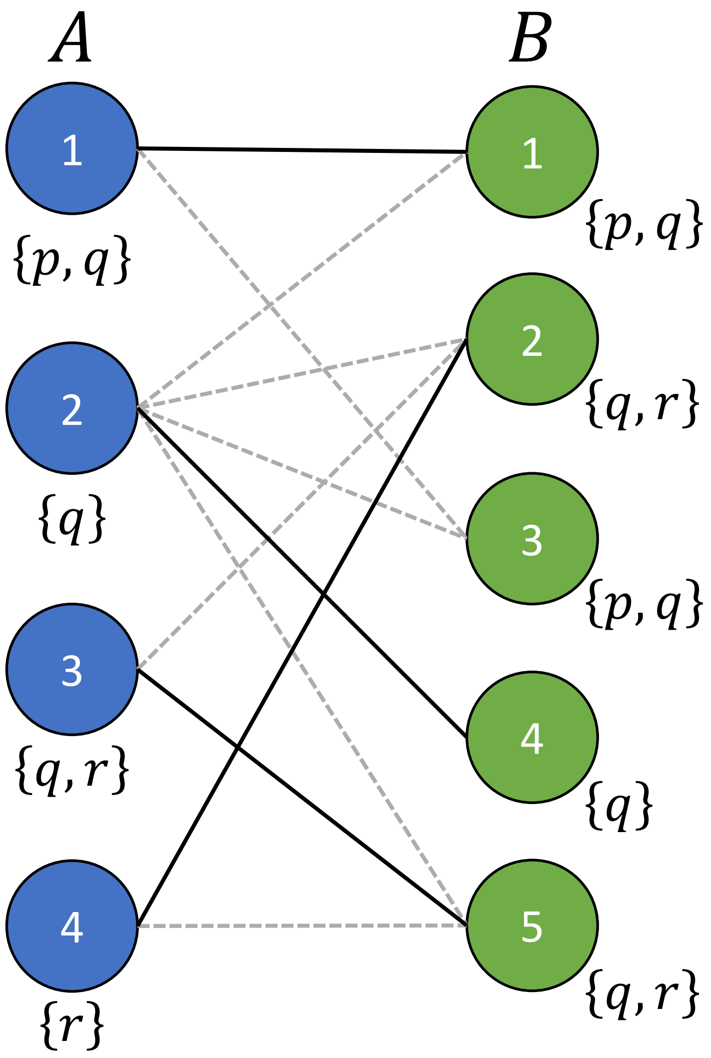

$M_1 \stackrel{\text{def}}{=} \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 4,5\rangle\}$

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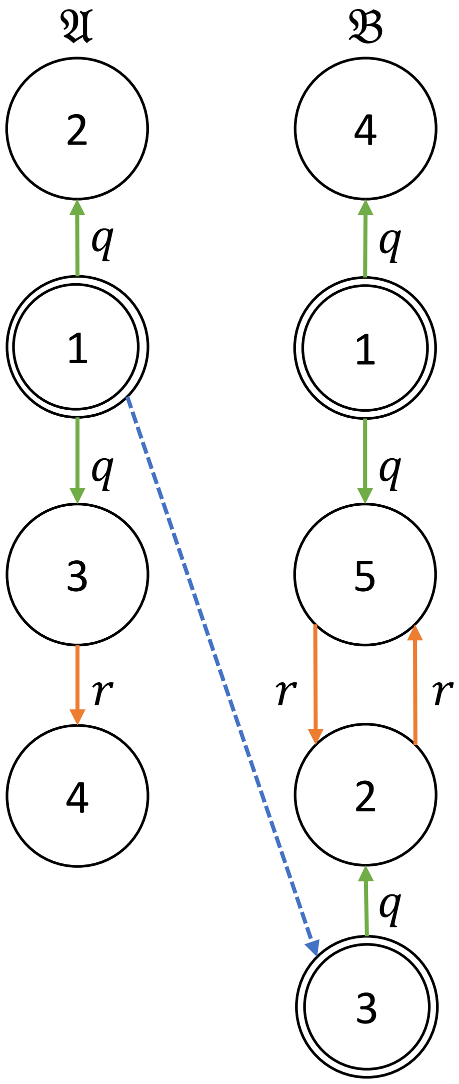
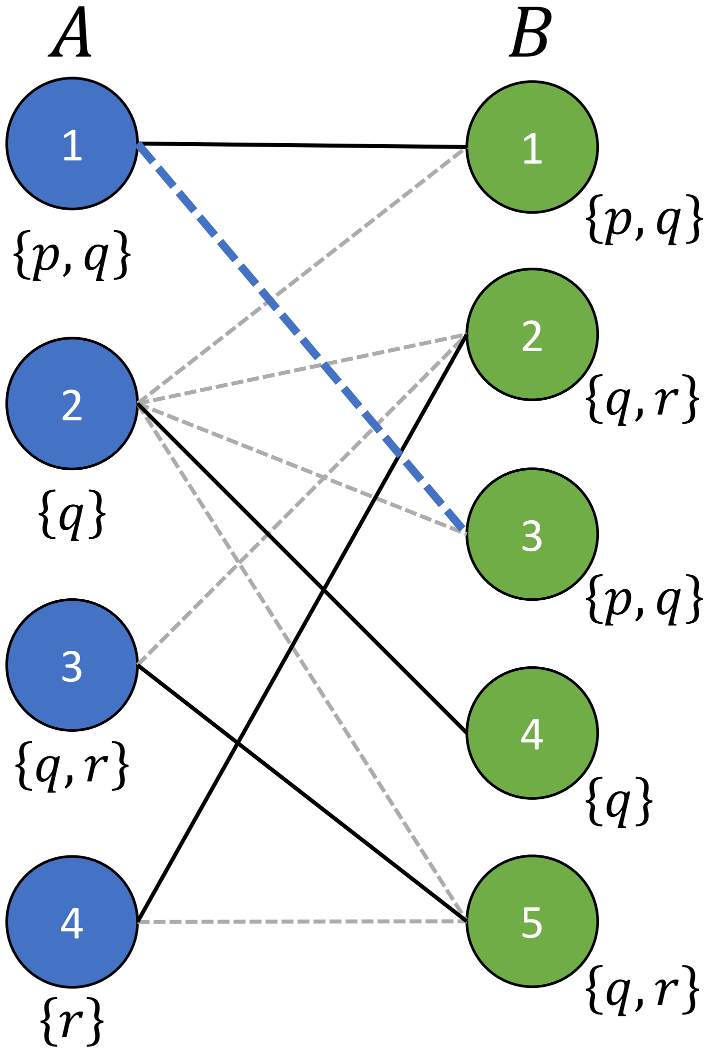
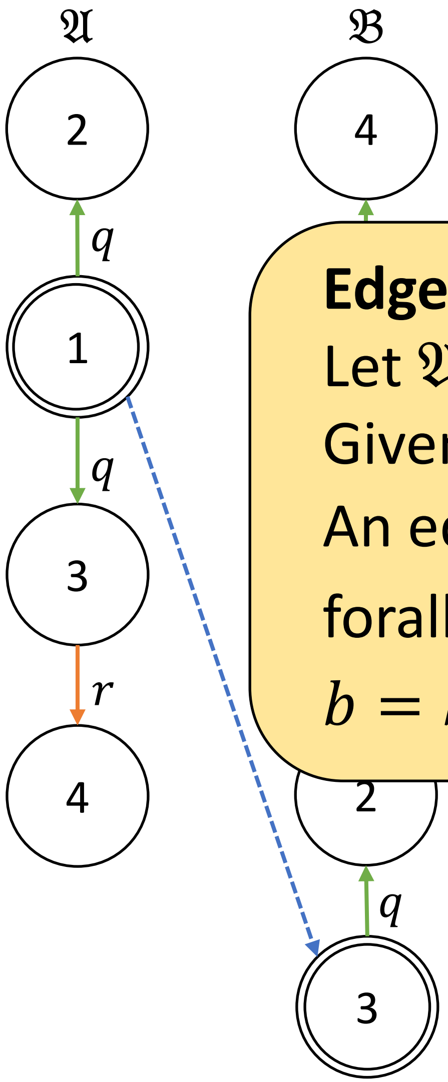
Consistency

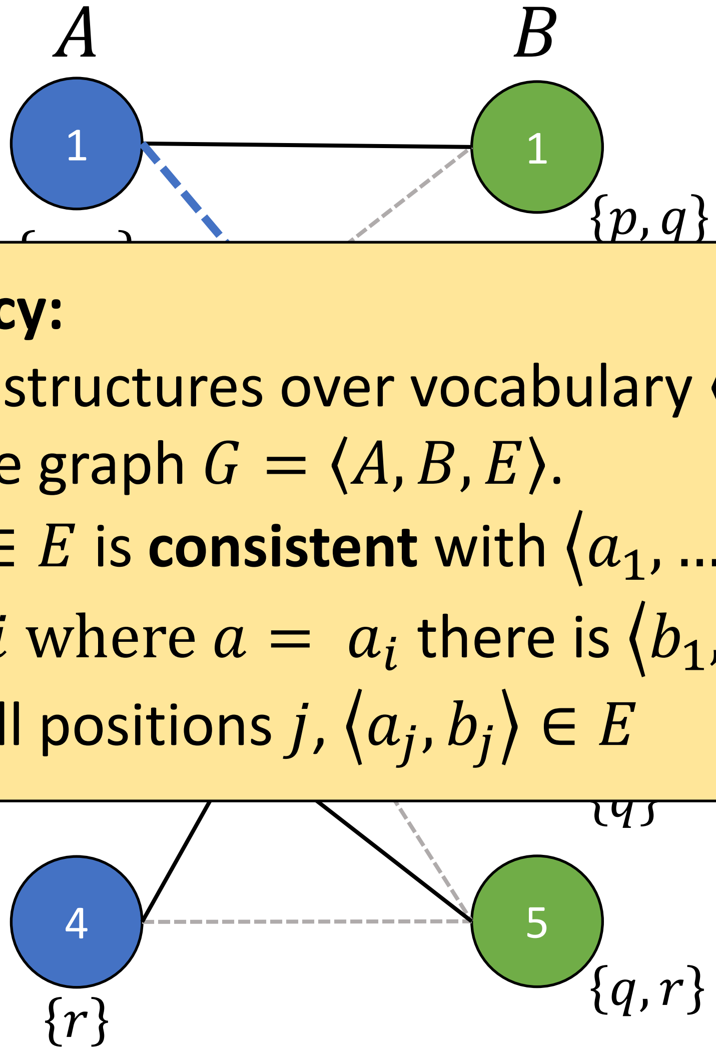




Consider edge ⟨1,3⟩:

Consistency





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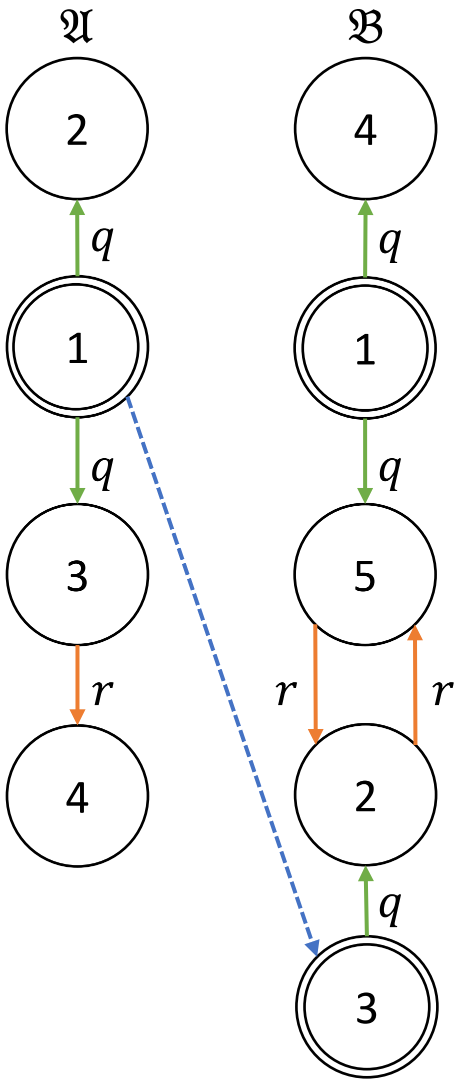
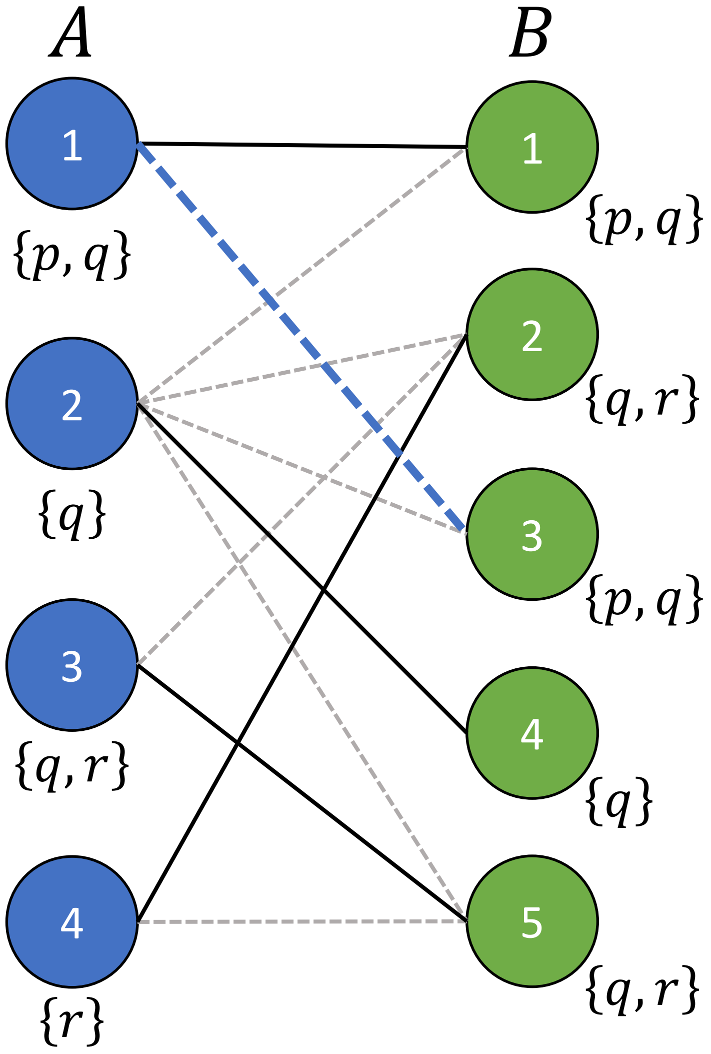
Edge Consistency:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over vocabulary $\langle Q, ar \rangle$.
Given a bipartite graph $G = \langle A, B, E \rangle$.
An edge $\langle a, b \rangle \in E$ is **consistent** with $\langle a_1, \dots, a_{ar(q)} \rangle \in q^{\mathfrak{A}}$ when forall positions i where $a = a_i$ there is $\langle b_1, \dots, b_{ar(q)} \rangle \in q^{\mathfrak{B}}$ s.t. $b = b_i$ and forall positions j, $\langle a_j, b_j \rangle \in E$

Consistency





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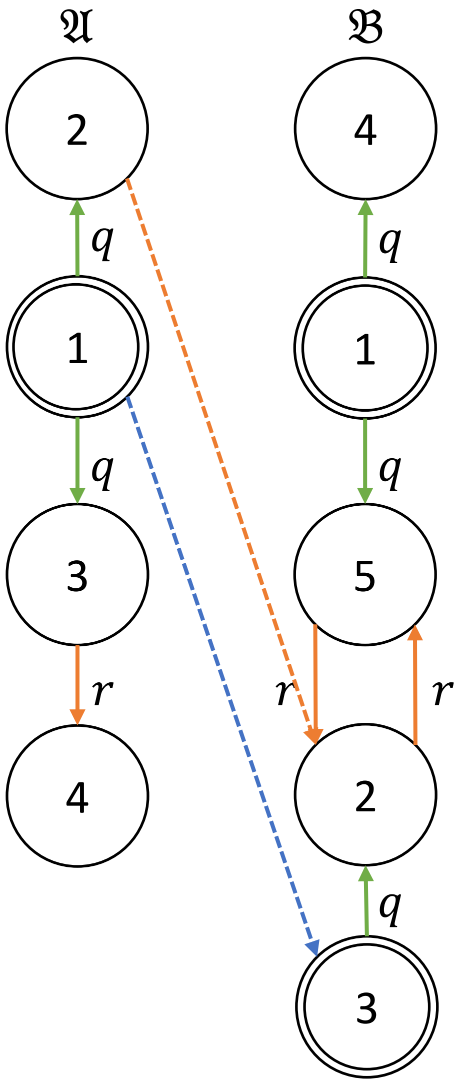

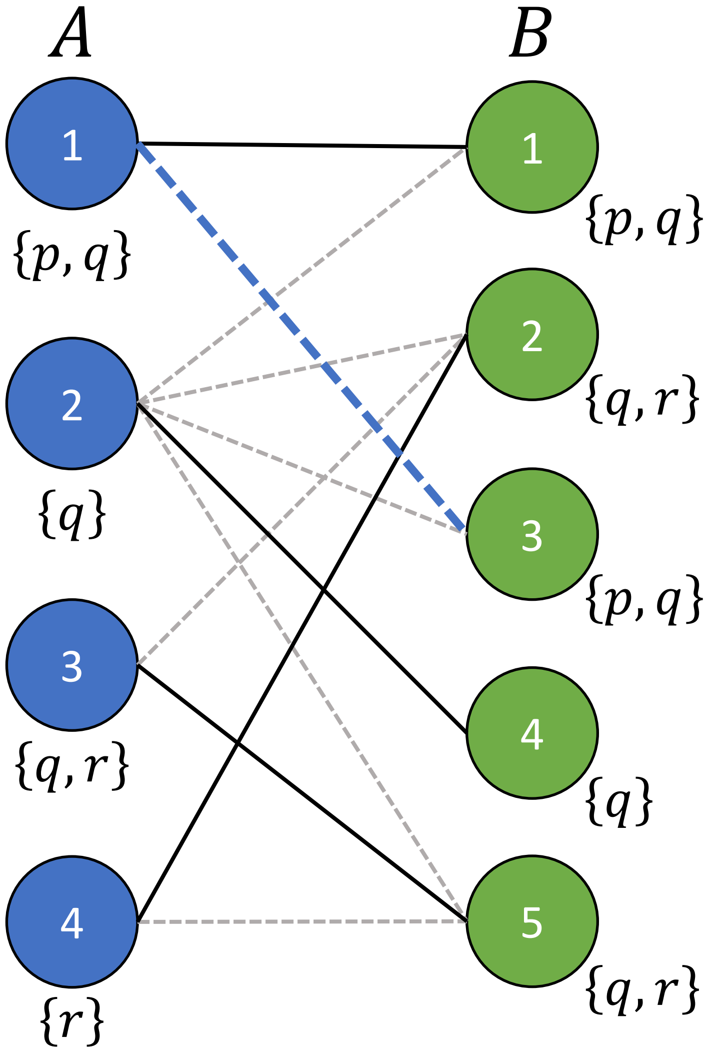

Consider edge ⟨1,3⟩:

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  - $\exists q(3,2) \in \mathfrak{B} \wedge \langle 2,2\rangle \in G$

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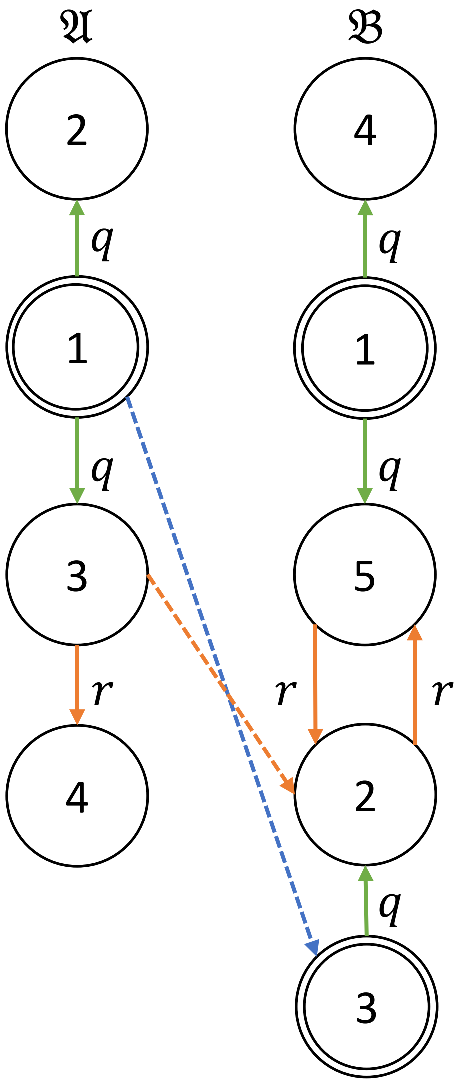

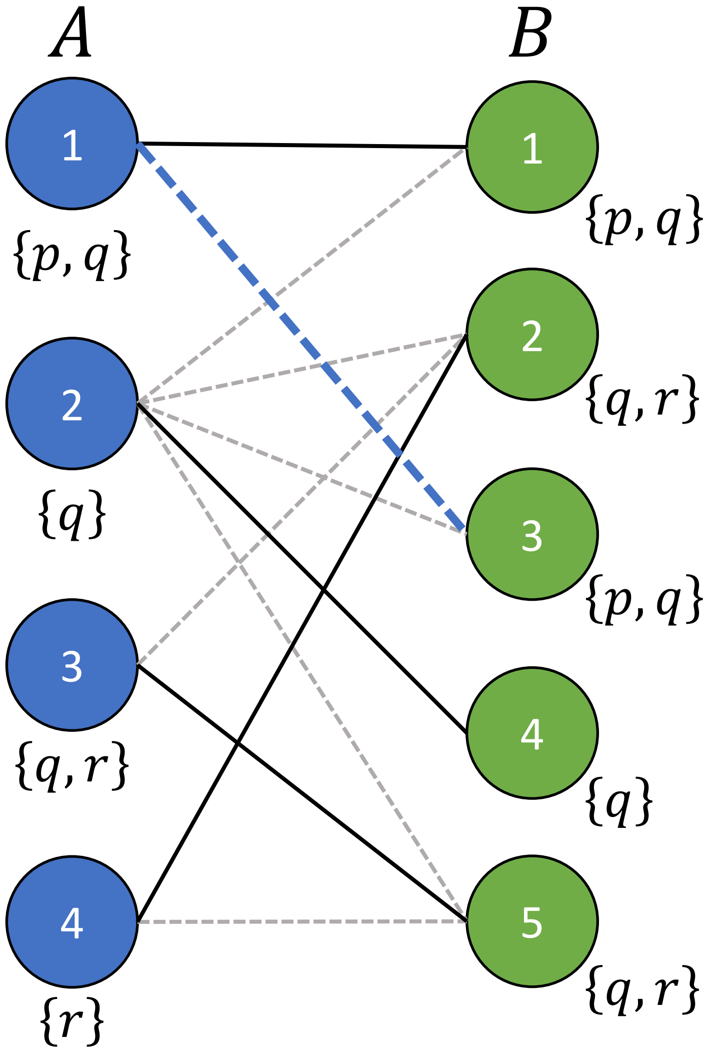

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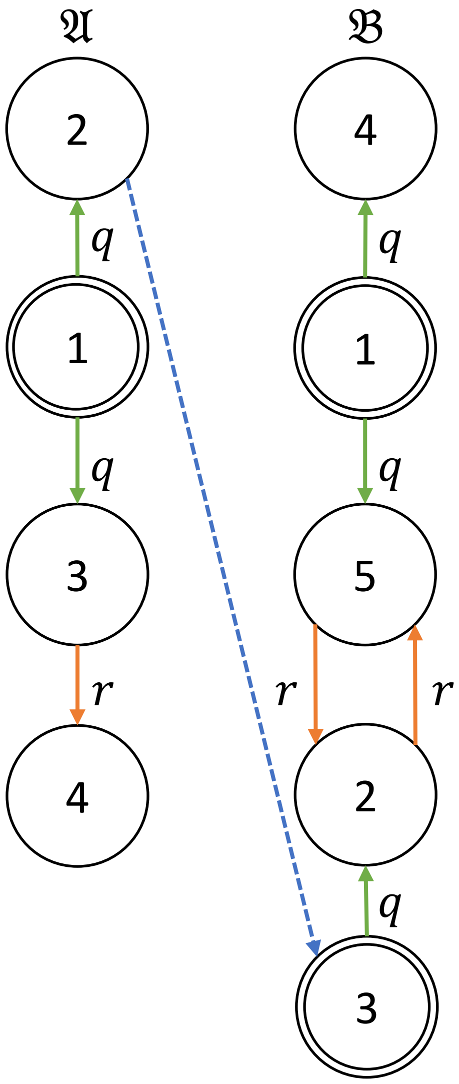
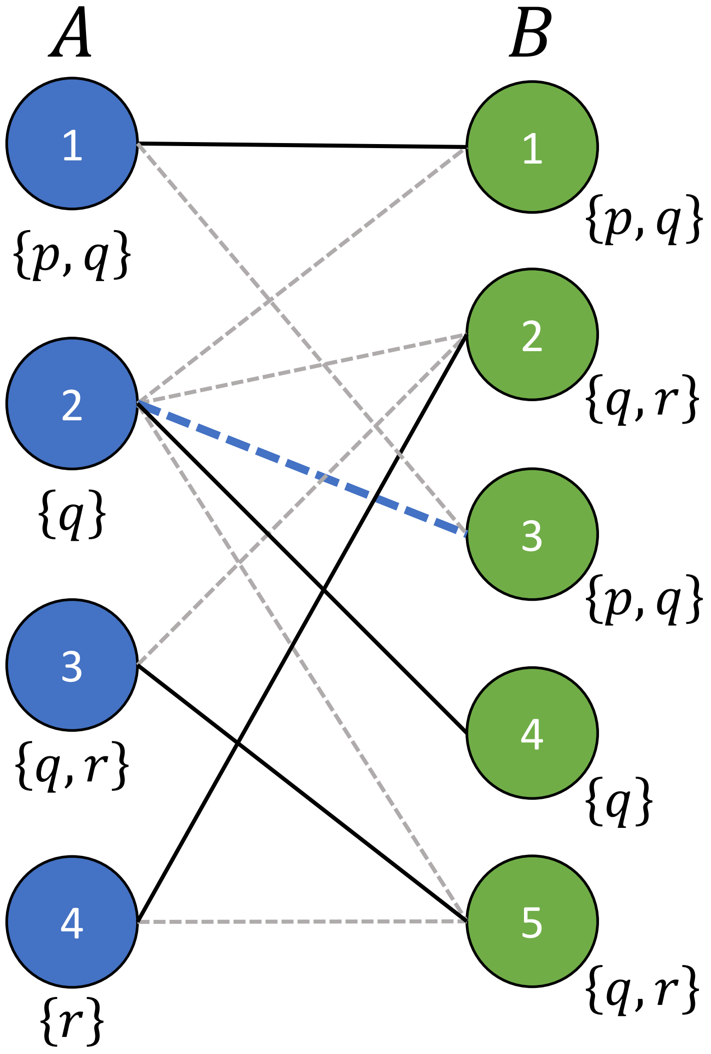
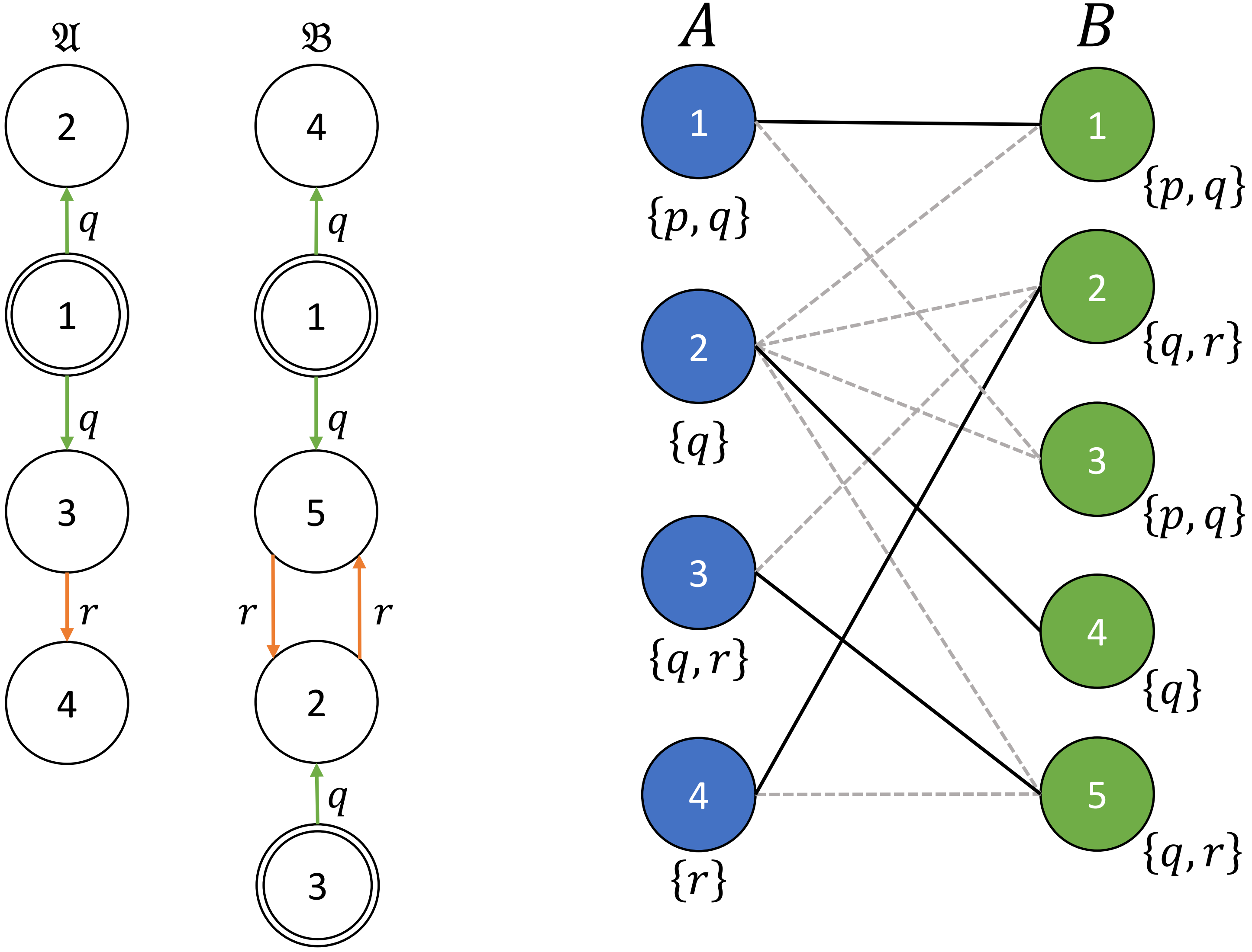




Consider edge $\langle 2,3 \rangle$:

- Inconsistent with $q(1,2)$
  - $\nexists q(*, 3) \in \mathfrak{B}$

Maximum Consistent Sub-Graph



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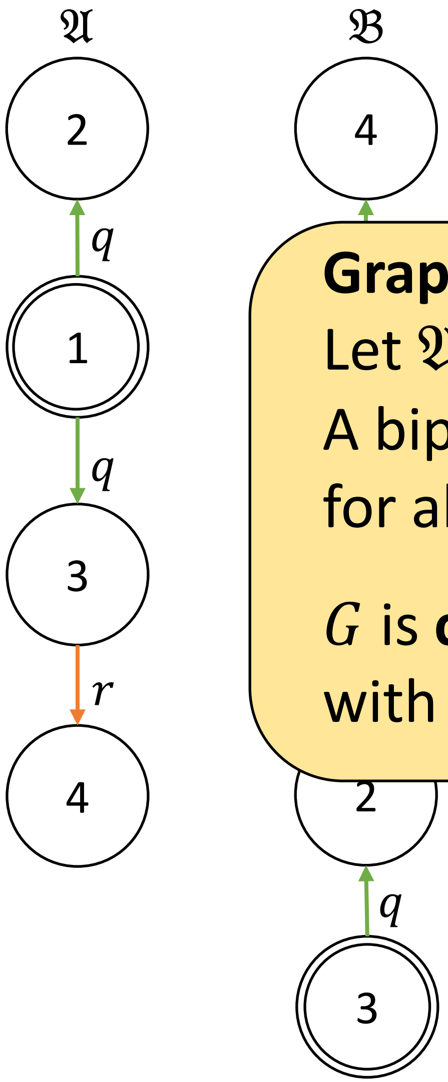
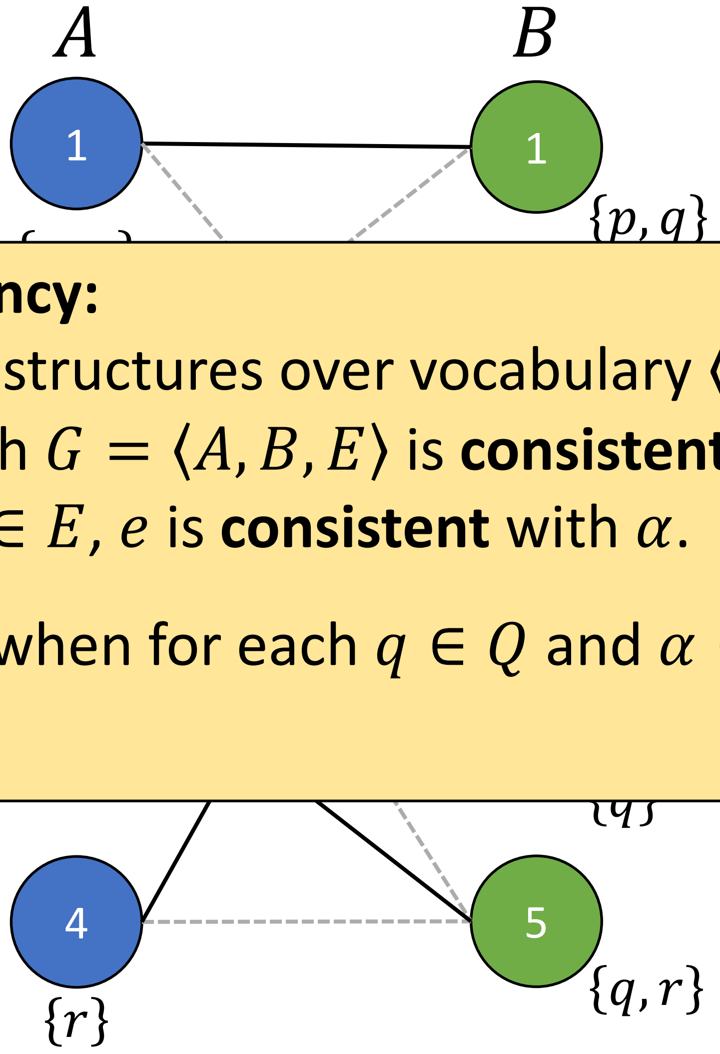
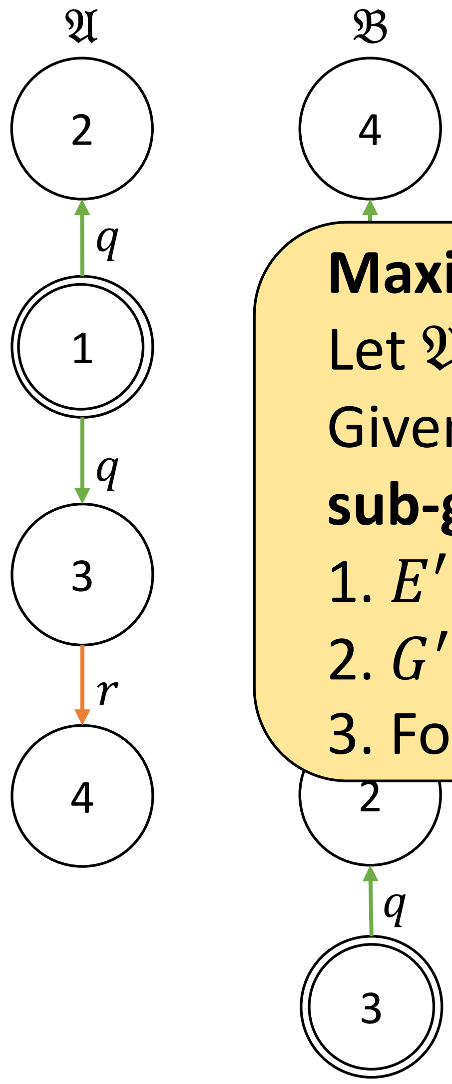


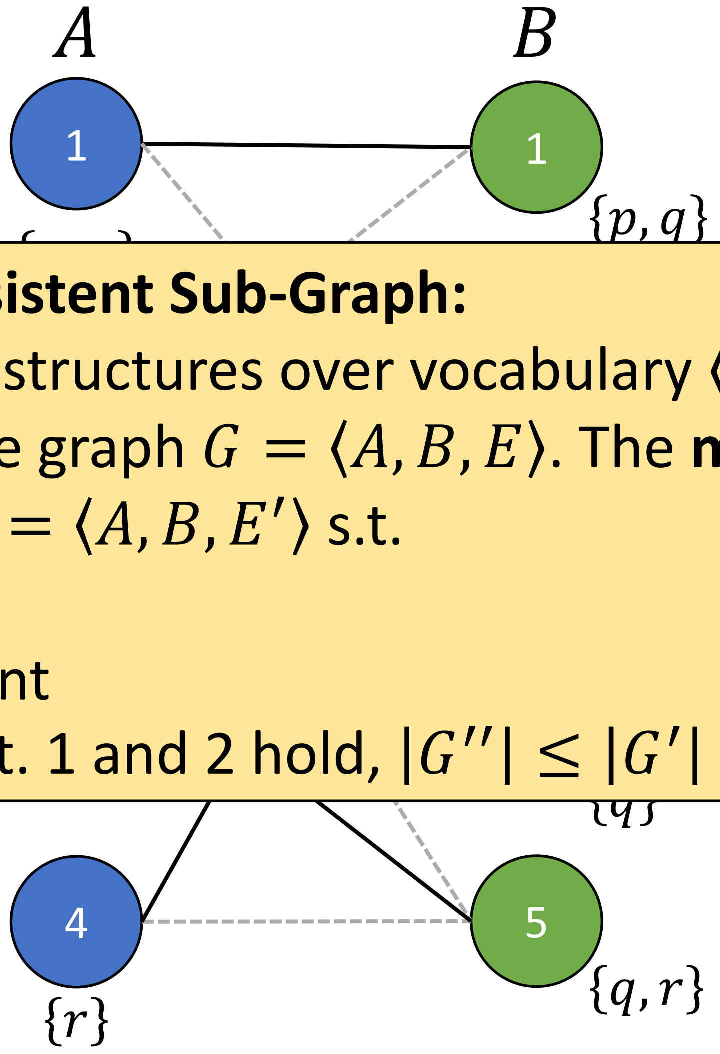


Graph Consistency:
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over vocabulary $\langle Q, ar \rangle$.
A bipartite graph $G = \langle A, B, E \rangle$ is **consistent** with $\alpha \in q^{\mathfrak{A}}$ when for all edges, $e \in E$, e is **consistent** with α.

G is **consistent** when for each $q \in Q$ and $\alpha \in q^{\mathfrak{A}}$, G is **consistent** with α.

Maximum Consistent Sub-Graph





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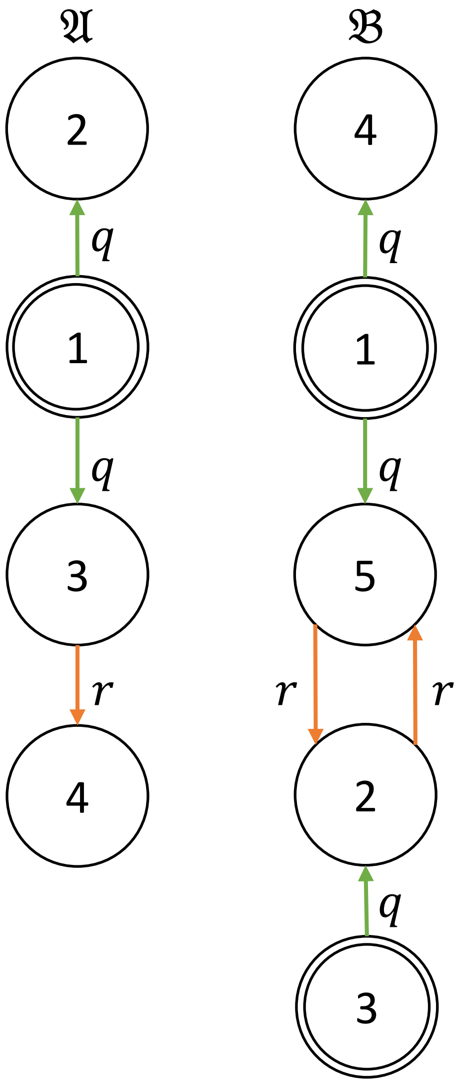
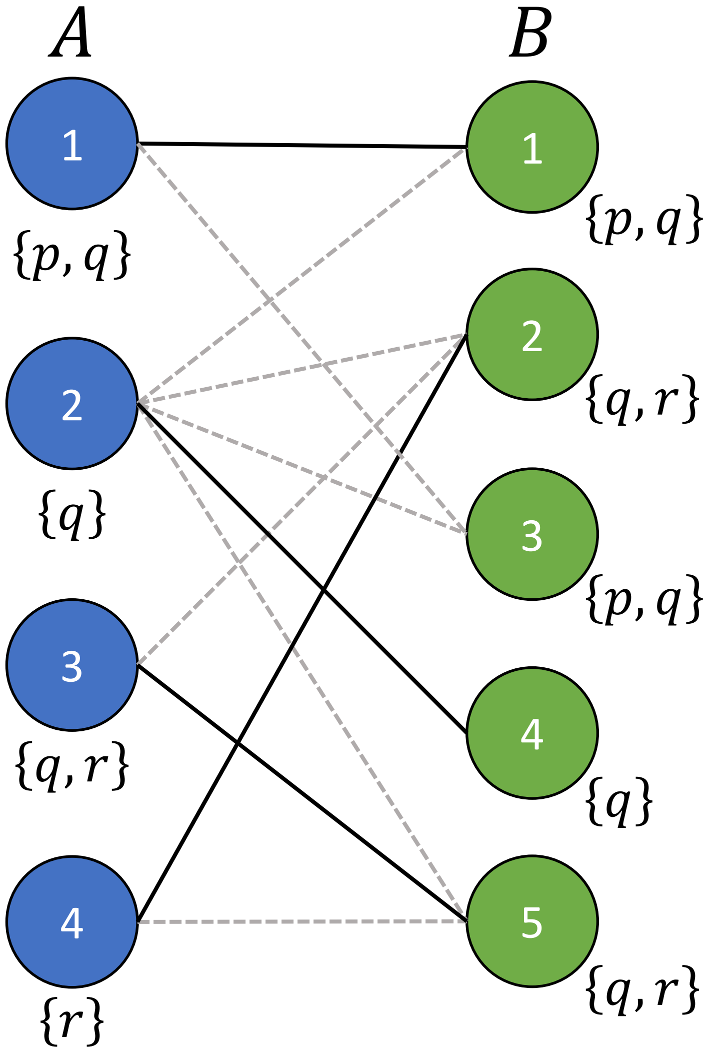
Given a bipartite graph $G = \langle A, B, E\rangle$. The **maximum consistent sub-graph** is $G' = \langle A, B, E'\rangle$ s.t.

1. $E' \subseteq E$
2. G' is consistent
3. For any G'' s.t. 1 and 2 hold, $|G''| \leq |G'|$

Maximum Consistent Sub-Graph





Goals:

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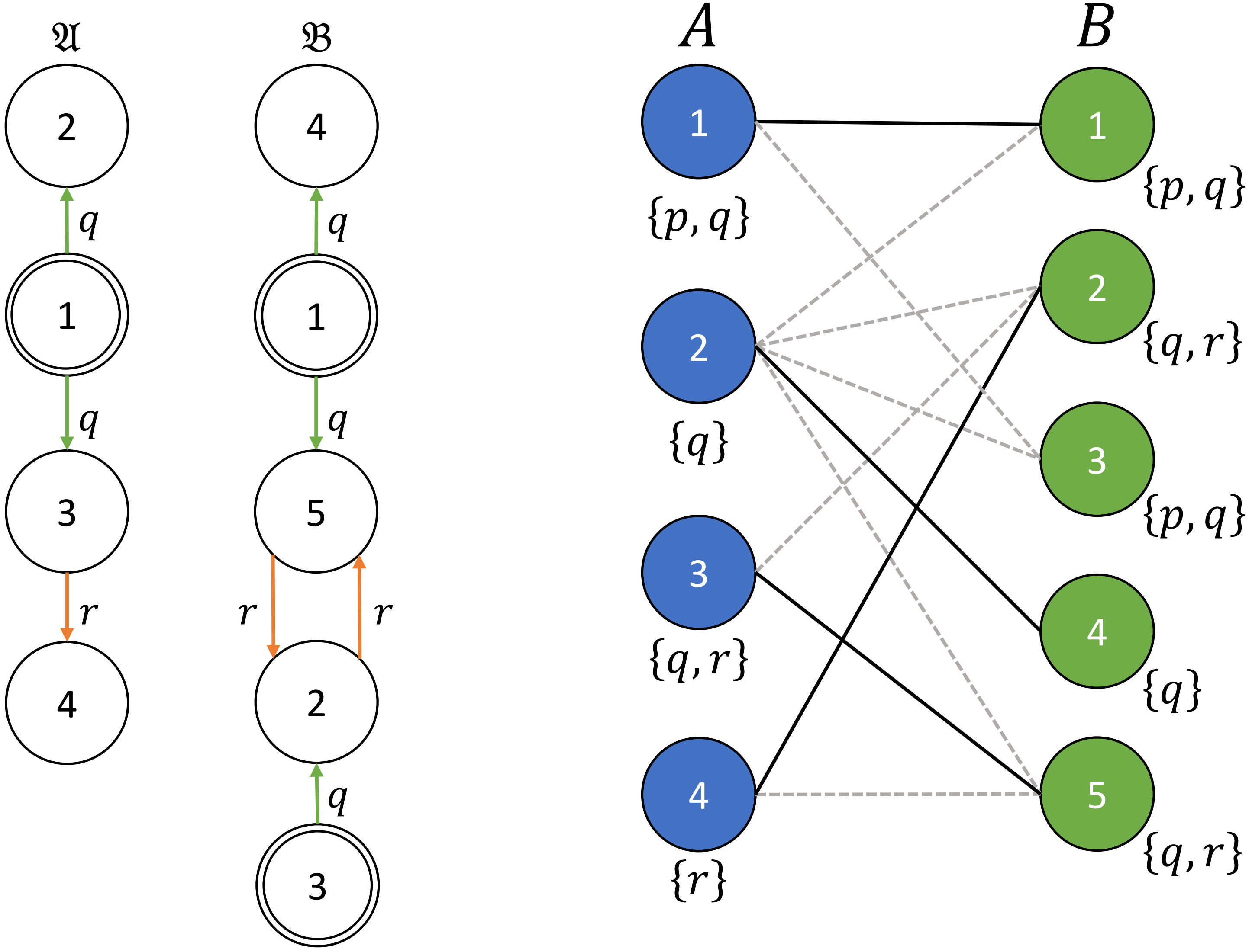

Goals:

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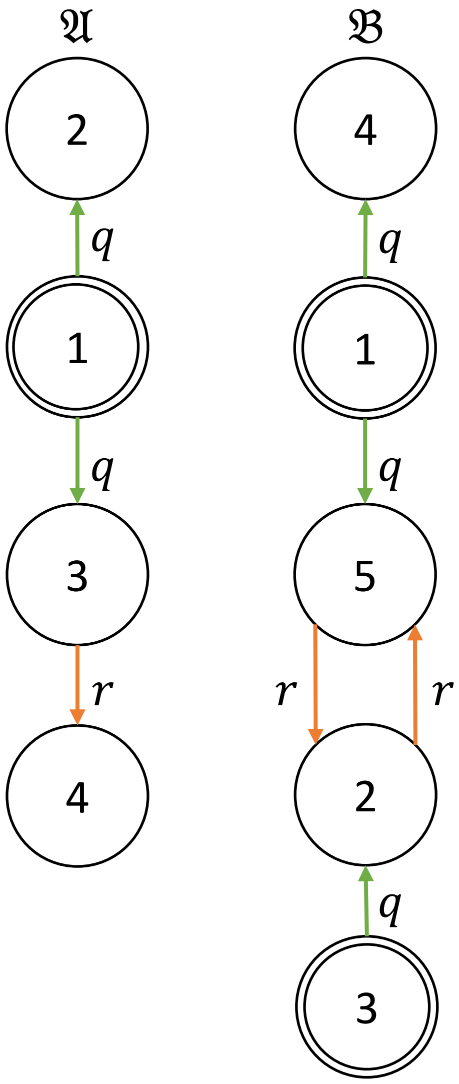
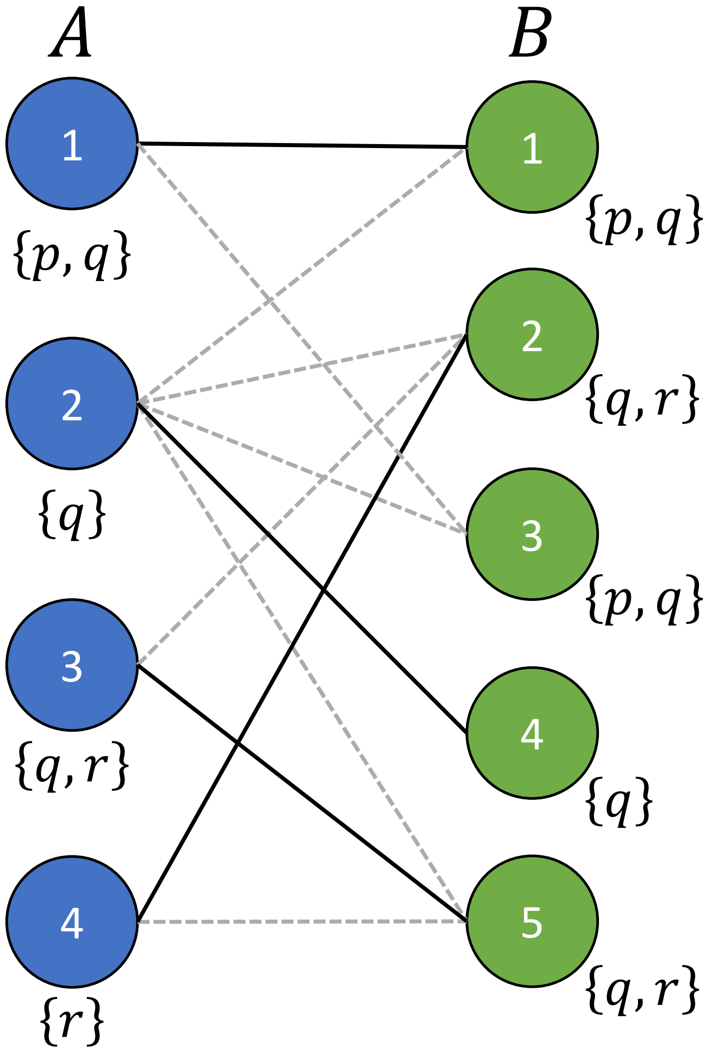
Maximum Consistent Sub-Graph





Goals:

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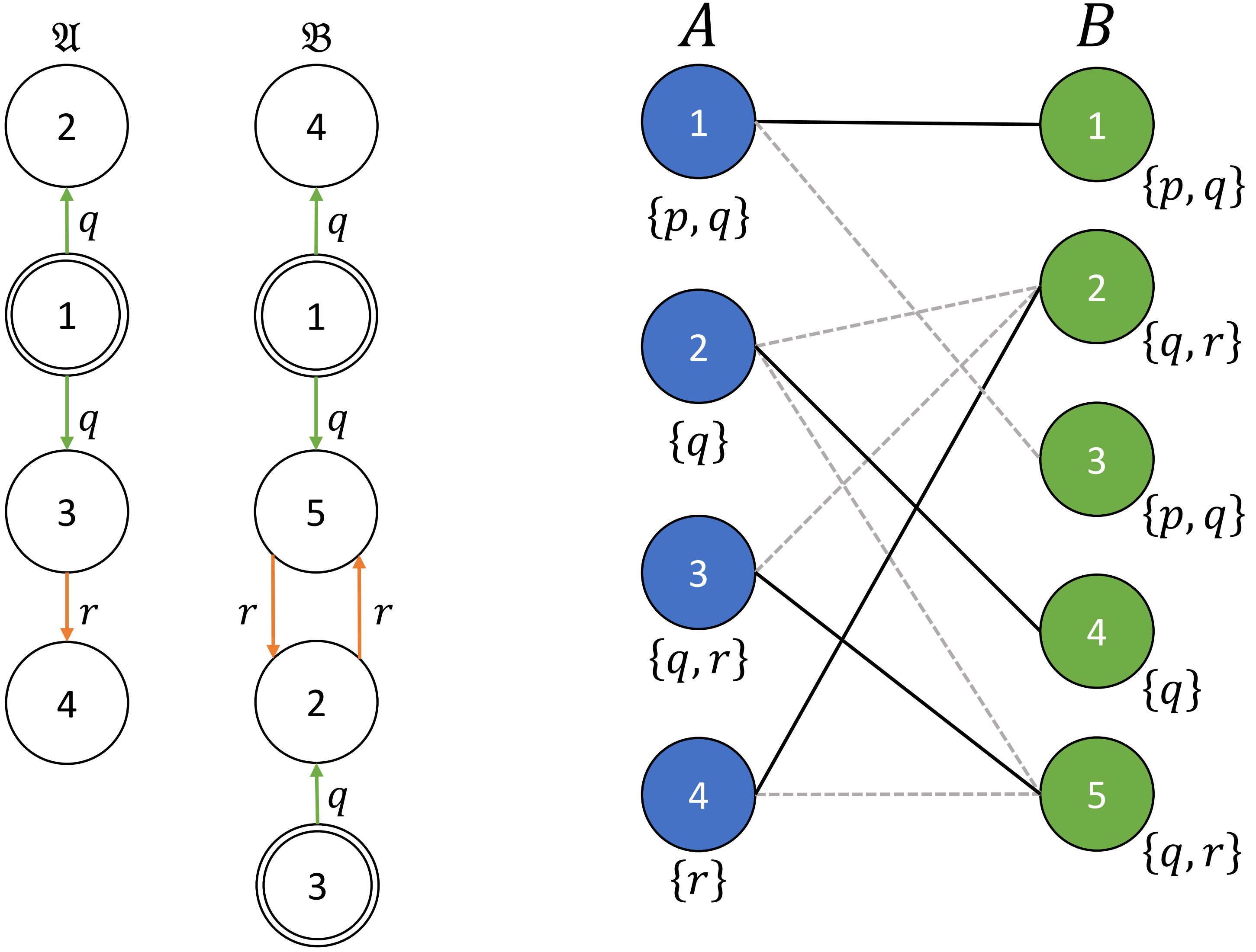
- Preserve embeddings

Maximum Consistent Sub-Graph

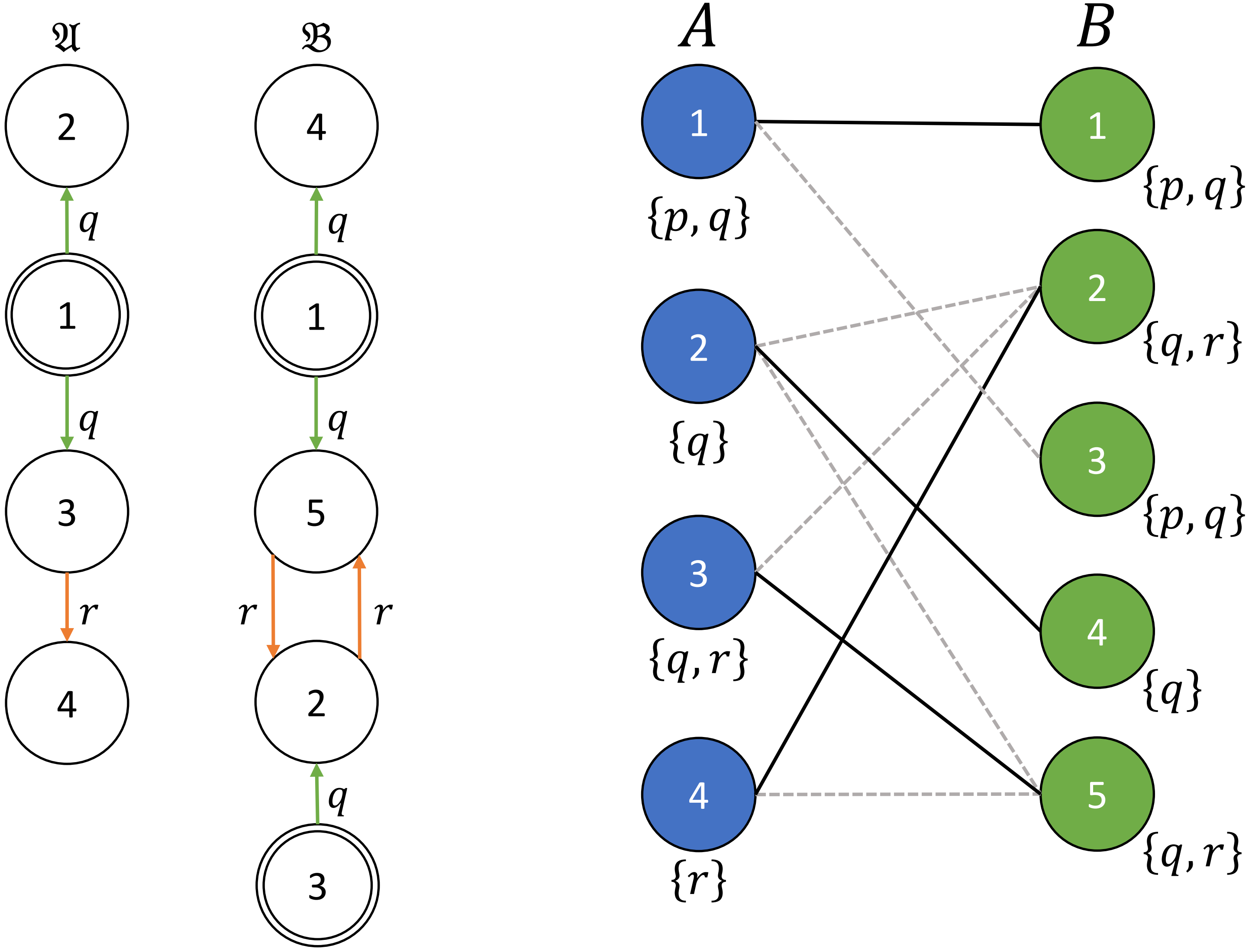


Goals:

- Remove inconsistent edges
- Preserve embeddings
- Efficiently Computable $O(E^2)$
  - Fixpoint Algorithm[1]

[Russel and Norvig. 2009][1]

Maximum Consistent Sub-Graph



$M_1 \stackrel{\text{def}}{=} \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 4,5\rangle\}$

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$M_3 \stackrel{\text{def}}{=} \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 4,5\rangle\}$

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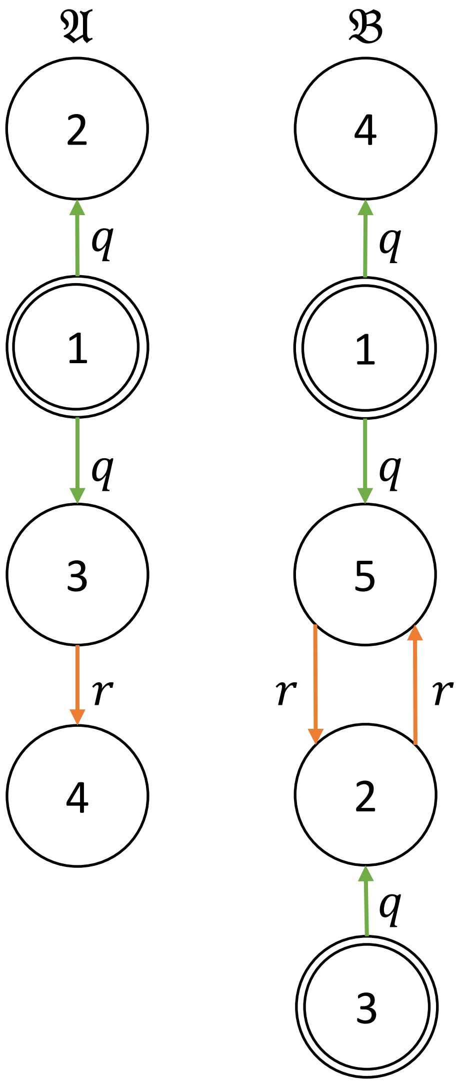
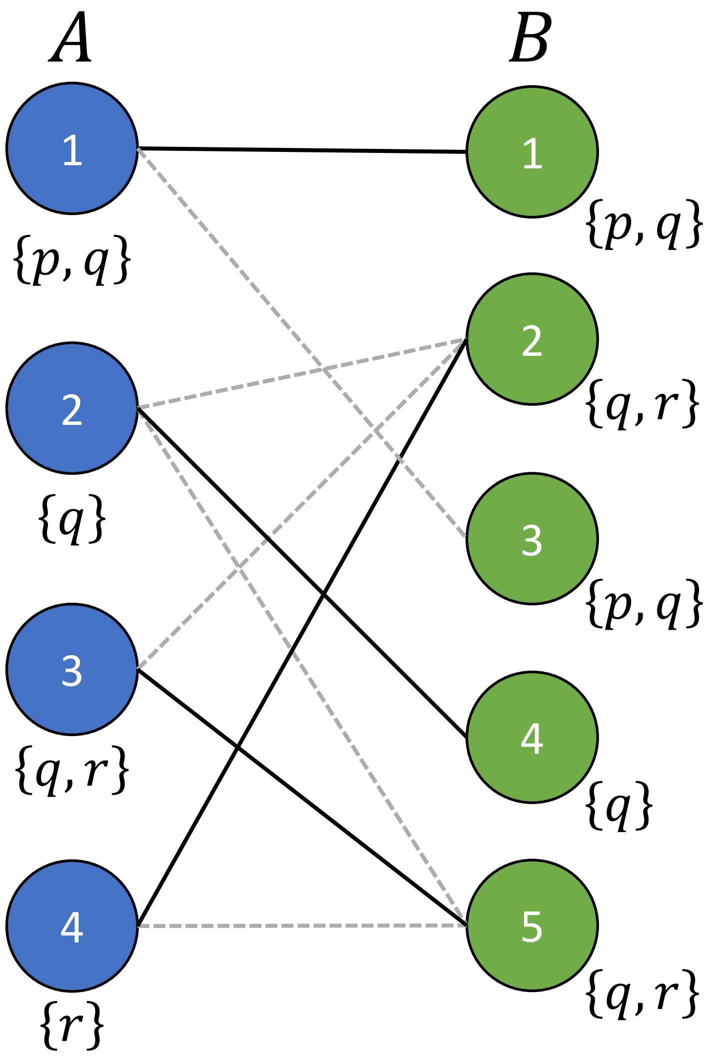
Match Embeds





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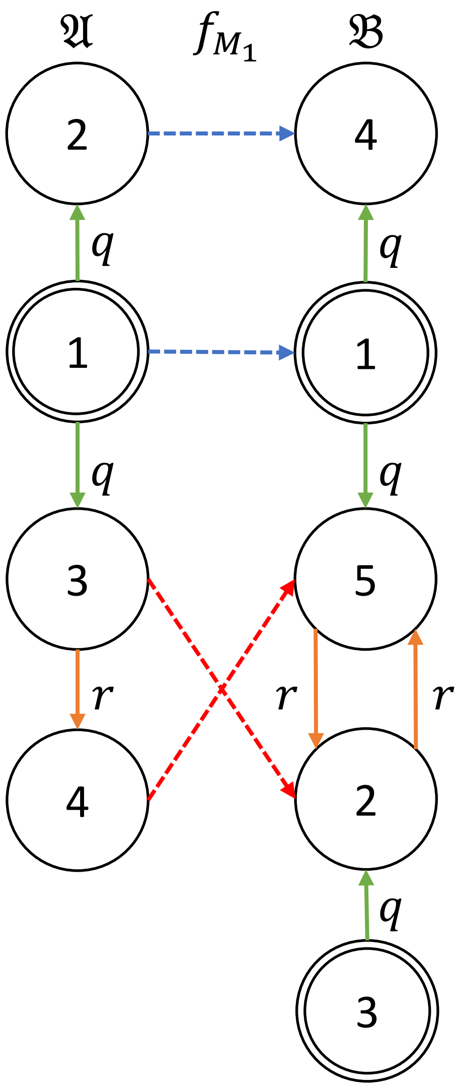

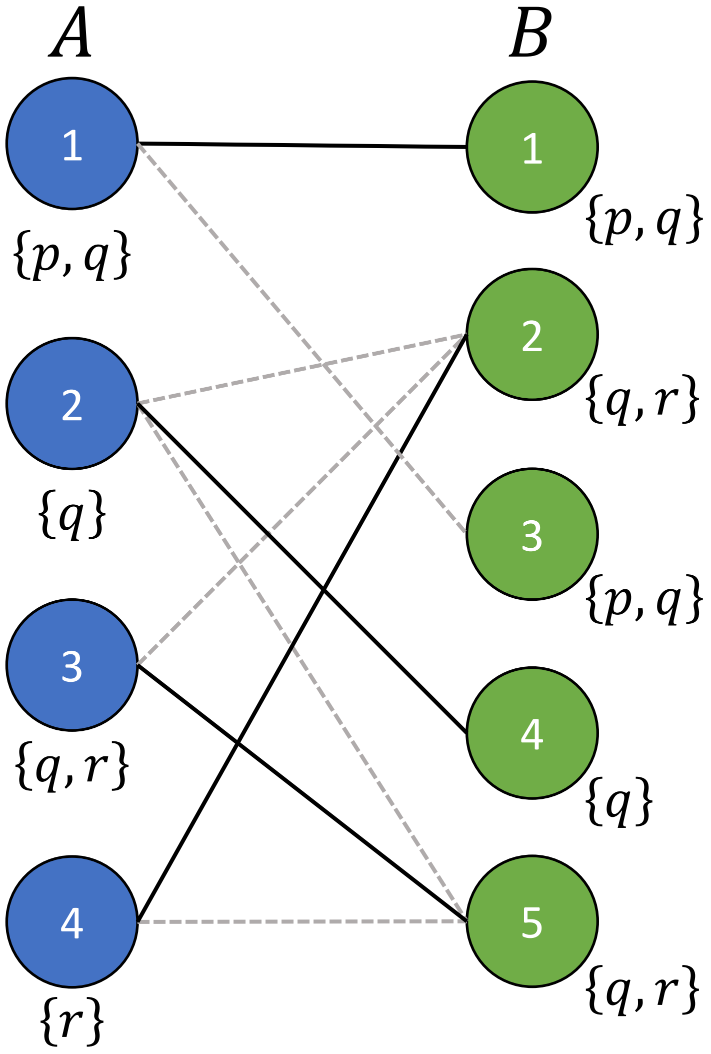

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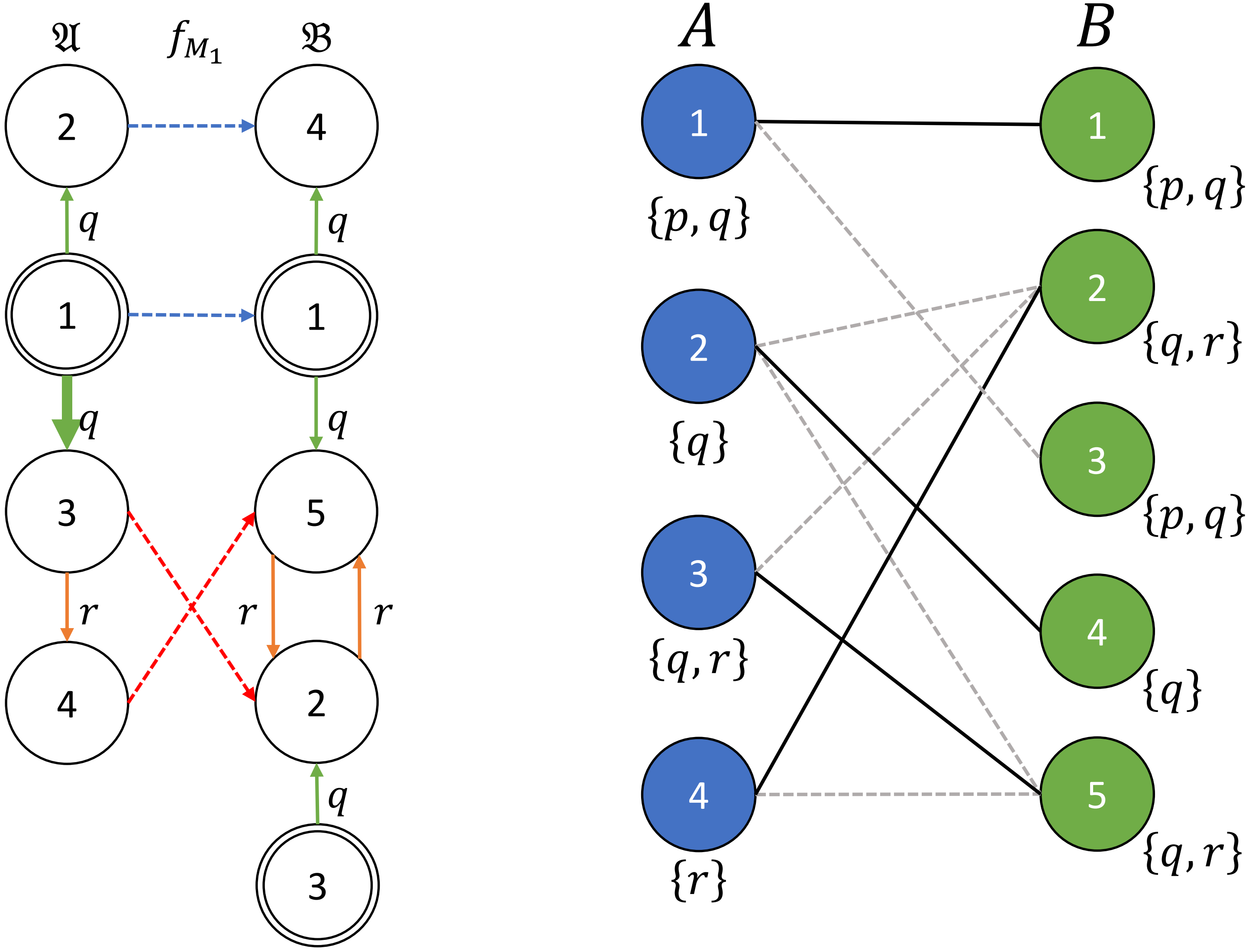
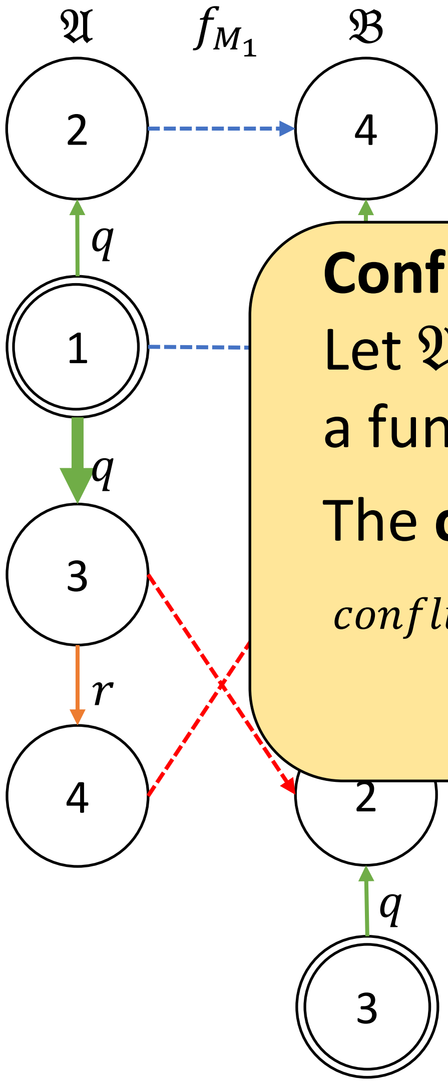


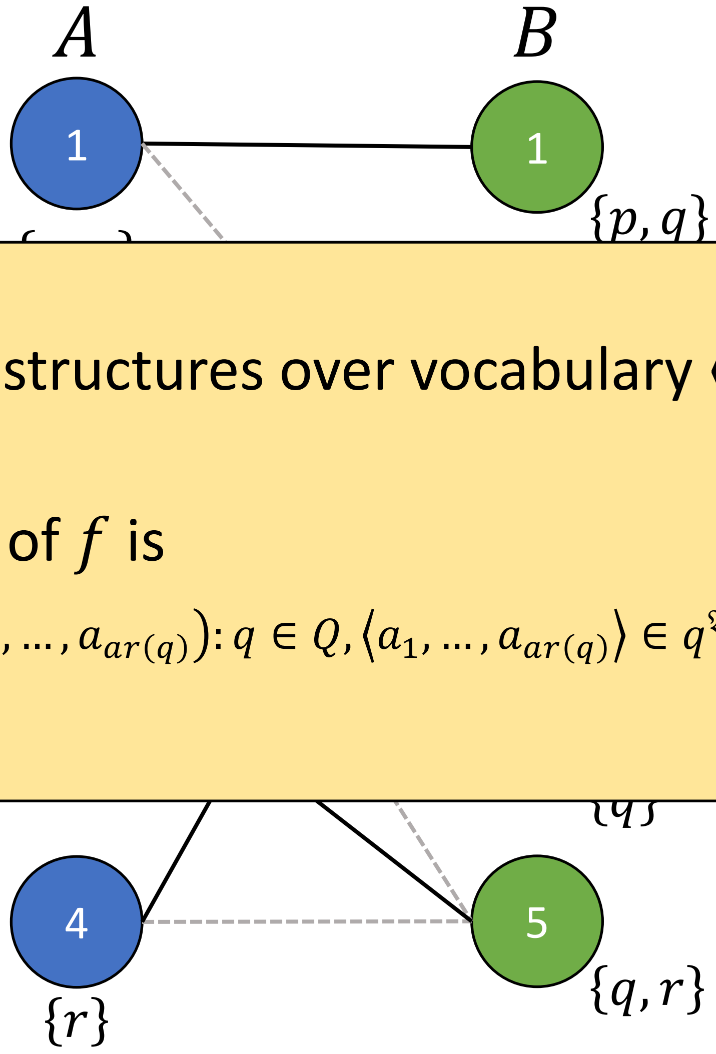
Compute Conflict Set

$$M_1 \stackrel{\text{def}}{=} \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 3,2\rangle, \langle 4,5\rangle\}$$

$$Conflict(f_{M_1}) \stackrel{\text{def}}{=} \{q(1,3)\}$$

Match Embeds





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$\{q(1,3)\}$

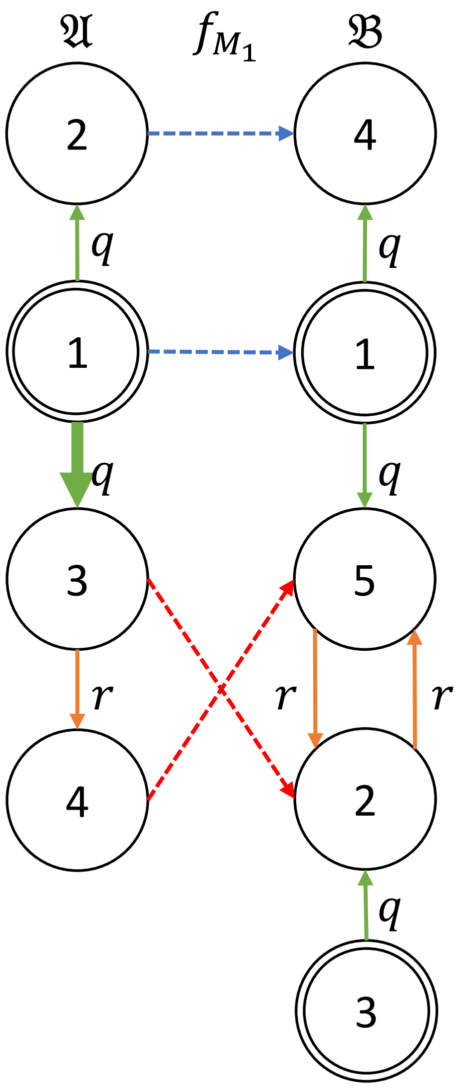
Conflict Set:

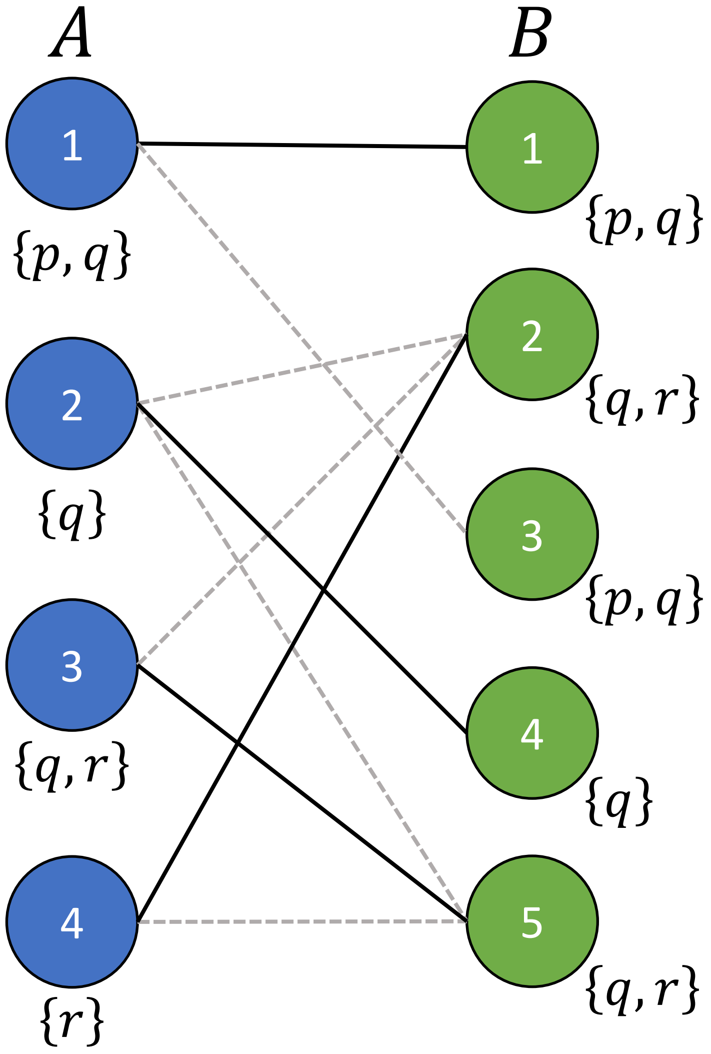
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over vocabulary $\langle Q, ar\rangle$ and $f: A \to B$ a function.

The **conflict set** of f is

$$conflict(f) \stackrel{\text{def}}{=} \{q(a_1, \dots, a_{ar(q)}) : q \in Q, \langle a_1, \dots, a_{ar(q)}\rangle \in q^{\mathfrak{A}}, \langle f(a_1), \dots, f(a_{ar(q)})\rangle \in q^{\mathfrak{B}}\}$$

Match Embeds





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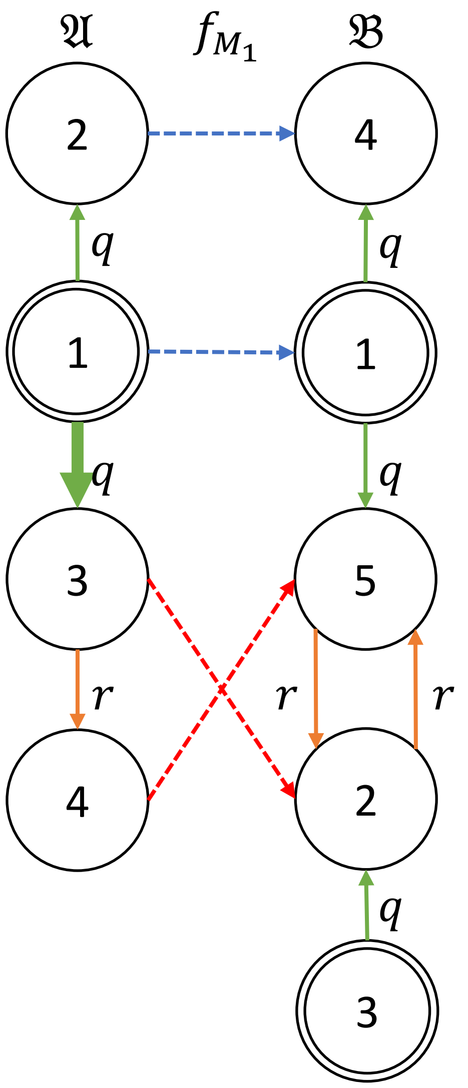

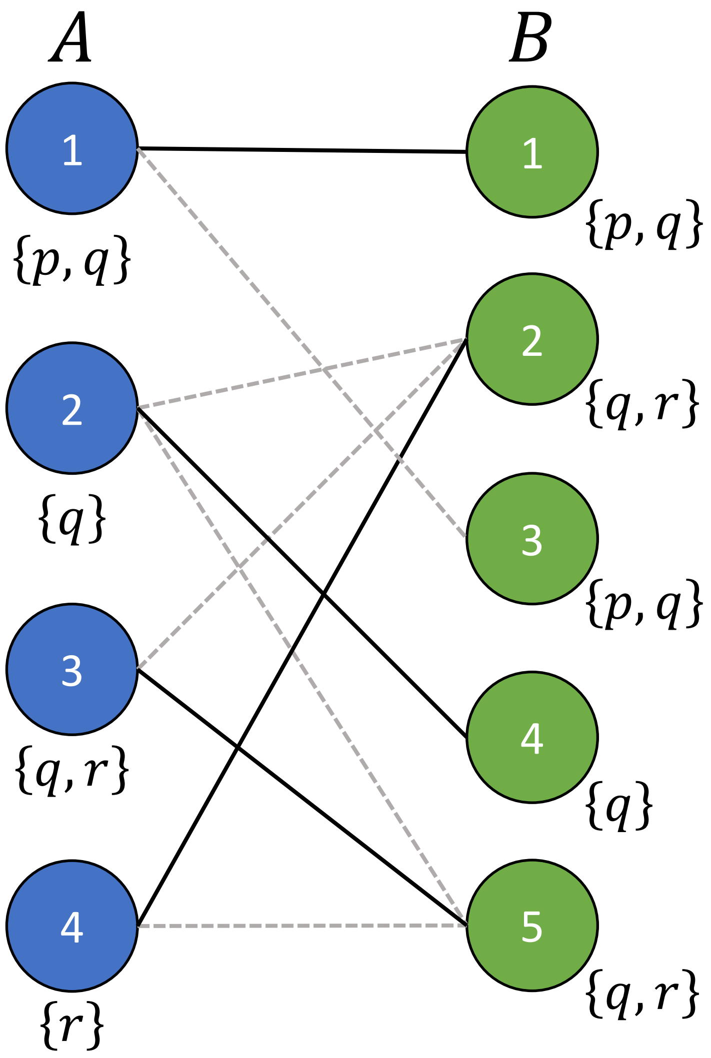

Compute Decisions

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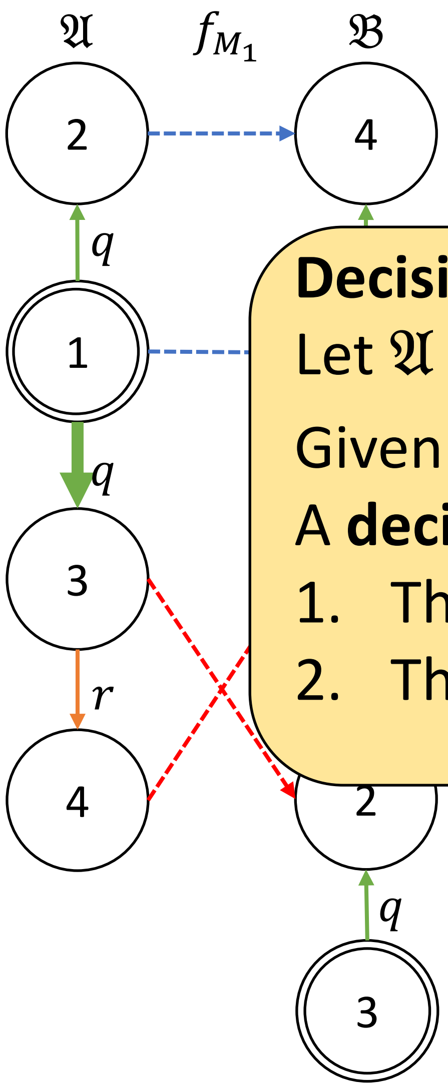
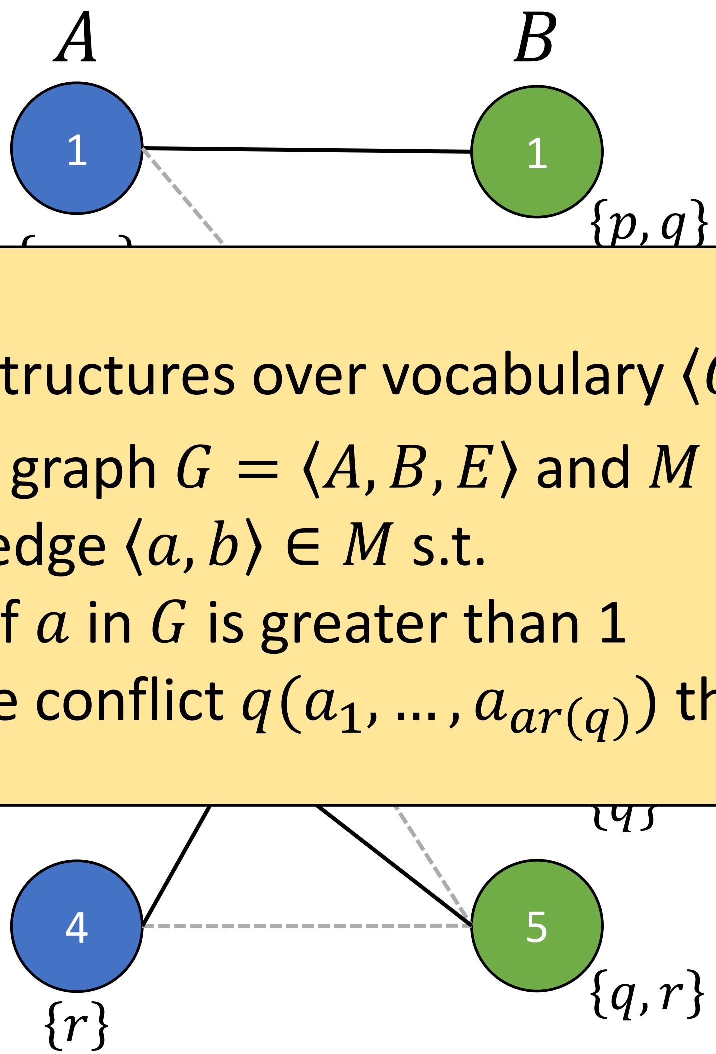
Match Embeds





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$\{q(1,3)\}$

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Decision:

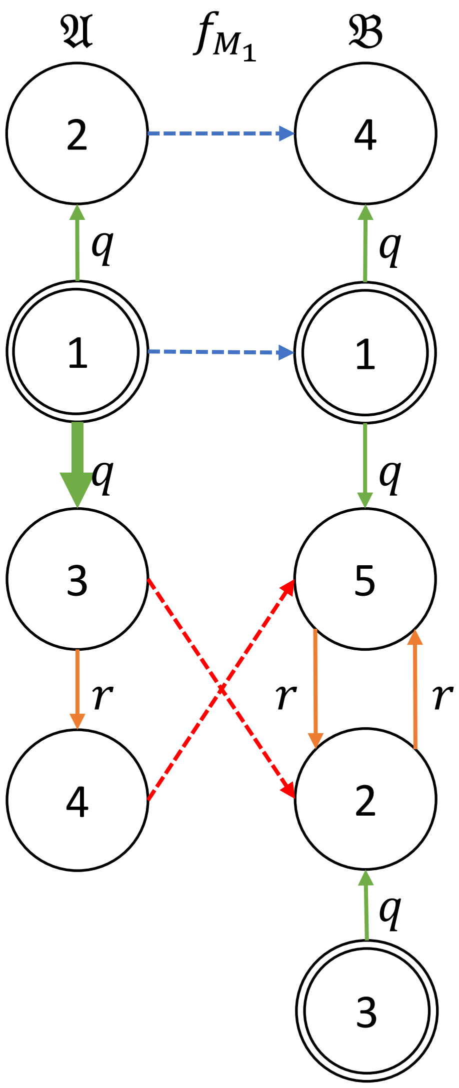
Let 𝔄 and 𝔅 be structures over vocabulary $\langle Q, ar\rangle$.

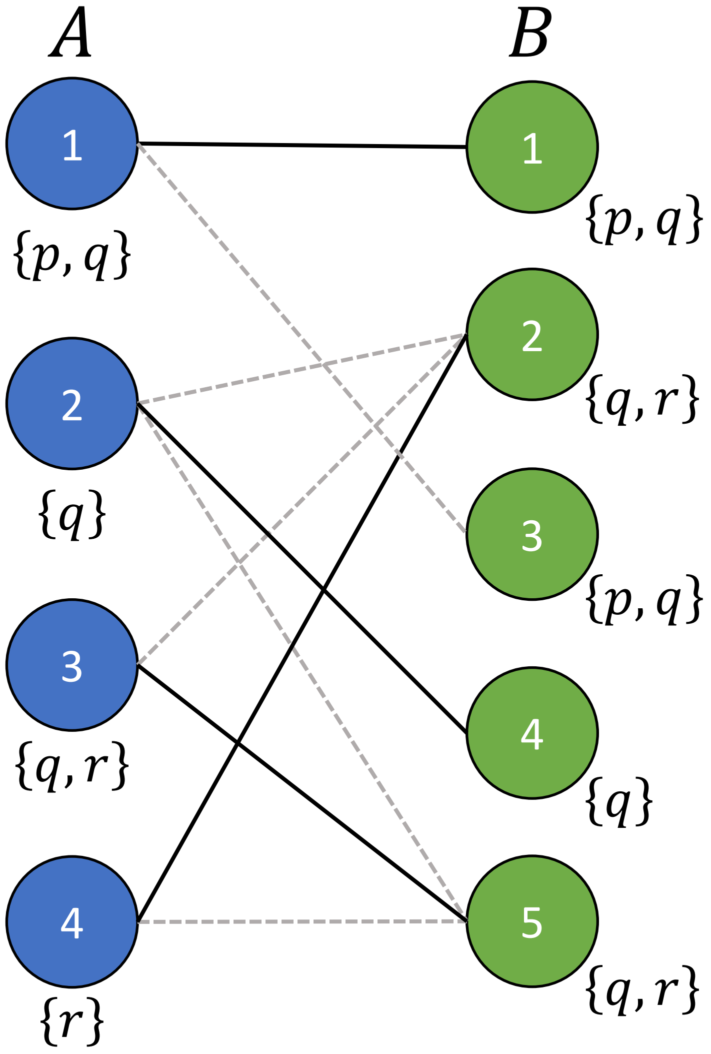
Given a bipartite graph $G = \langle A, B, E\rangle$ and M a matching on G.

A **decision** is an edge $\langle a, b\rangle \in M$ s.t.

1. The degree of a in G is greater than 1
2. There is some conflict $q(a_1, \ldots, a_{ar(q)})$ that involves a ($a = a_i$).

Match Embeds





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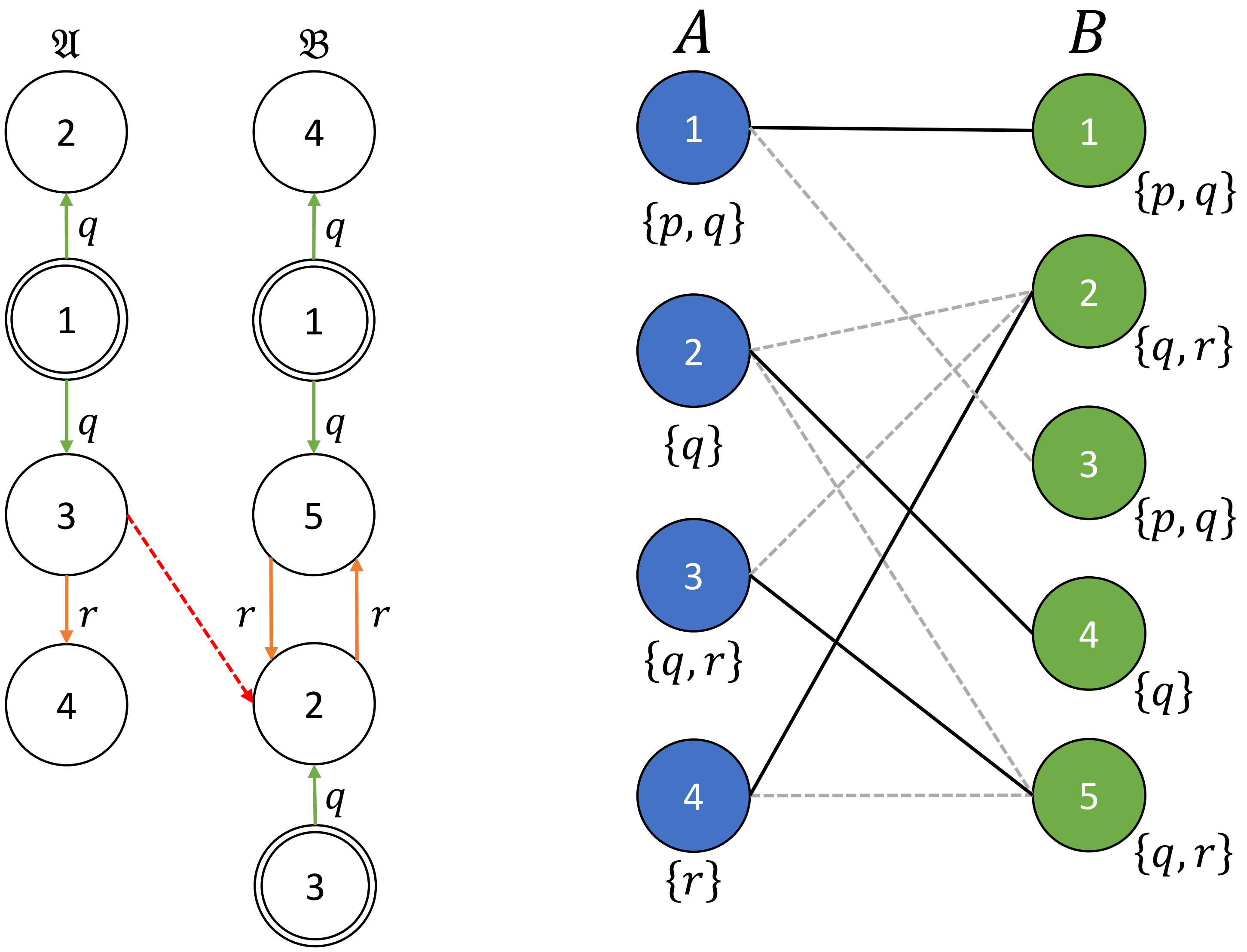
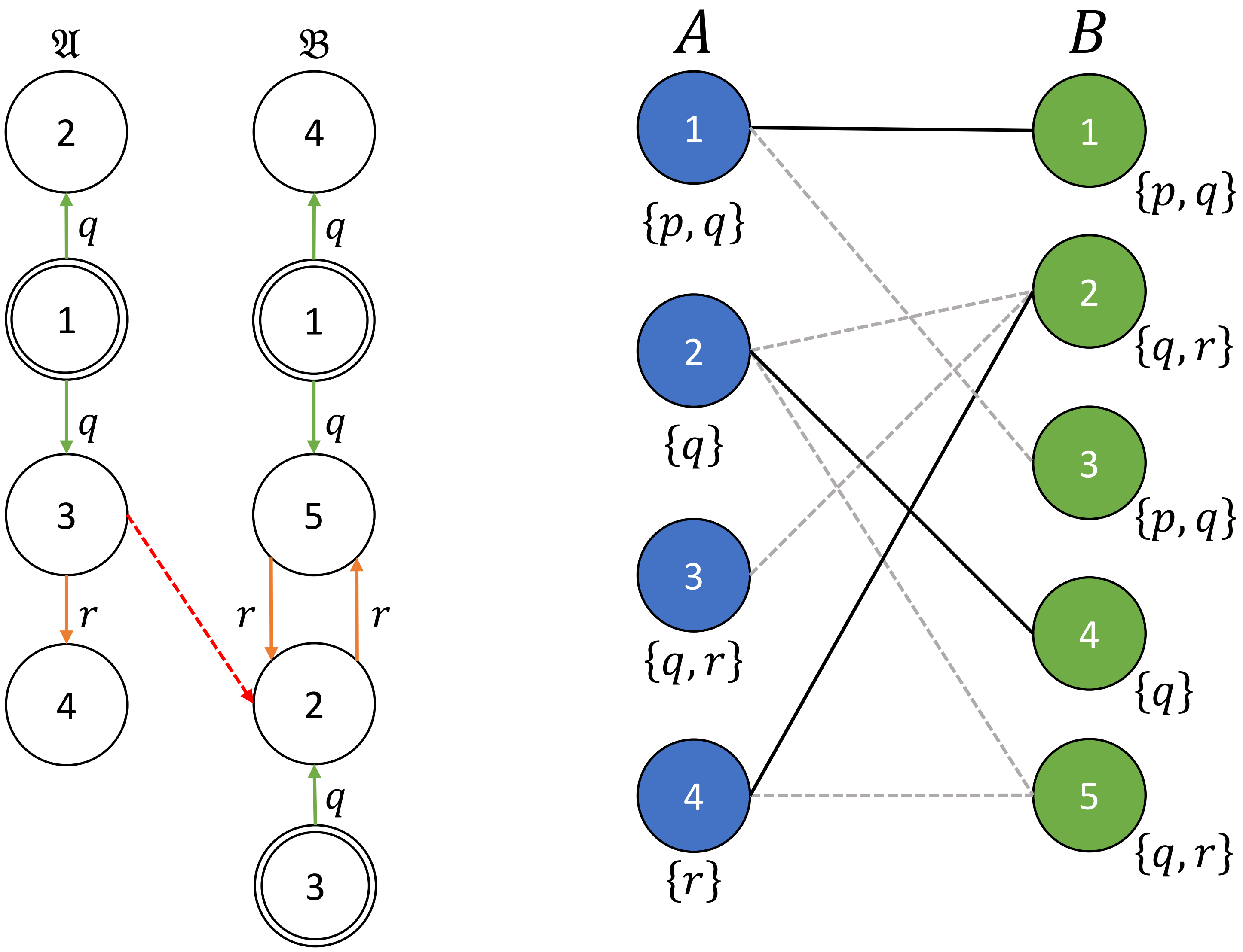


Decide $[3 \longmapsto 2]$

Match Embeds



Decide $[3 \longmapsto 2]$

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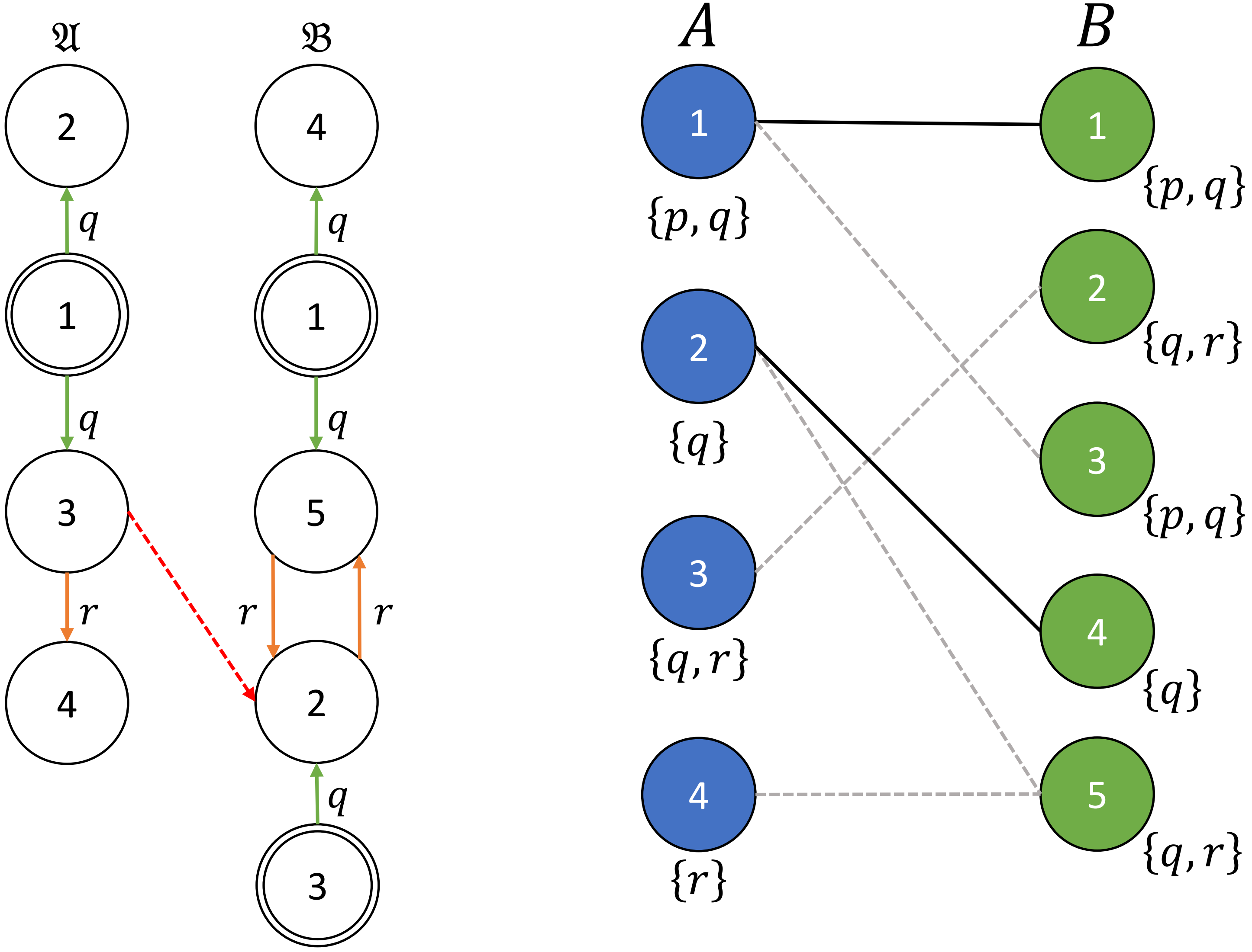

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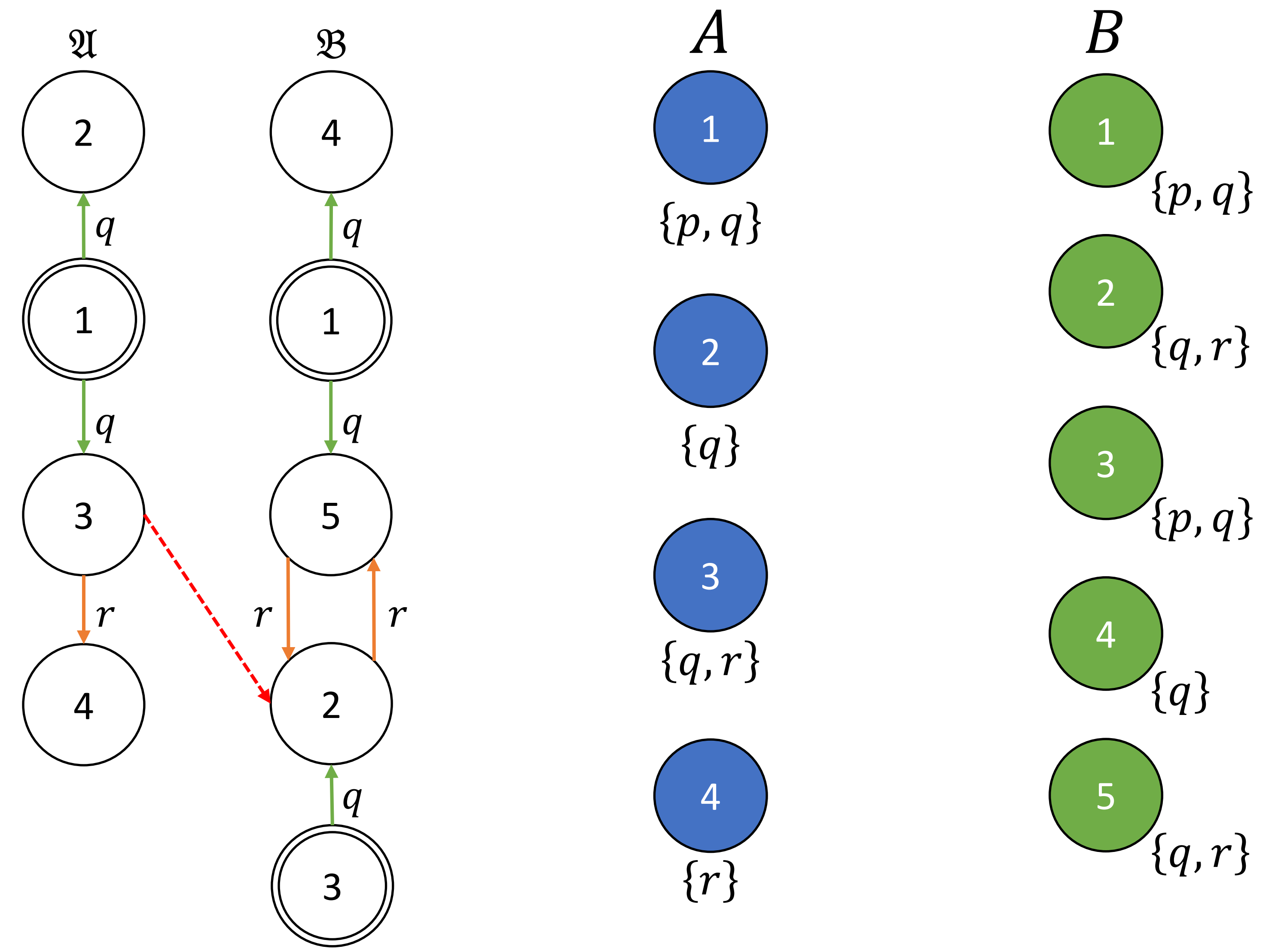

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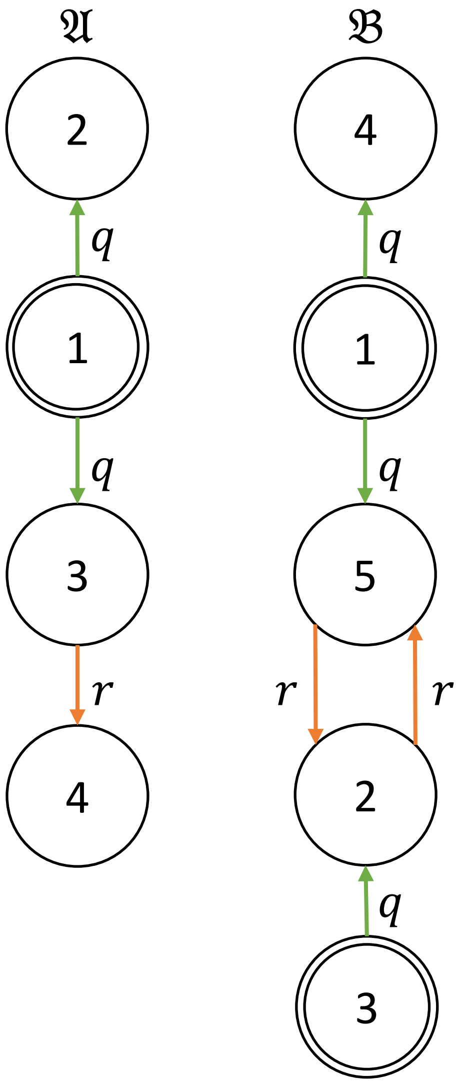

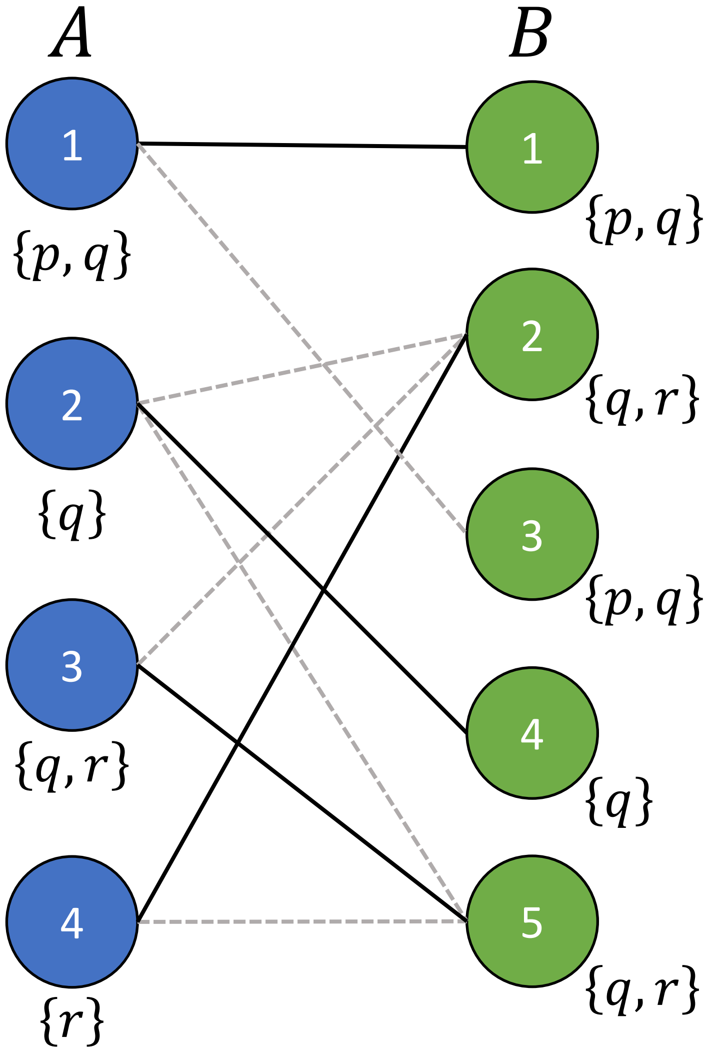

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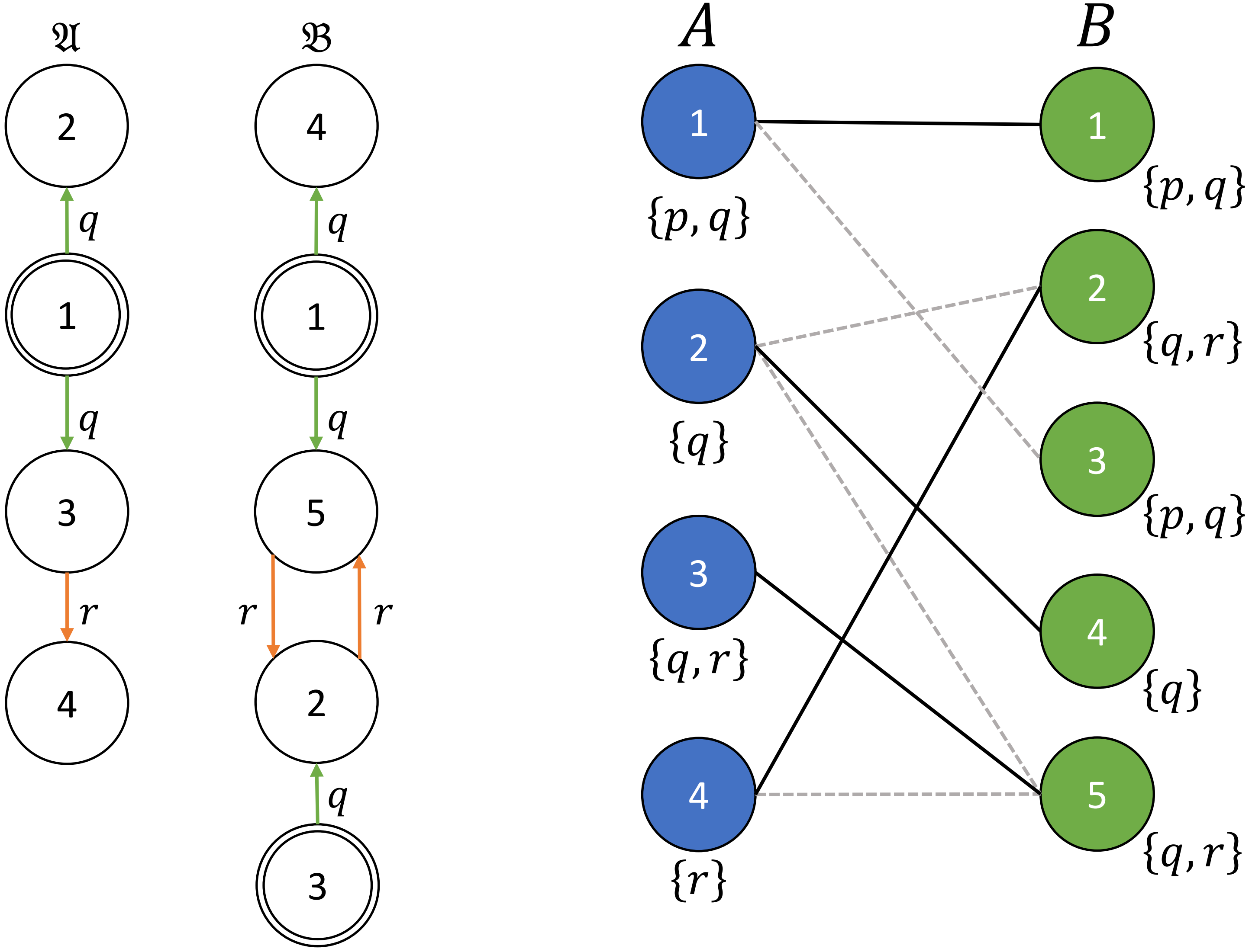

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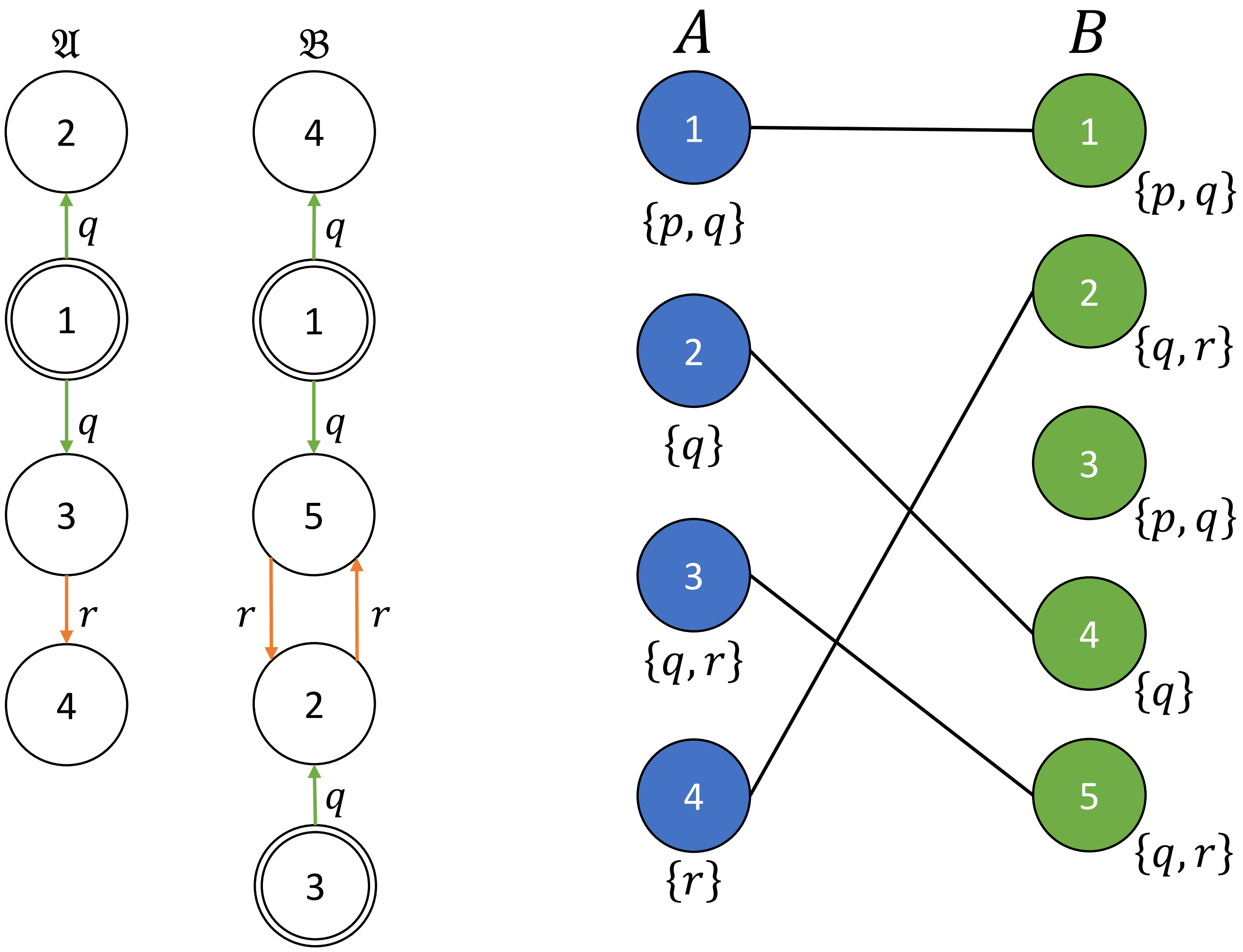

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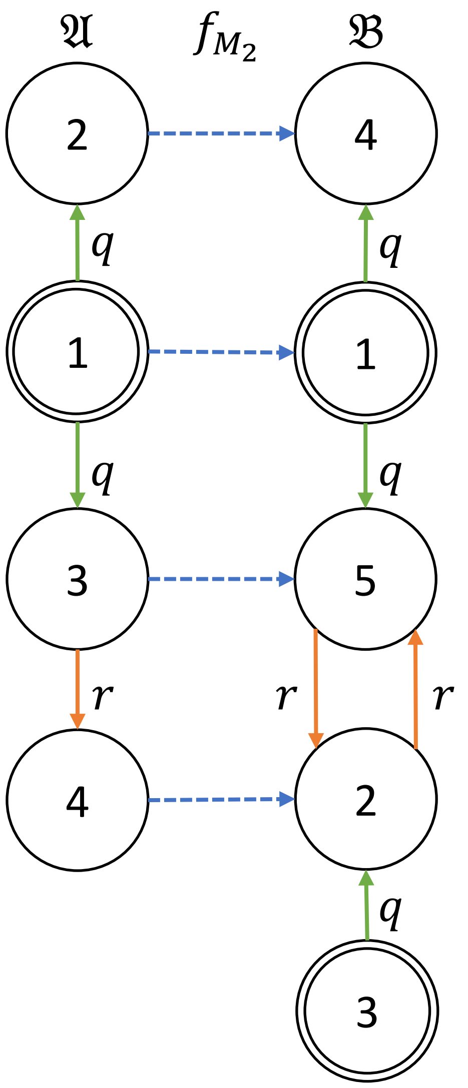

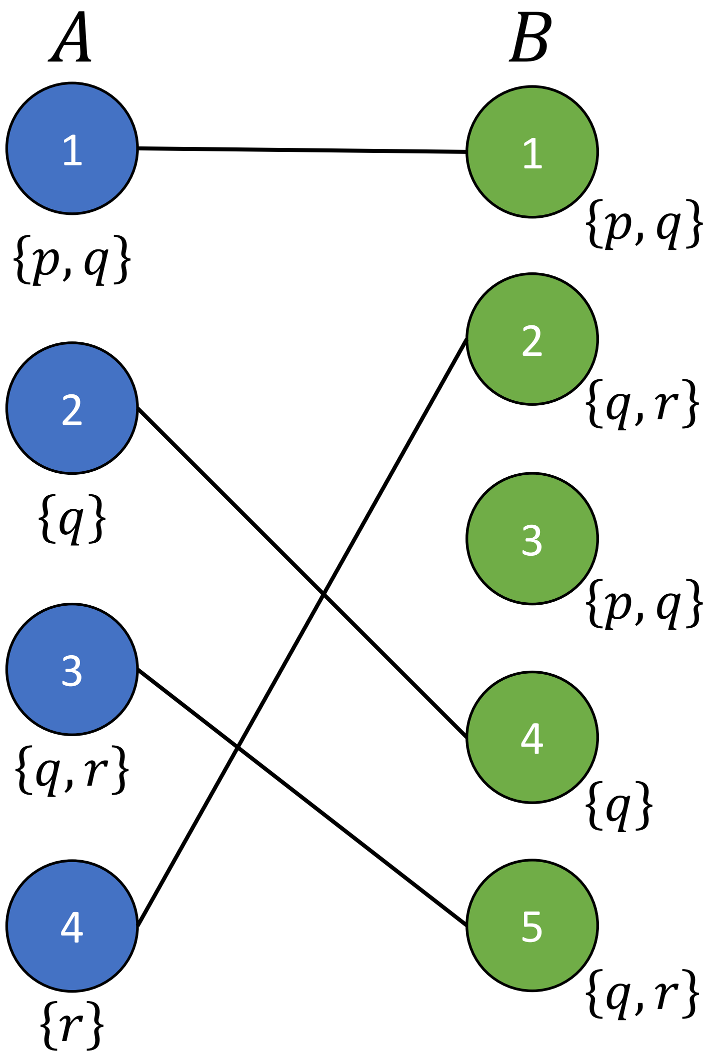

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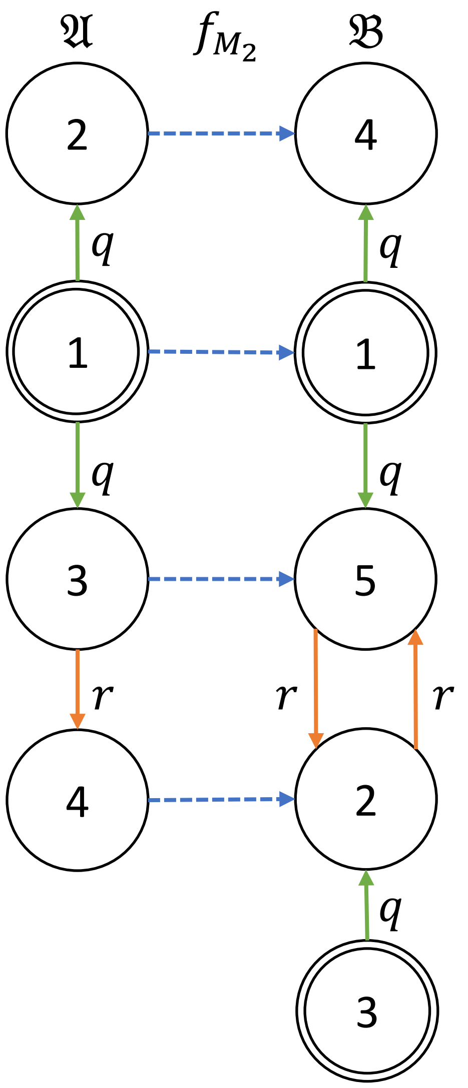

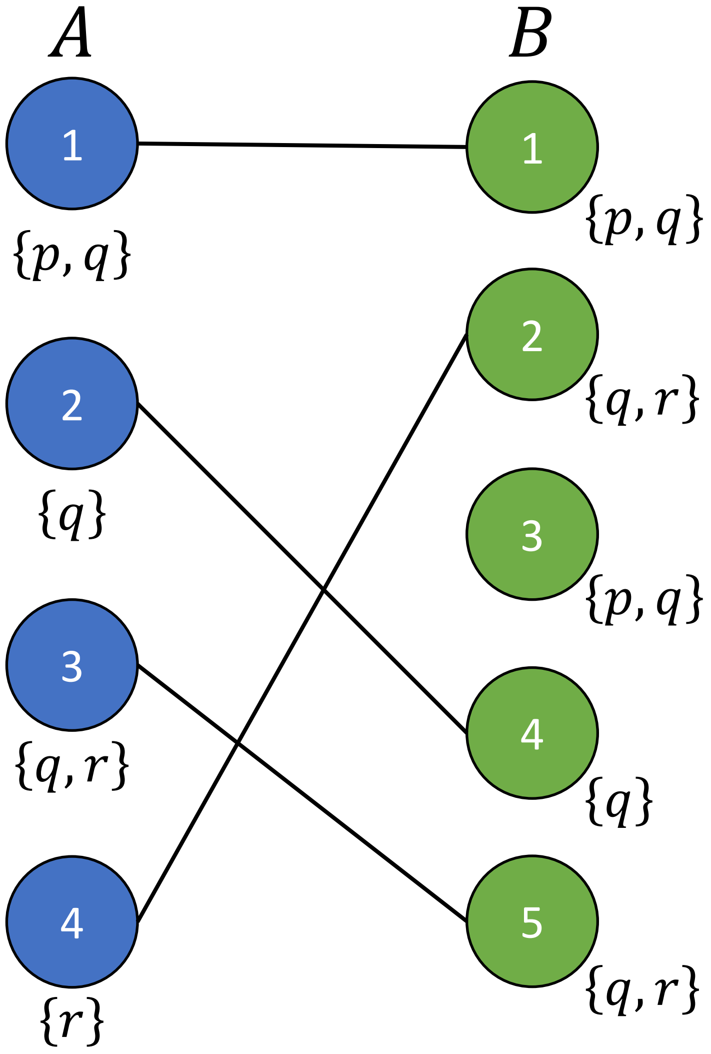

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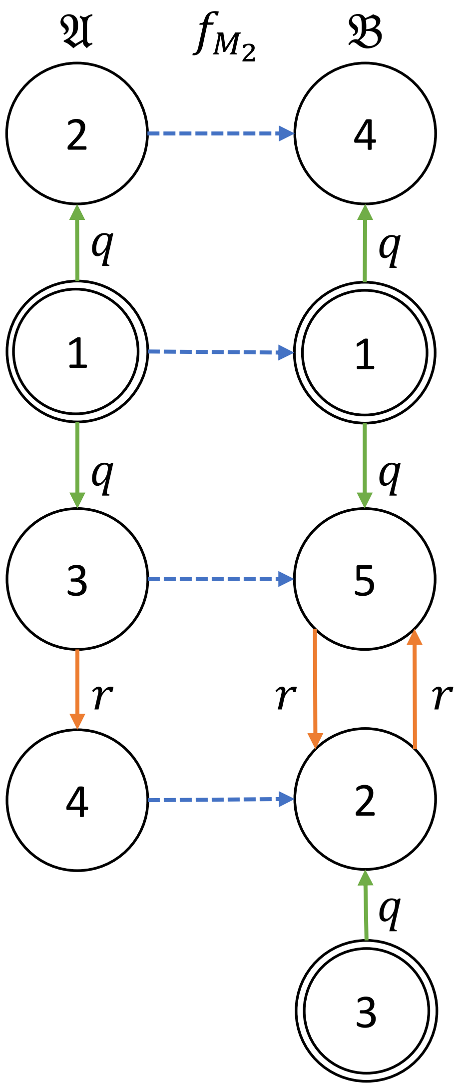

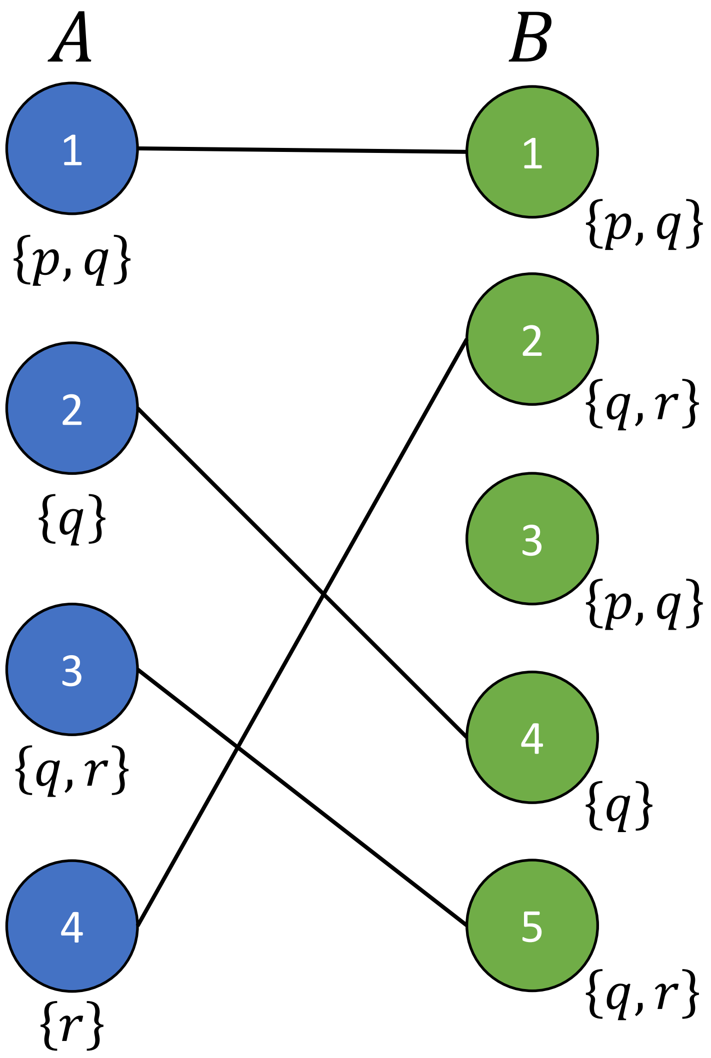

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Match Embeds for Program verification

- Practical procedure for deciding structure embedding problem

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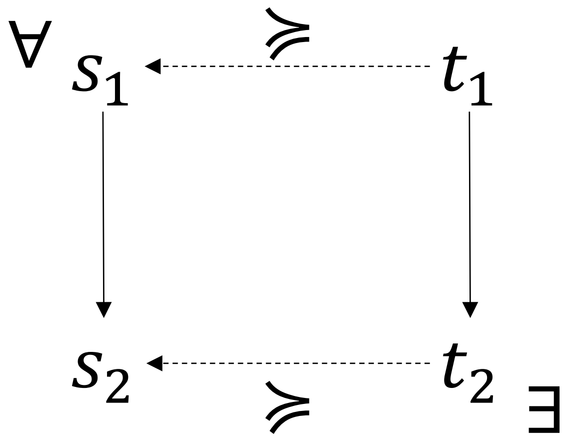
- For Predicate Automata prune unnecessary branches:



Match Embeds for Program verification

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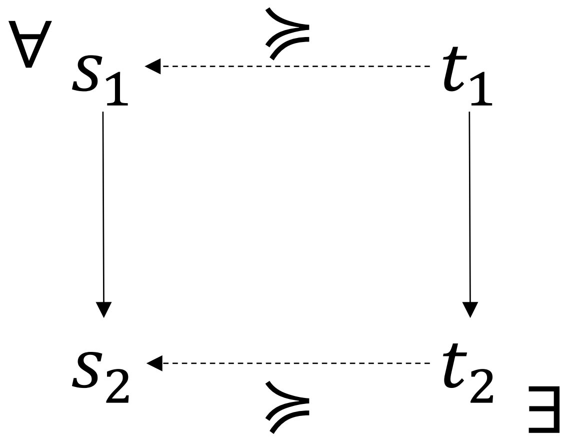

- Need to search for some already explored t_1 to prune s_1.

Multi-Source Single-Target Embeddings

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  - Use range queries on k-d tree and test returned structures

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- Let structures be over vocabulary $\langle Q, ar \rangle$
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  - $v(\mathfrak{A})_i = 1 \Leftrightarrow q_i^{\mathfrak{A}} \neq \emptyset \quad (q_i(\dots) \in \mathfrak{A})$

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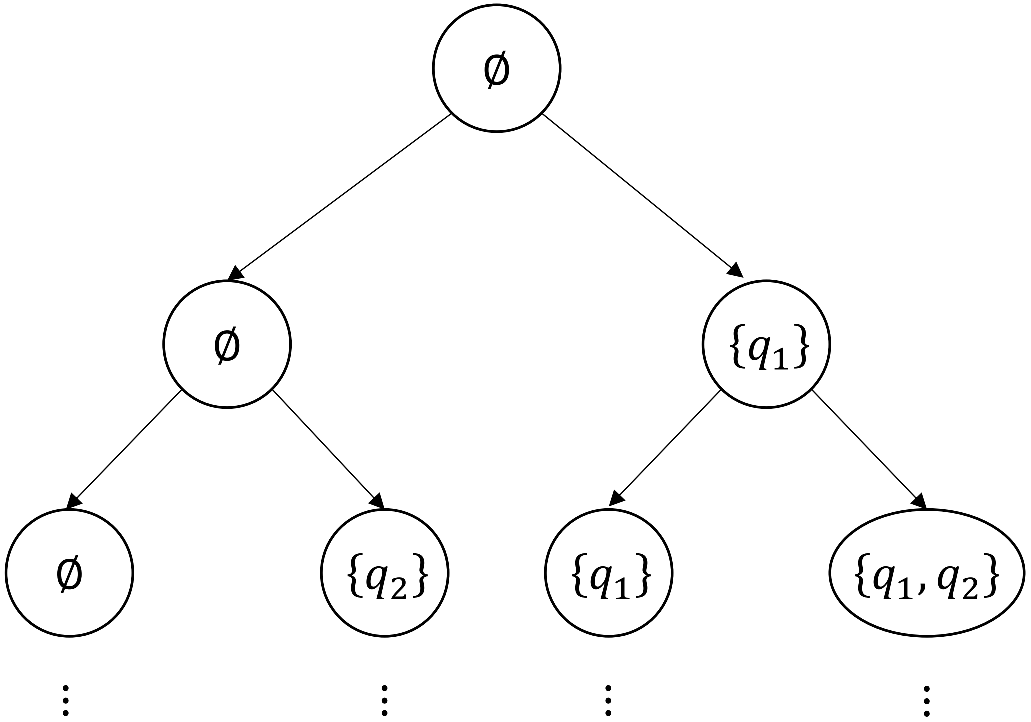
k-d Tree Structure



Multi-Source Single-Target Embeddings

- Range Query: $\mathfrak{A} = \langle q_2(1), q_2(2) \rangle$
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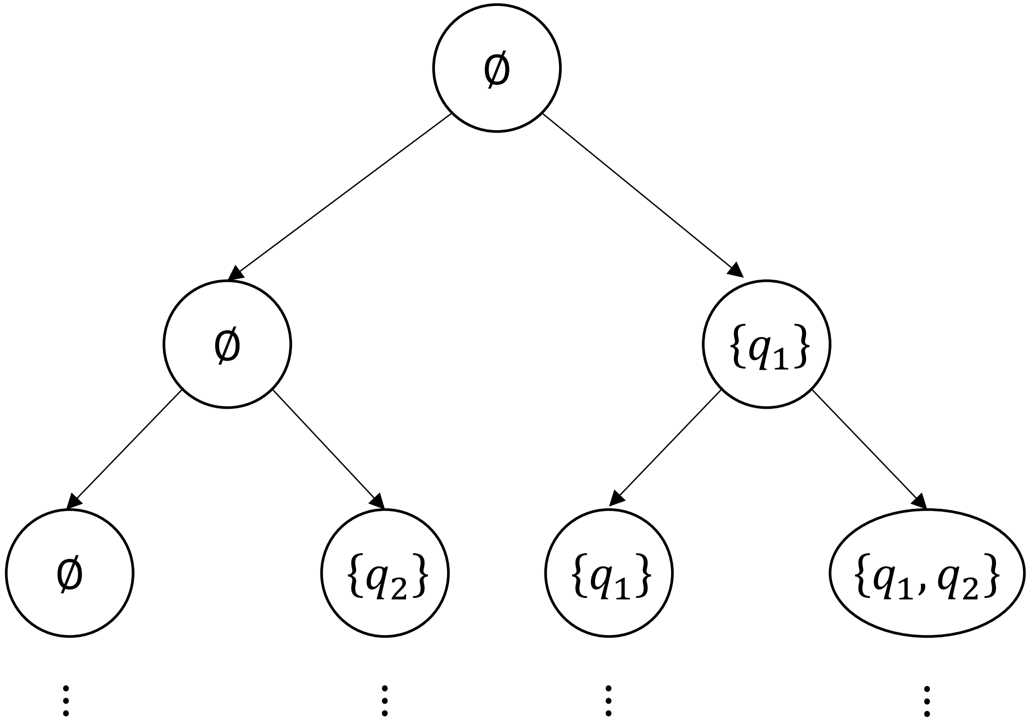
k-d Tree Structure



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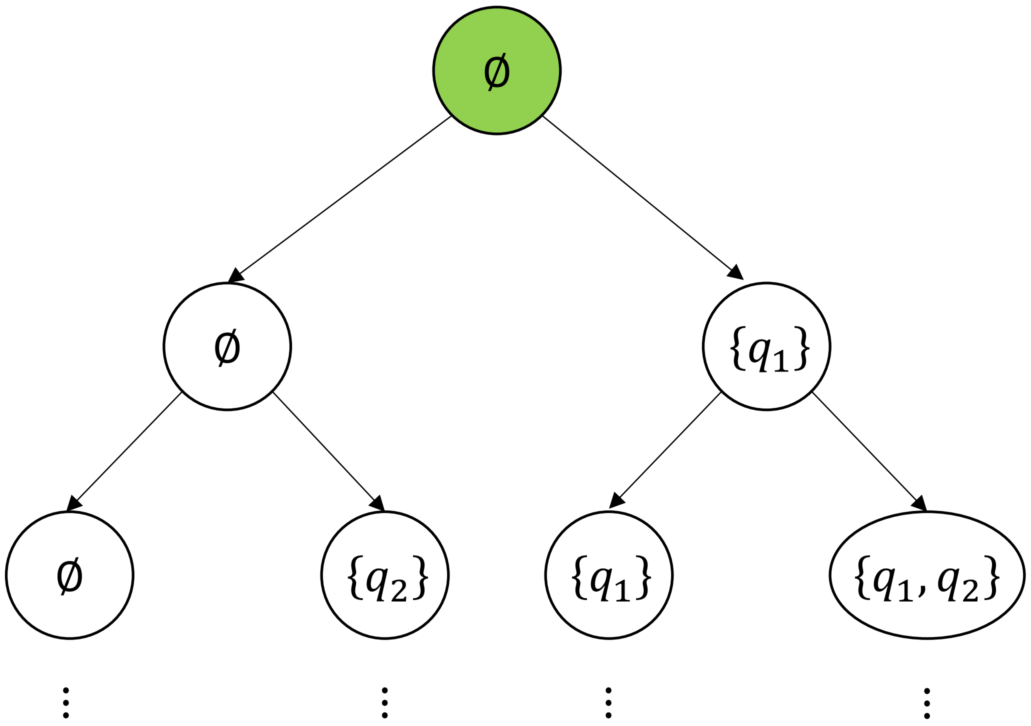
$\boldsymbol{k}$-$\boldsymbol{d}$ Tree Structure



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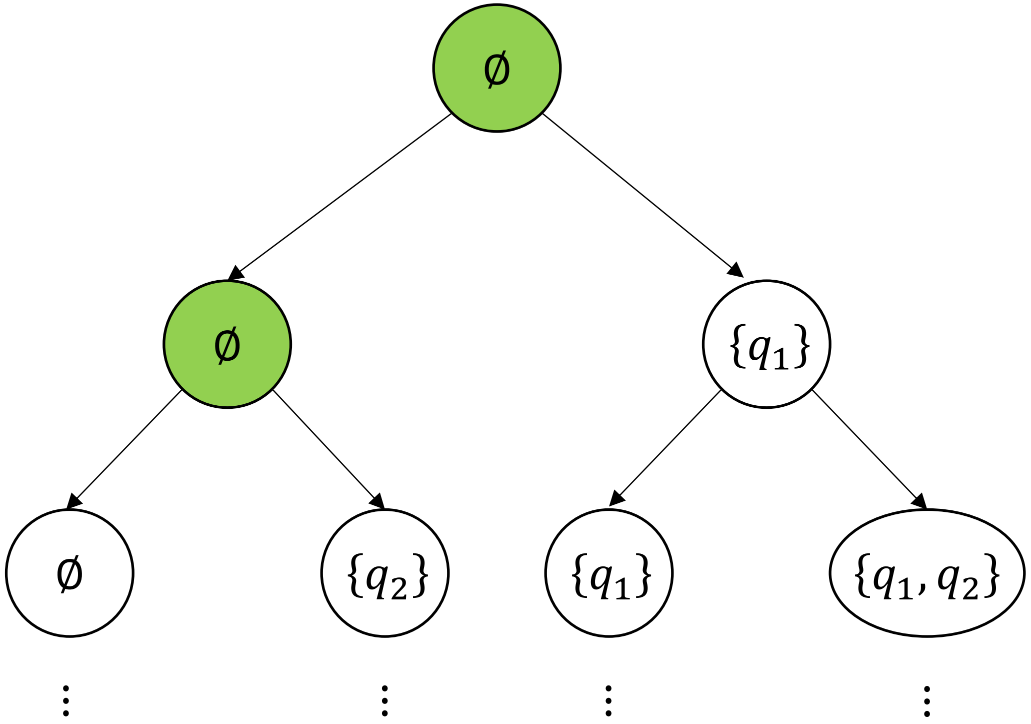

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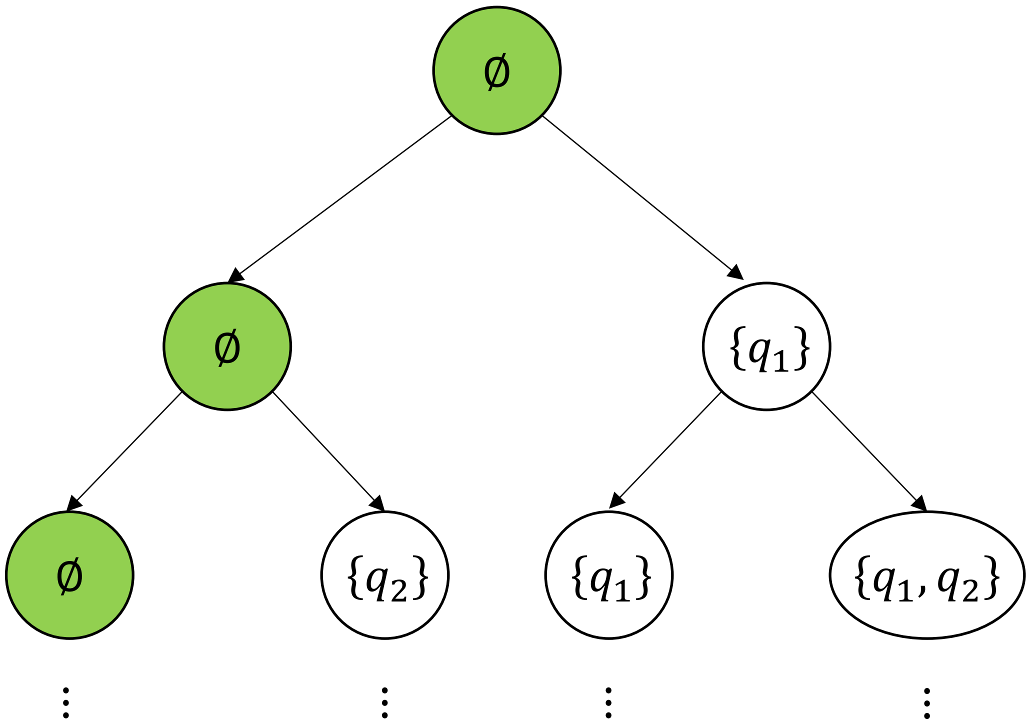

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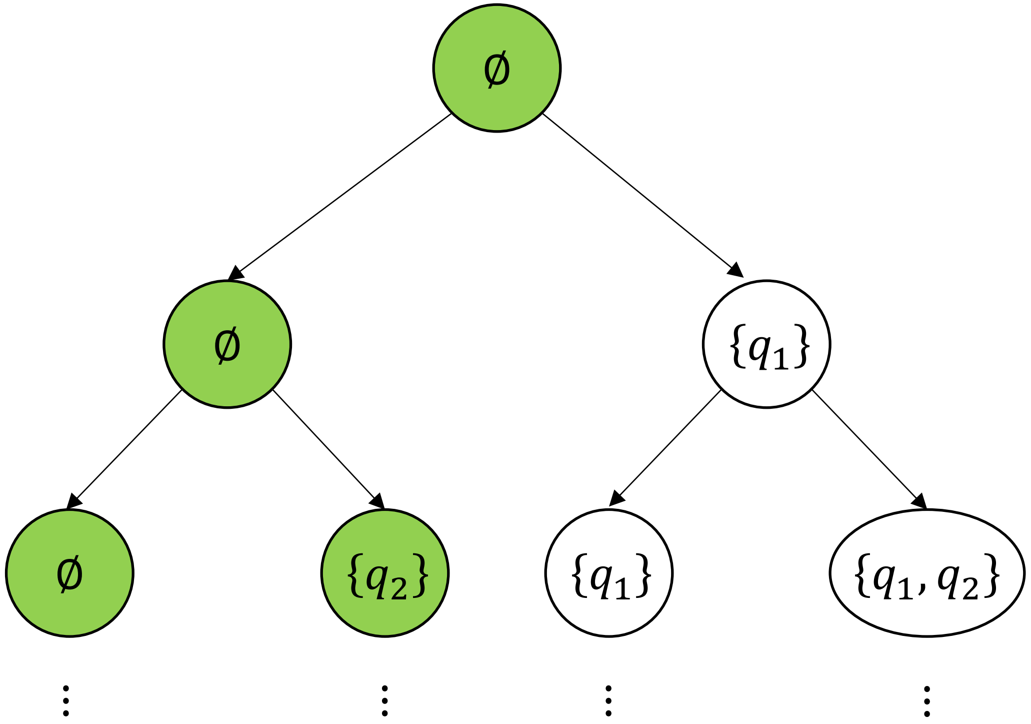
  2. Check left tree
  3. At level i check right tree if $q_{i+1}^{\mathfrak{A}} \neq \emptyset$

k-d Tree Structure



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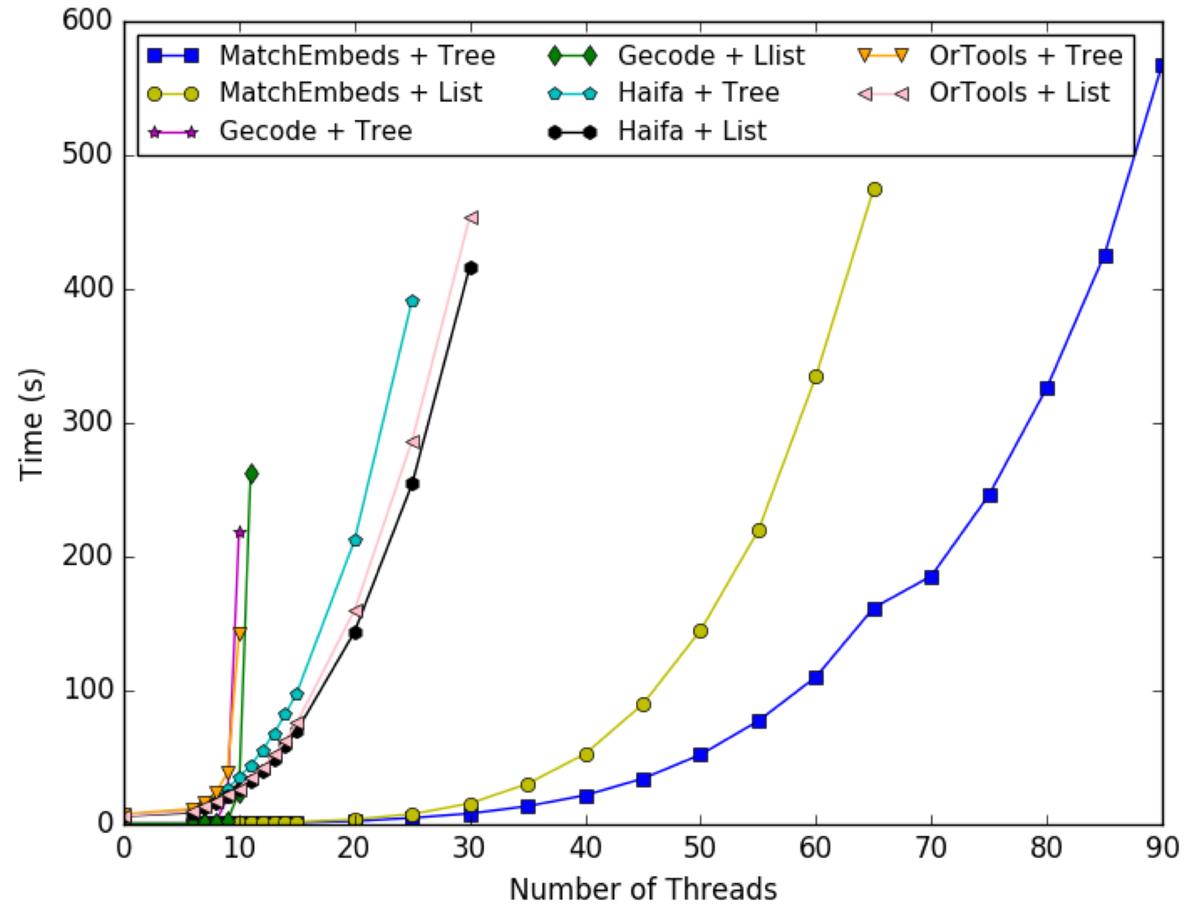
  create constraint $\left\langle X_{a_1}, \dots, X_{a_{ar(q_i)}} \right\rangle \in q_i^{\mathfrak{B}}$

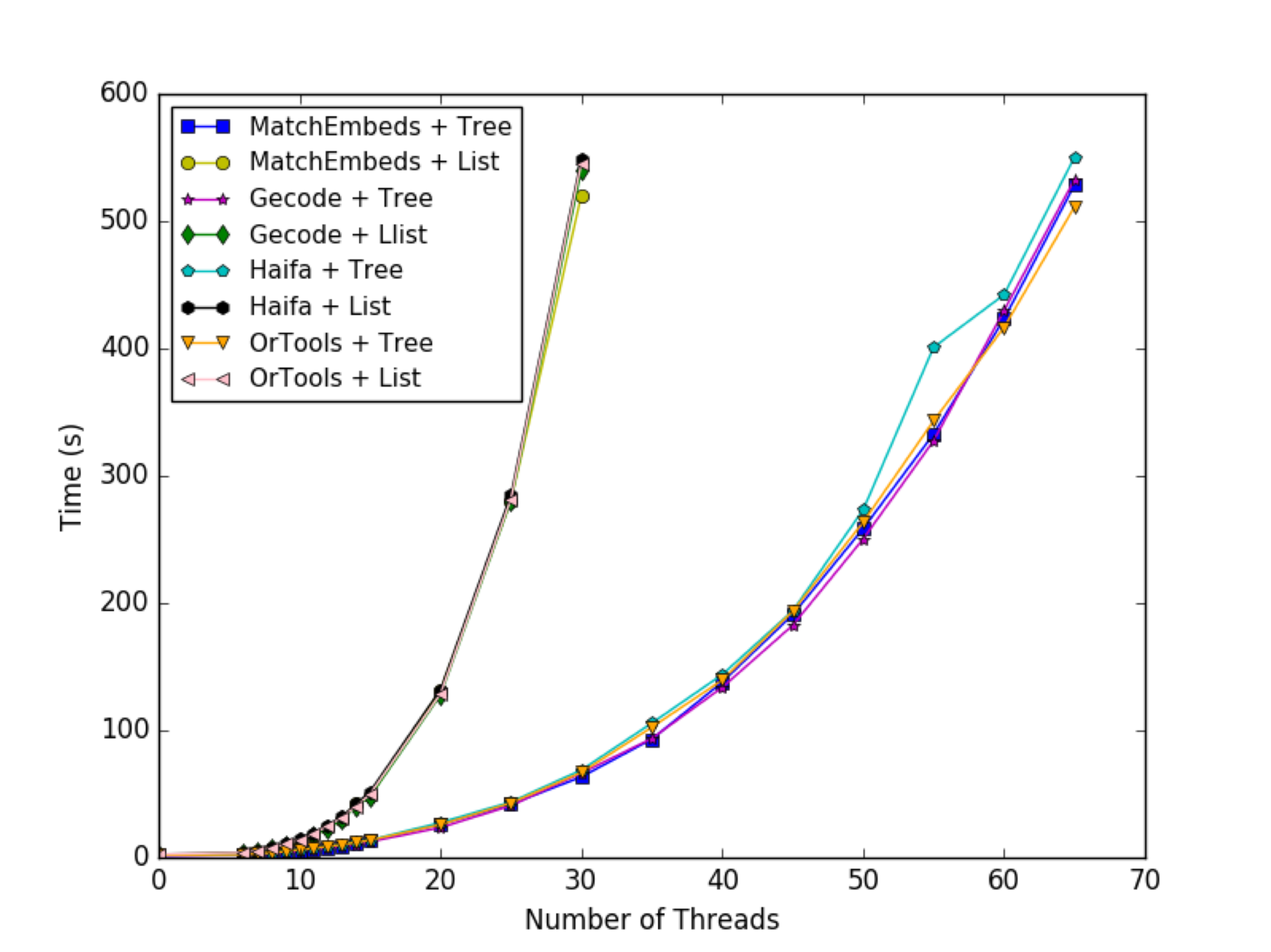
Experiment Count Threads

```
main():
  count = 0
  for i = 1 to N:
    fork thread
  assert(count ≤ N)

thread():
  count = count+1
```



Experiment Secret Sharing

```
main():
 from = 0
 while (*)
  local secret = *
  assume(secret > 0)
  for i = 1 to N:
   to = secret
   fork thread
   while (to > 0): skip
  if (from > 0):
   assert(from == secret)

thread():
  local m = to
  to = 0
  from = m
```



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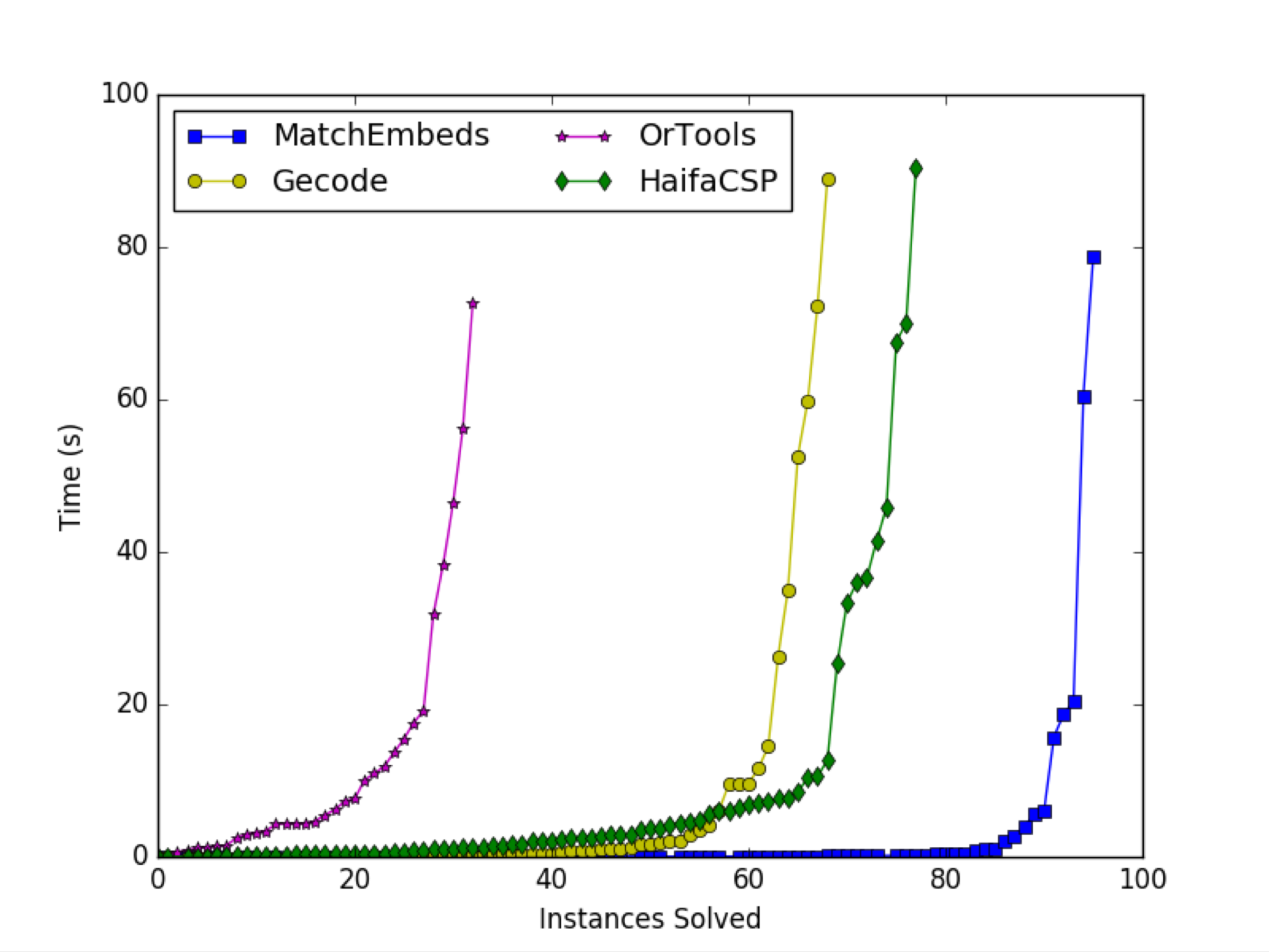
    - $e' \in [e, 4e]$

Experiment Random Graphs

- Generate 100 instances
  - 48 positive embeddings
  - 47 negative embeddings
  - 5 unsolved embeddings



Experiment Random Monadic Structures

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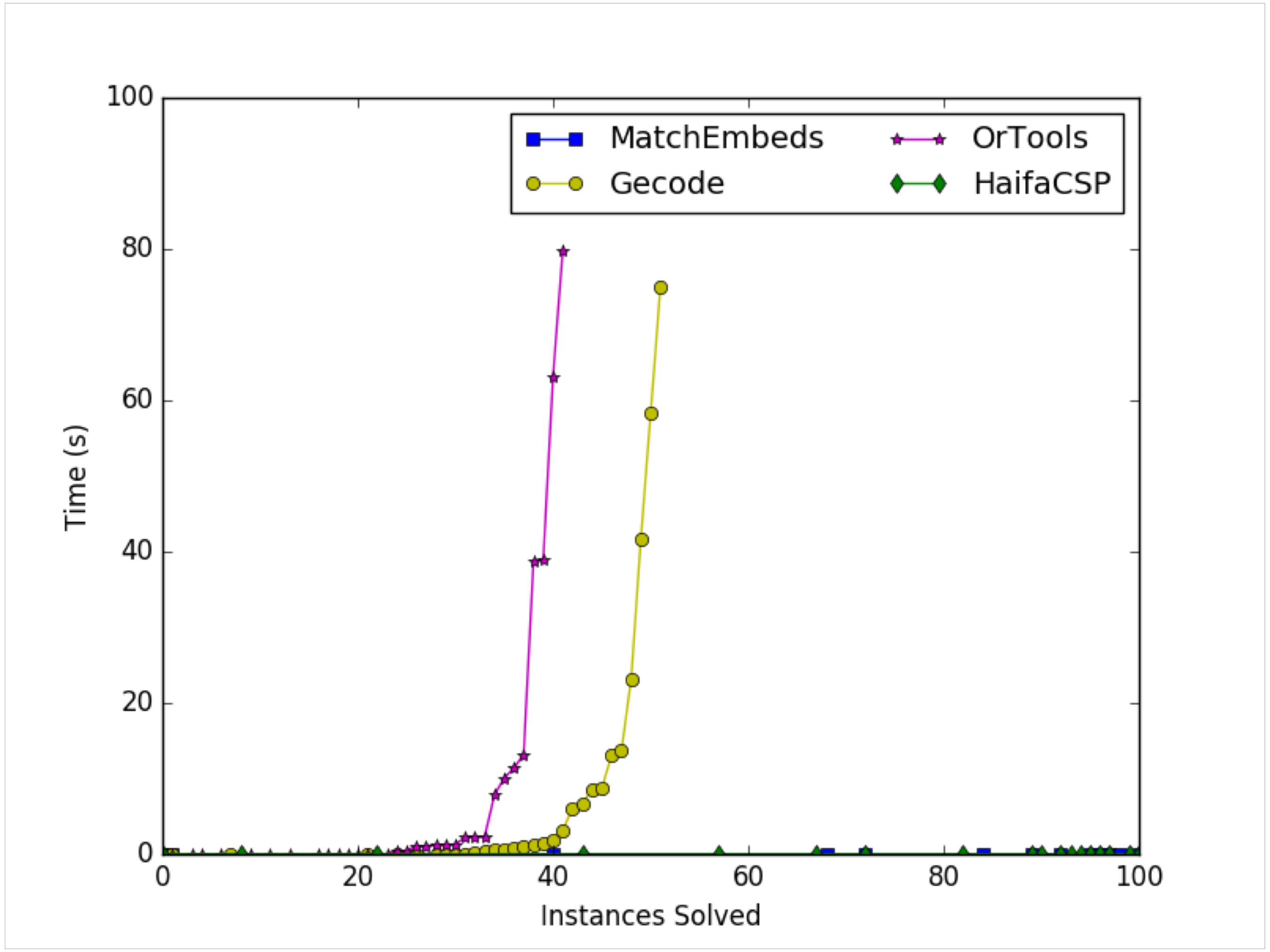

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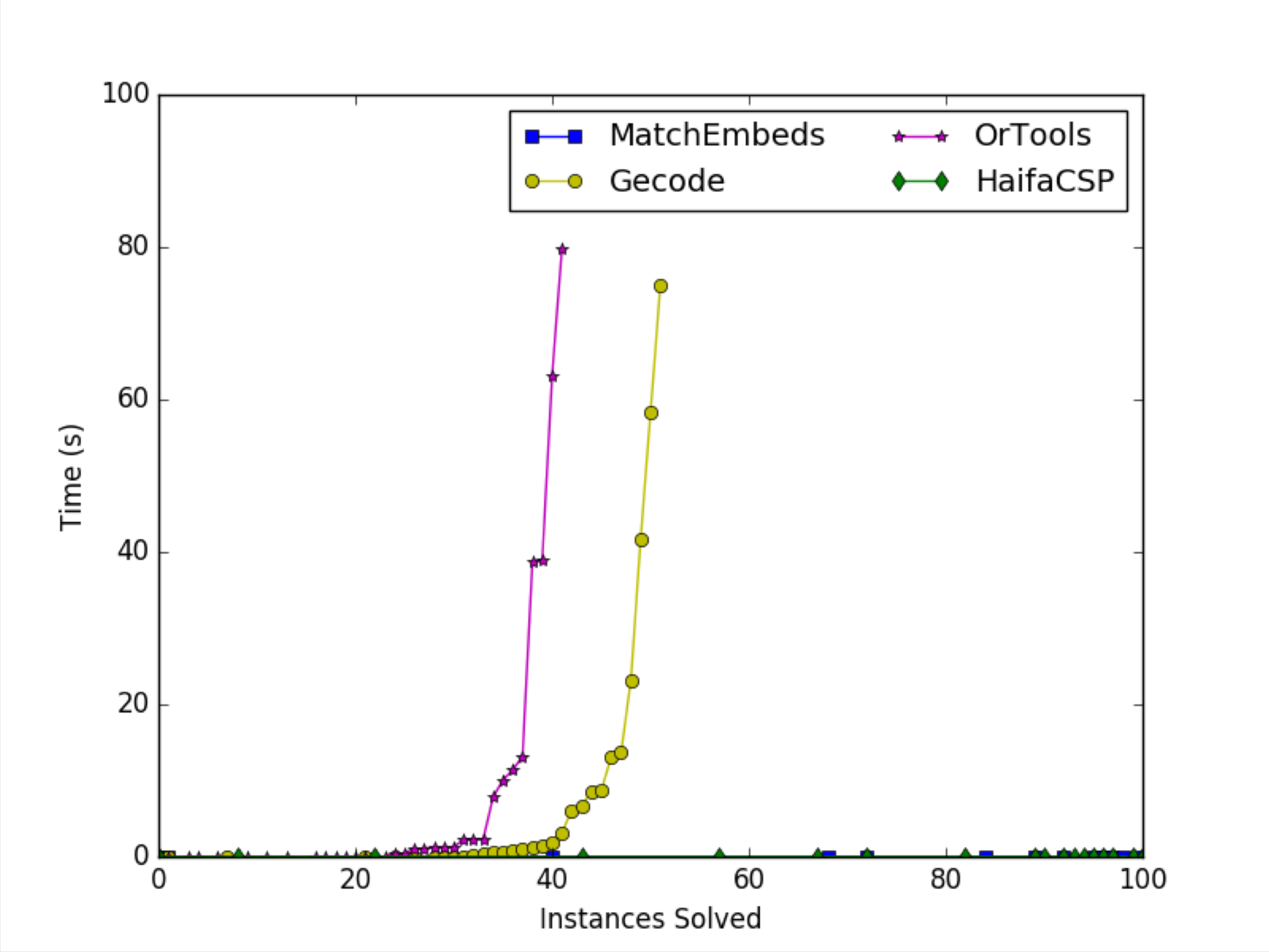
- Match Embeds & HaifaCSP[1]
  - Polytime monadic instances

[Régin, 1994][1]



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  - Focus: find all such isomorphisms
  - Exploit local structure rather than global structure
    - None known to take advantage of all difference constraint

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  - Improves Proof Spaces
    - Verify programs with 70 threads vs 20-30 threads
- k-d structure (multi-source embeddings)
  - Avoids unnecessary embeddings
  - Further Improves Proof Spaces
    - Verify programs with 20+ more threads.

References

[1] Kincaid, Z. Podelski, A., Farzan, A. *Proof Spaces for Unbounded Parallelism*. POPL, pgs. 407-420 (2015).

[2] Finkel, A. Schnoebelen, Ph. *Well Structured Transition Systems Everywhere.* Theoretical Computer Science Vol 256:1, pgs. 63-92 (2001).

[3] Hopcroft, J., Karp, R. *An* $n^{5/2}$ *Algorithm for Maximum Matchings in Bipartite Graphs*. SIAM Journal of Computing, Vol. 2, No. 5 : pgs. 225-231 (1973).

[4] Régin, J.C.: *A filtering Algorithm for Constraints of Difference in CSPs*. In: AAAI. pgs. 362-367 (1994)

[5] Russell, S.J., Norvig, P. *Artificial Intelligence - a Modern Approach*, 3rd Edition. Prentice Hall series in Artificial Intelligence, Prentice Hall (2009)

Extra Slides

Proof Spaces

[Kincaid et. al. 2015]

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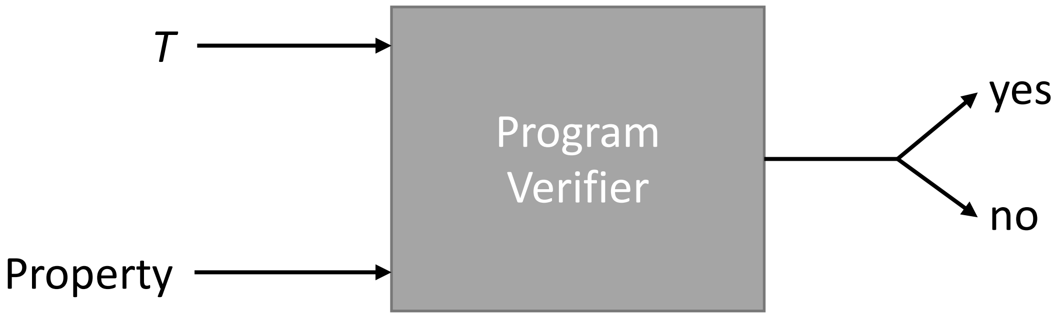

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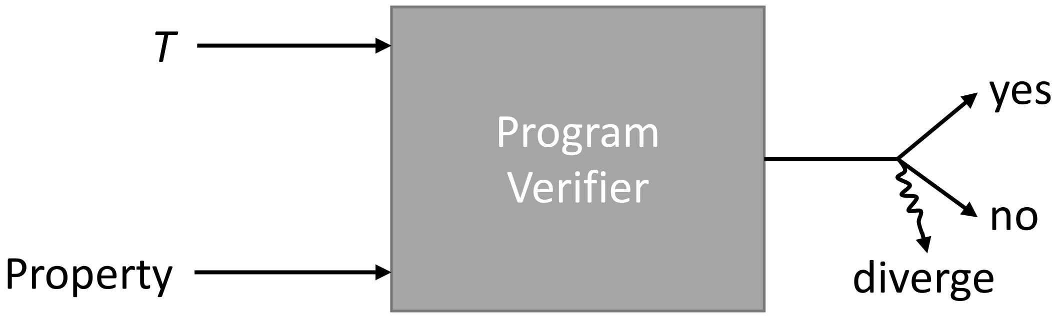

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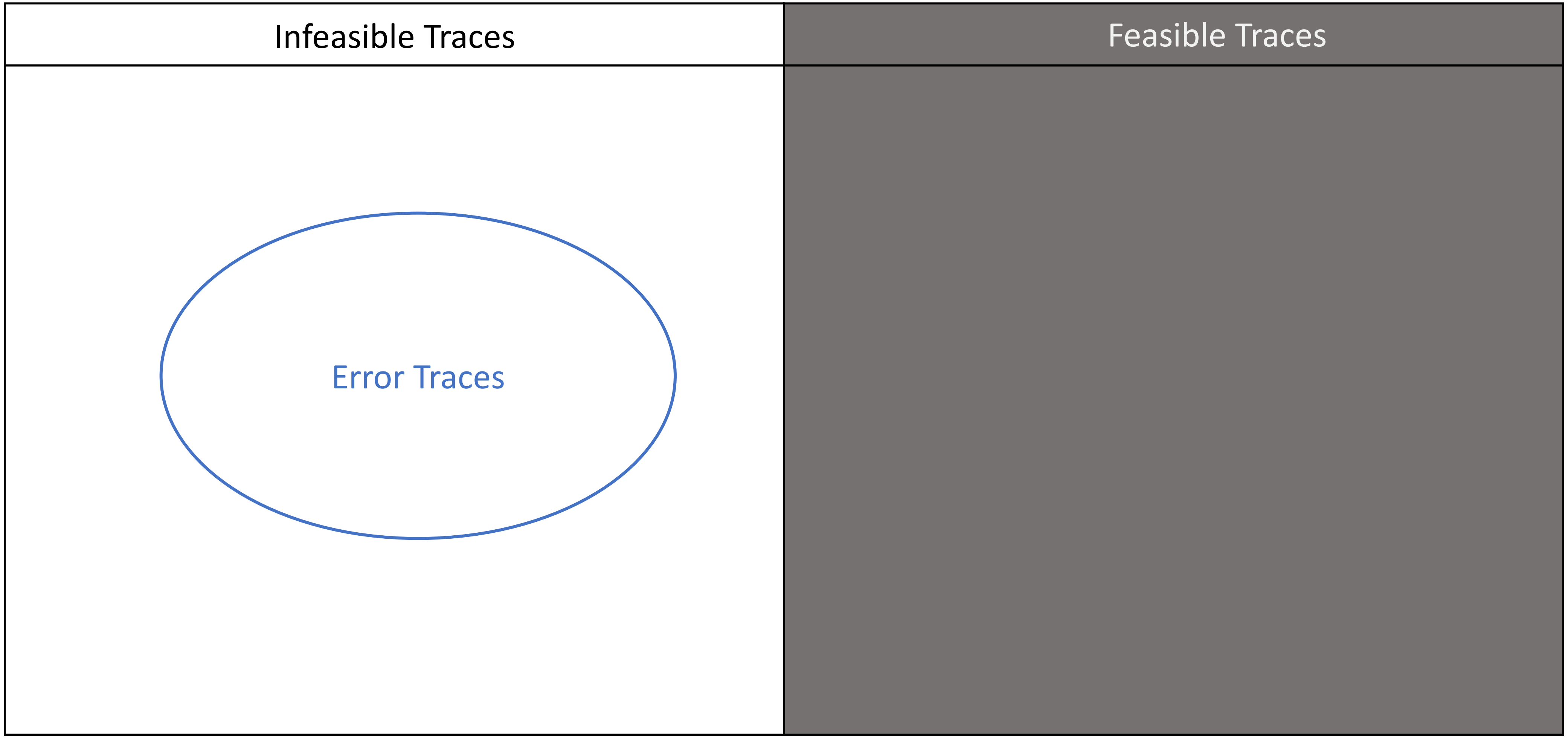

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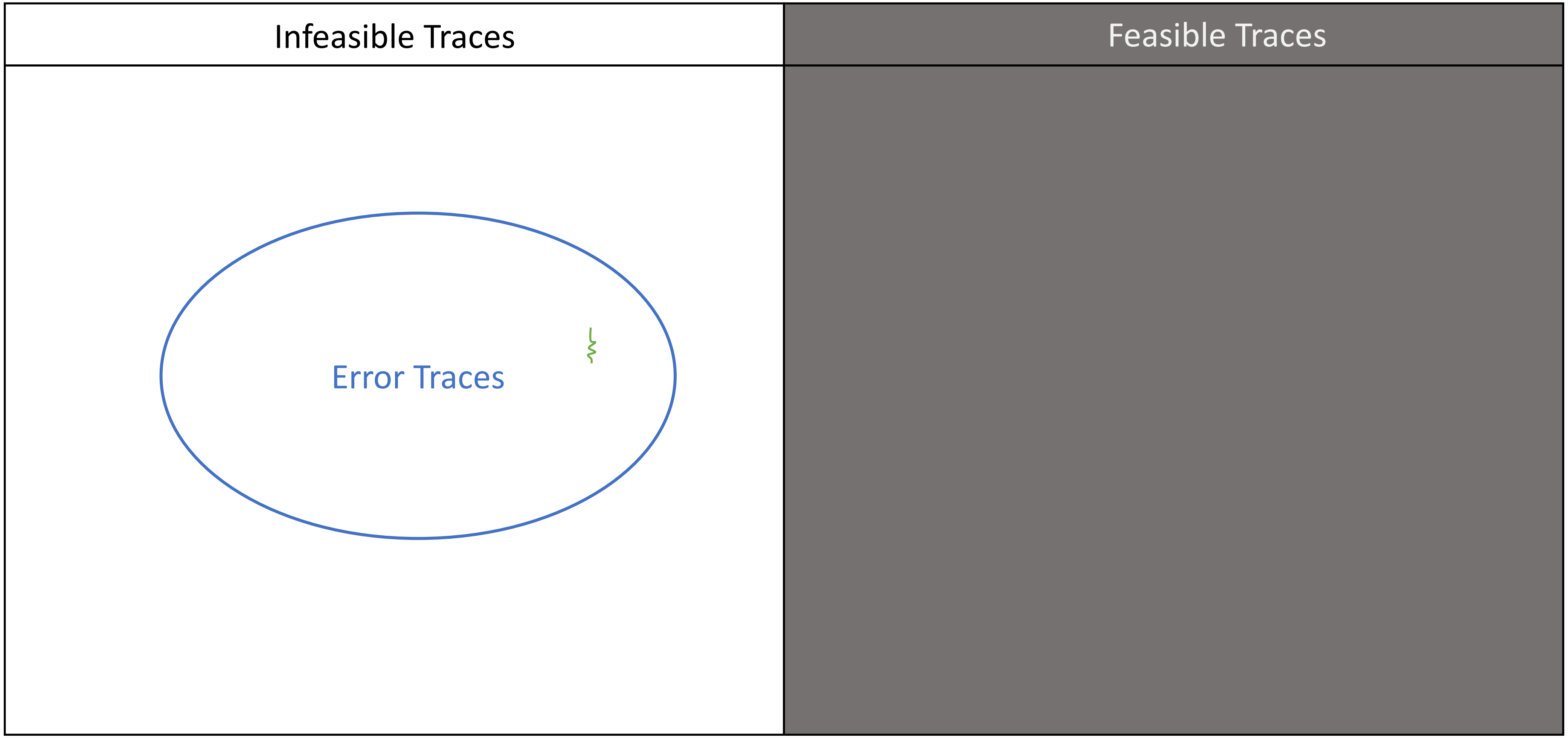

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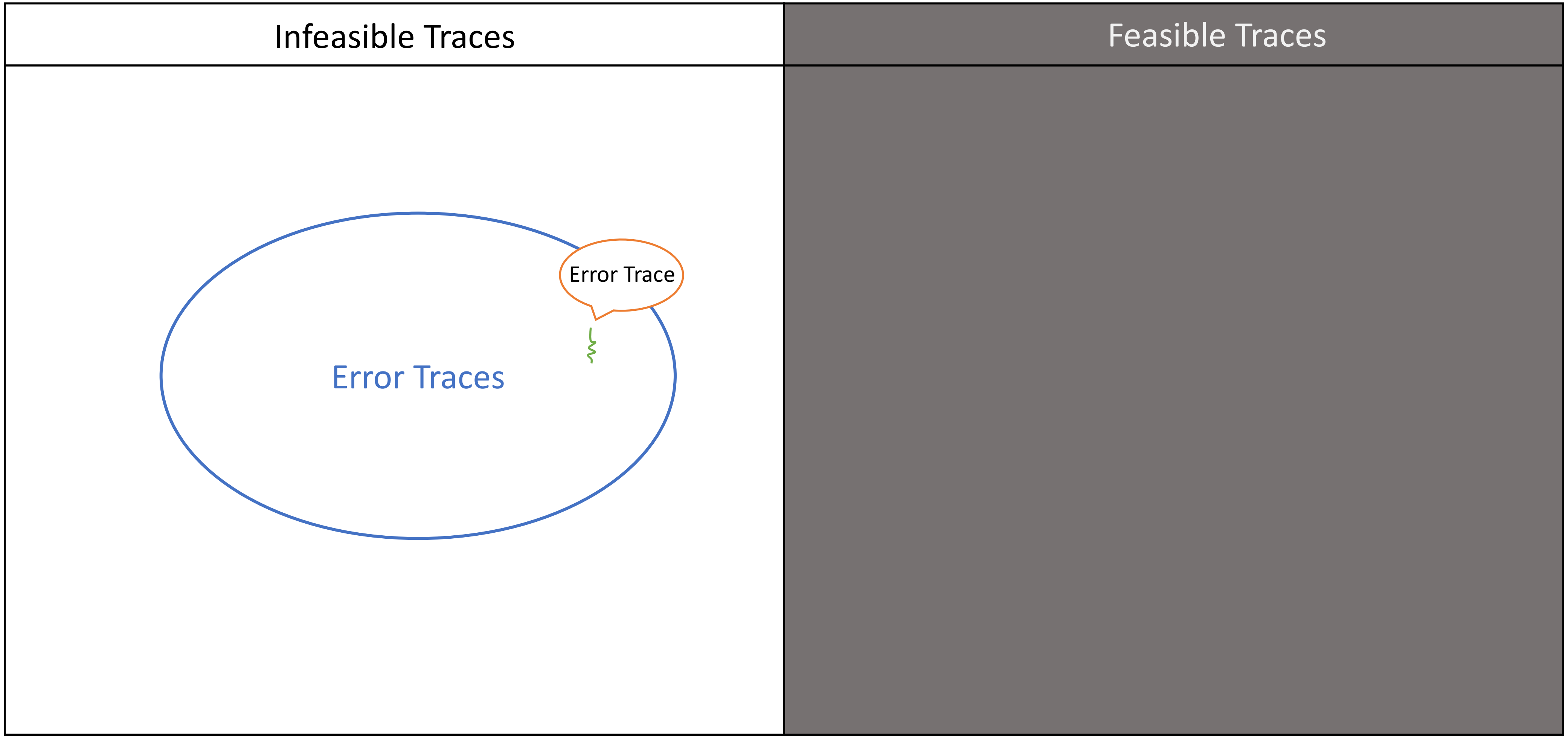

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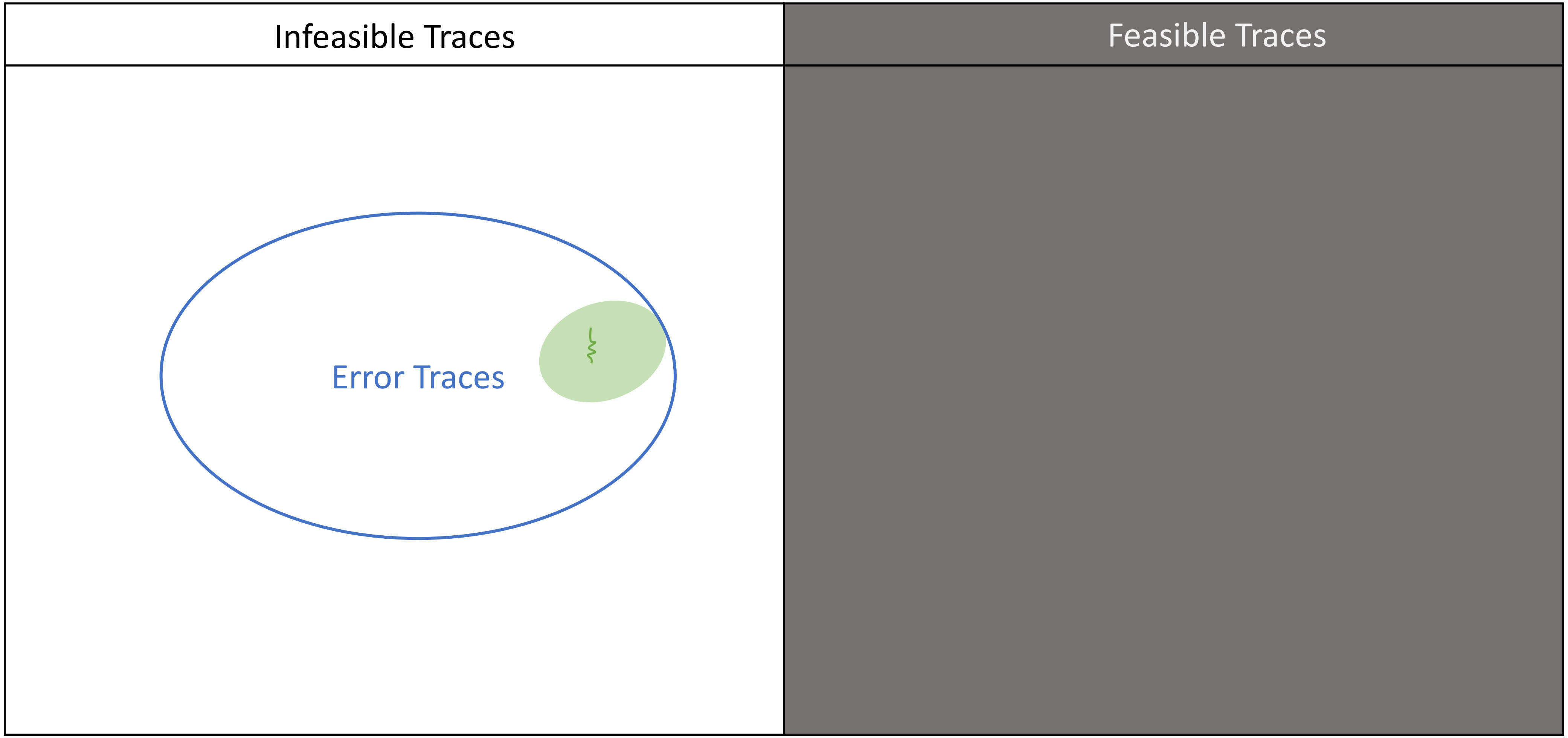

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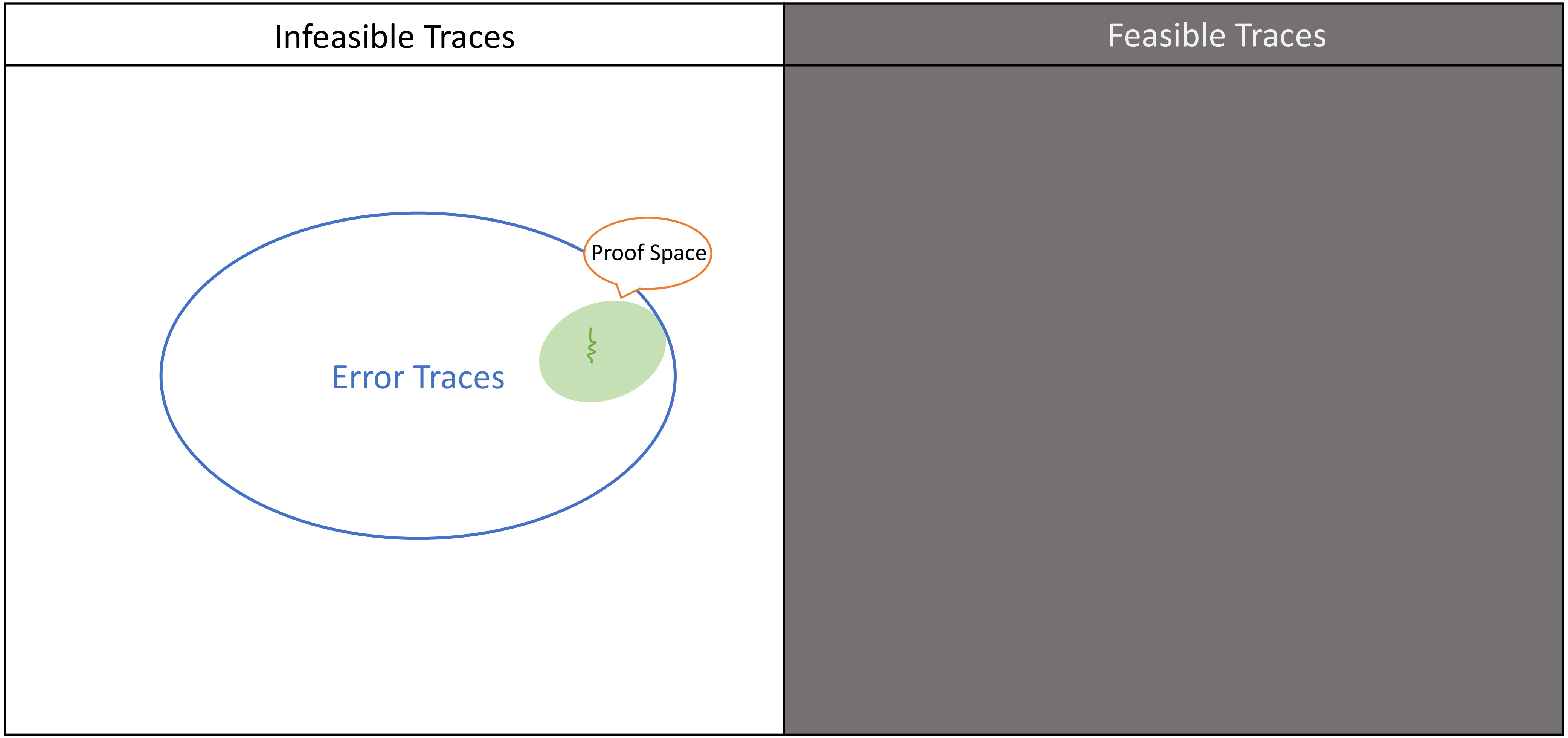

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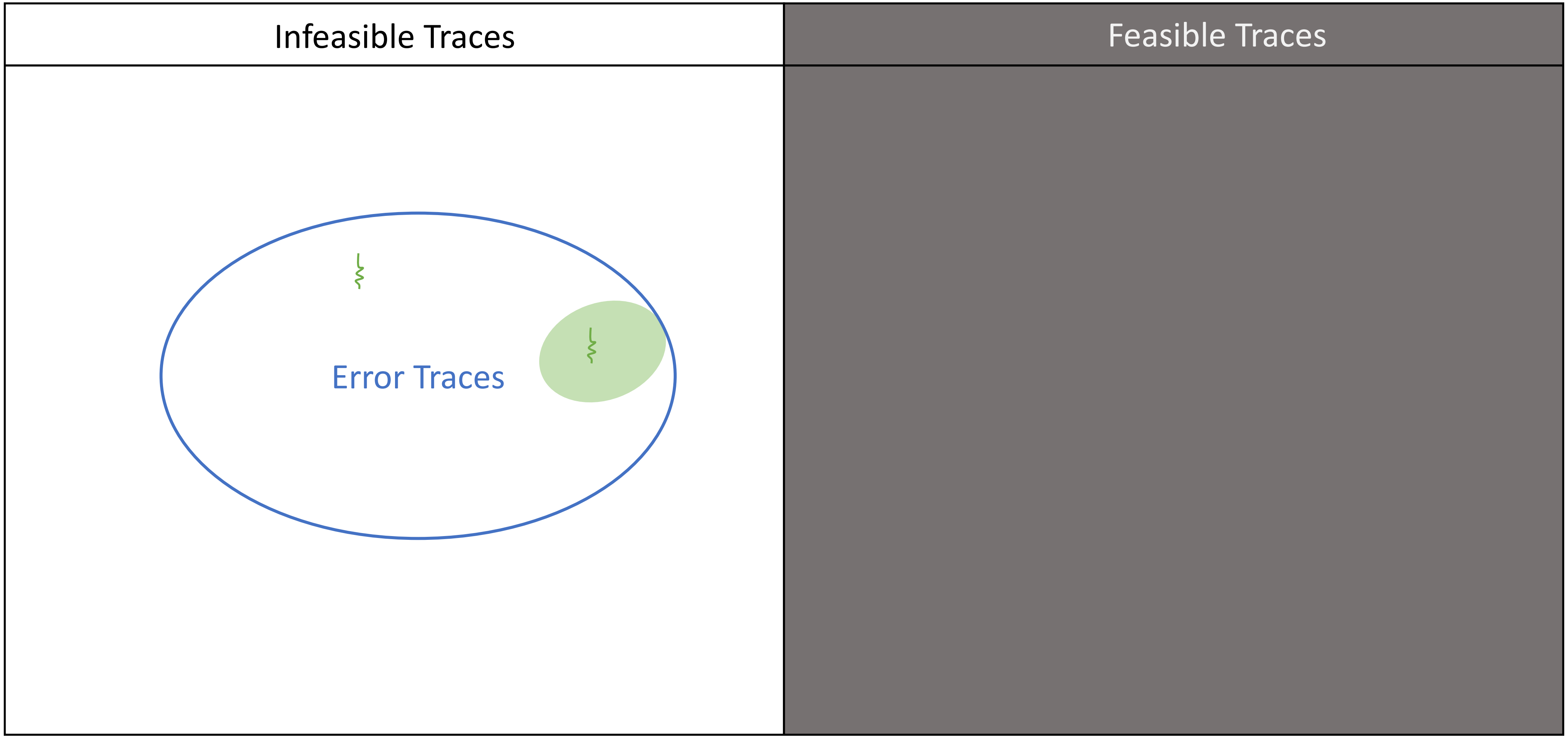

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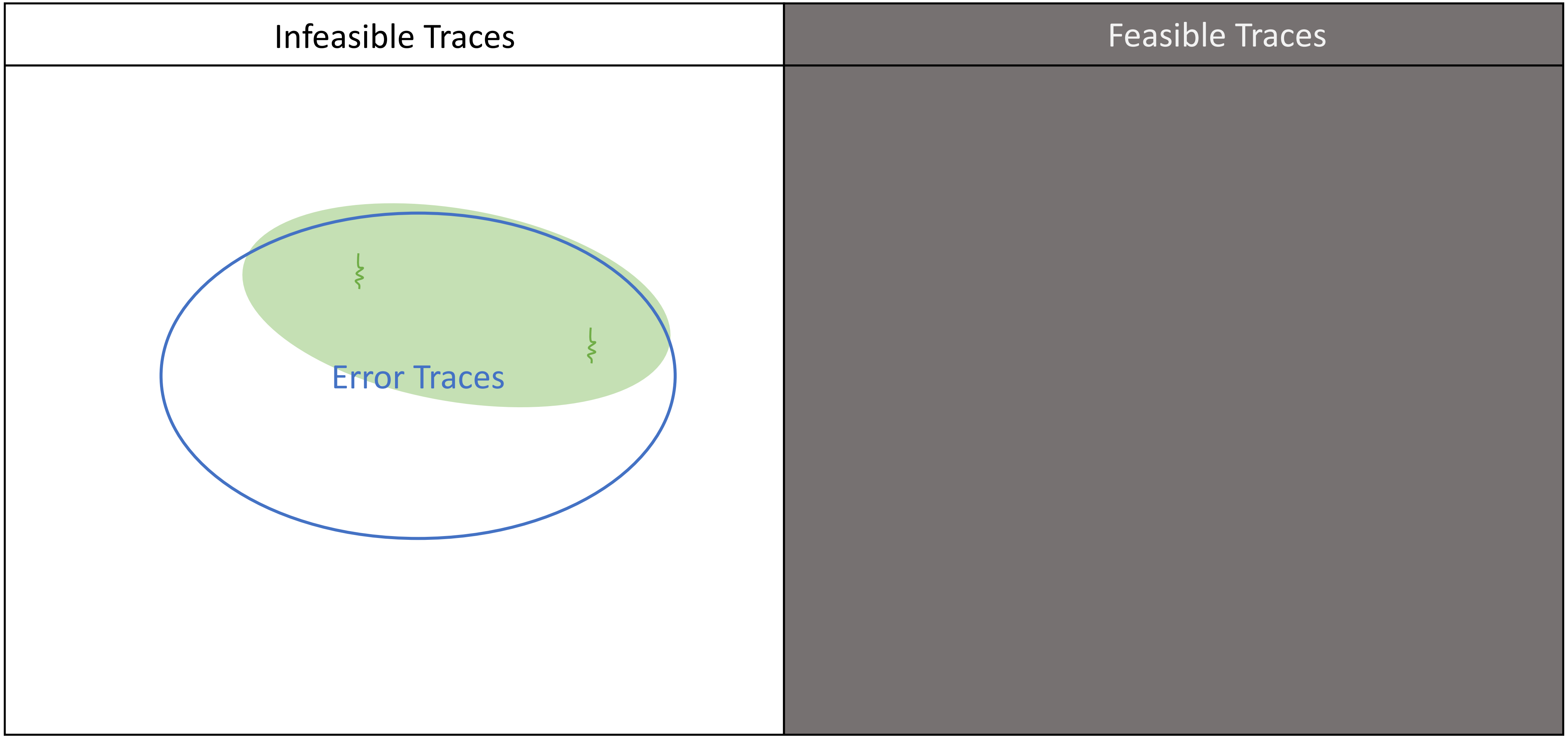

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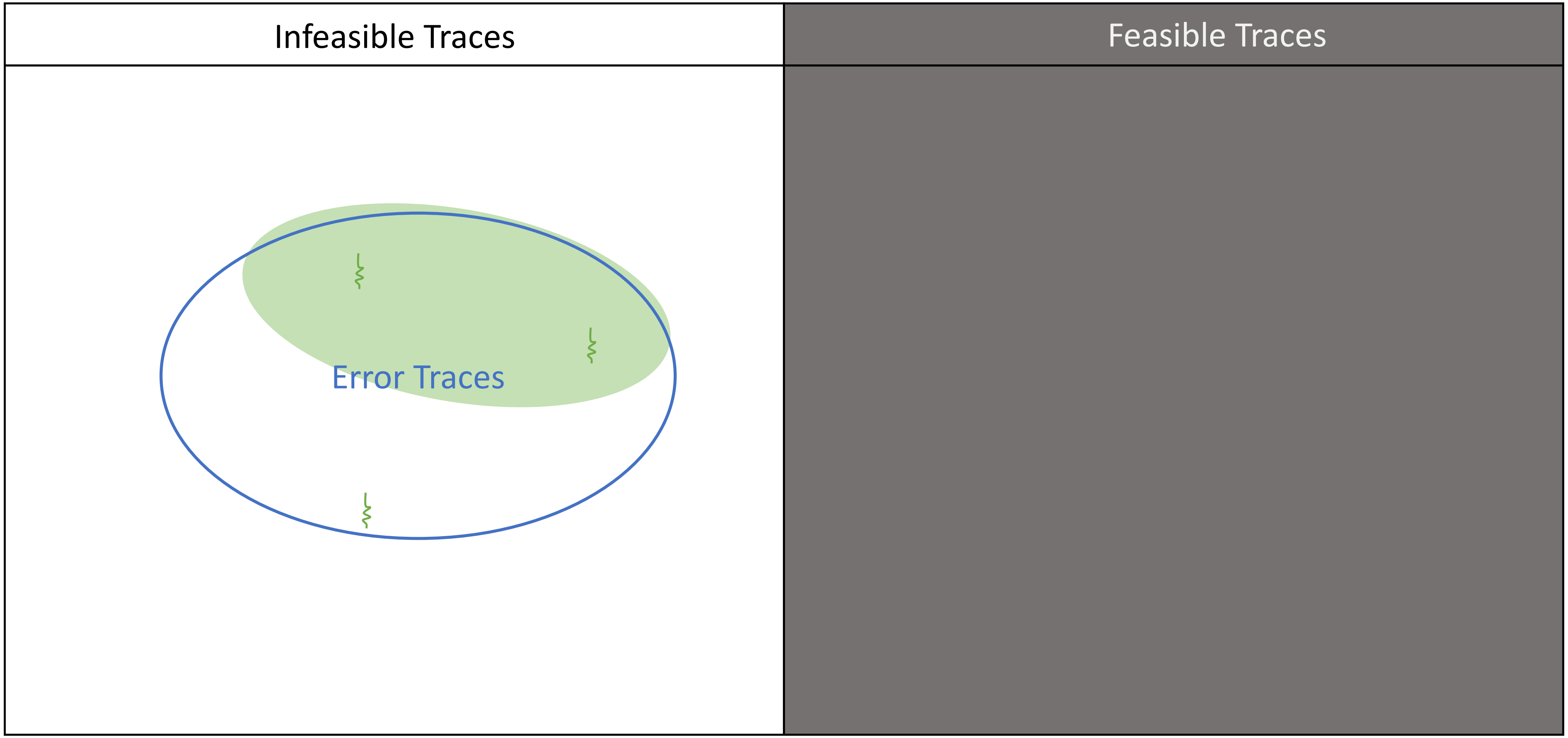

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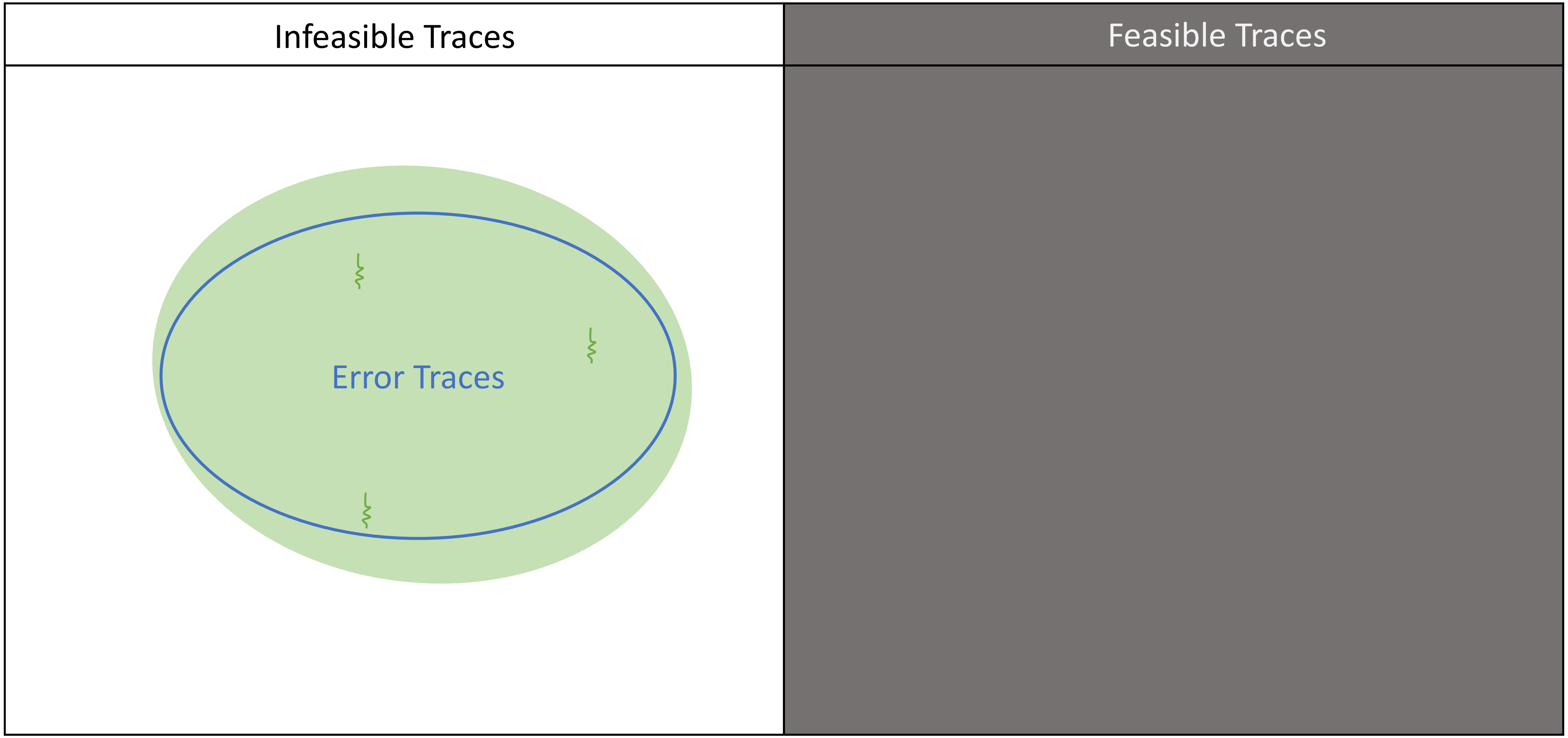

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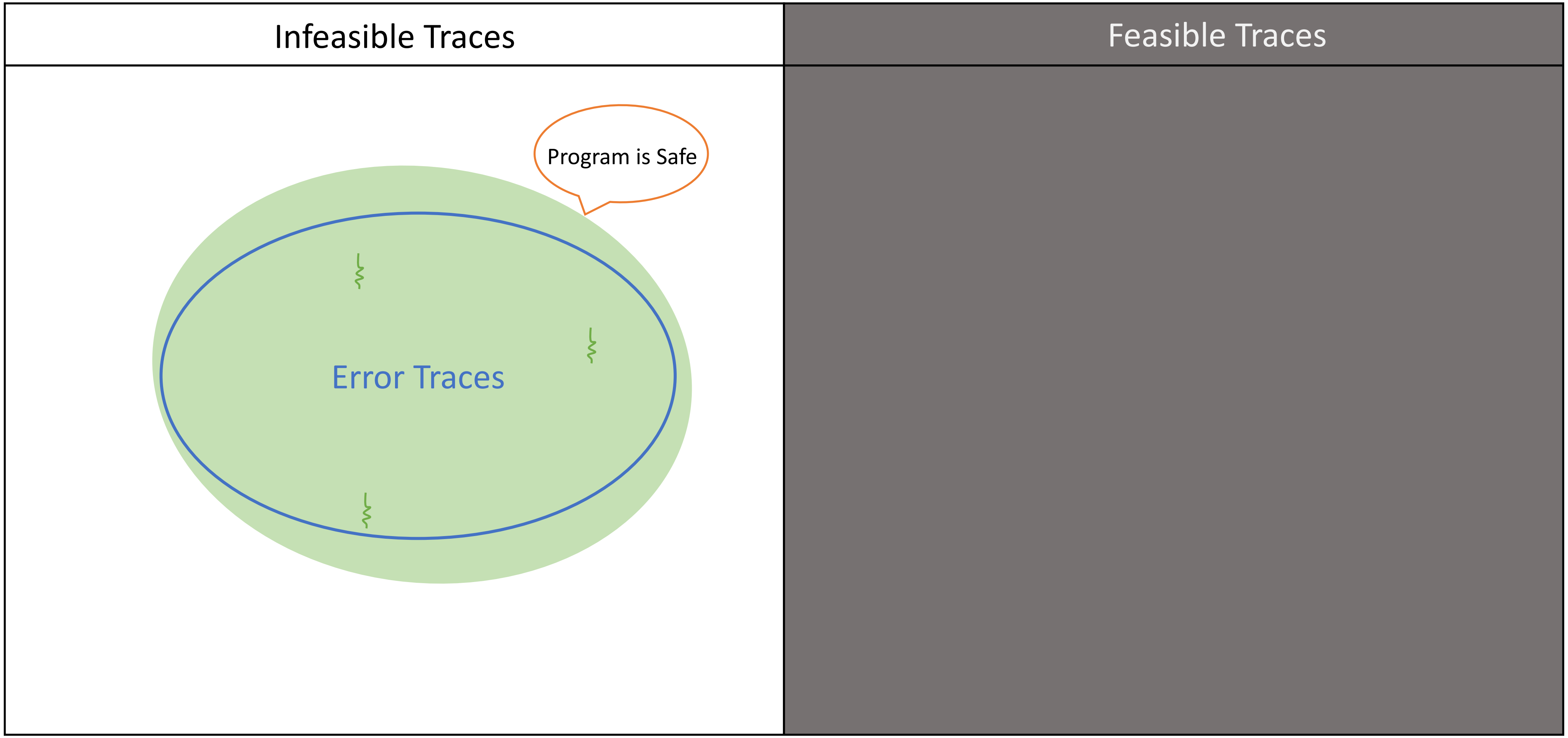

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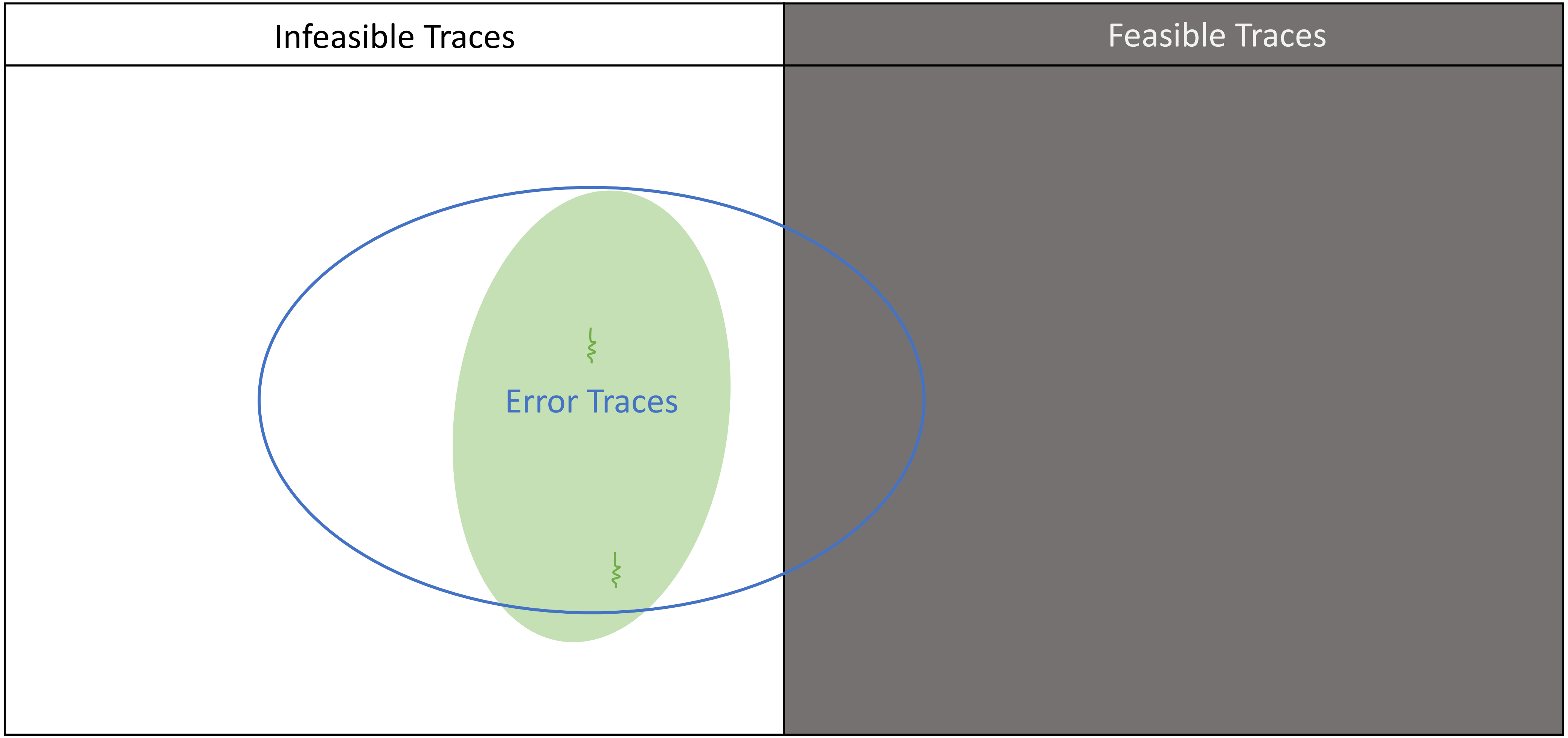

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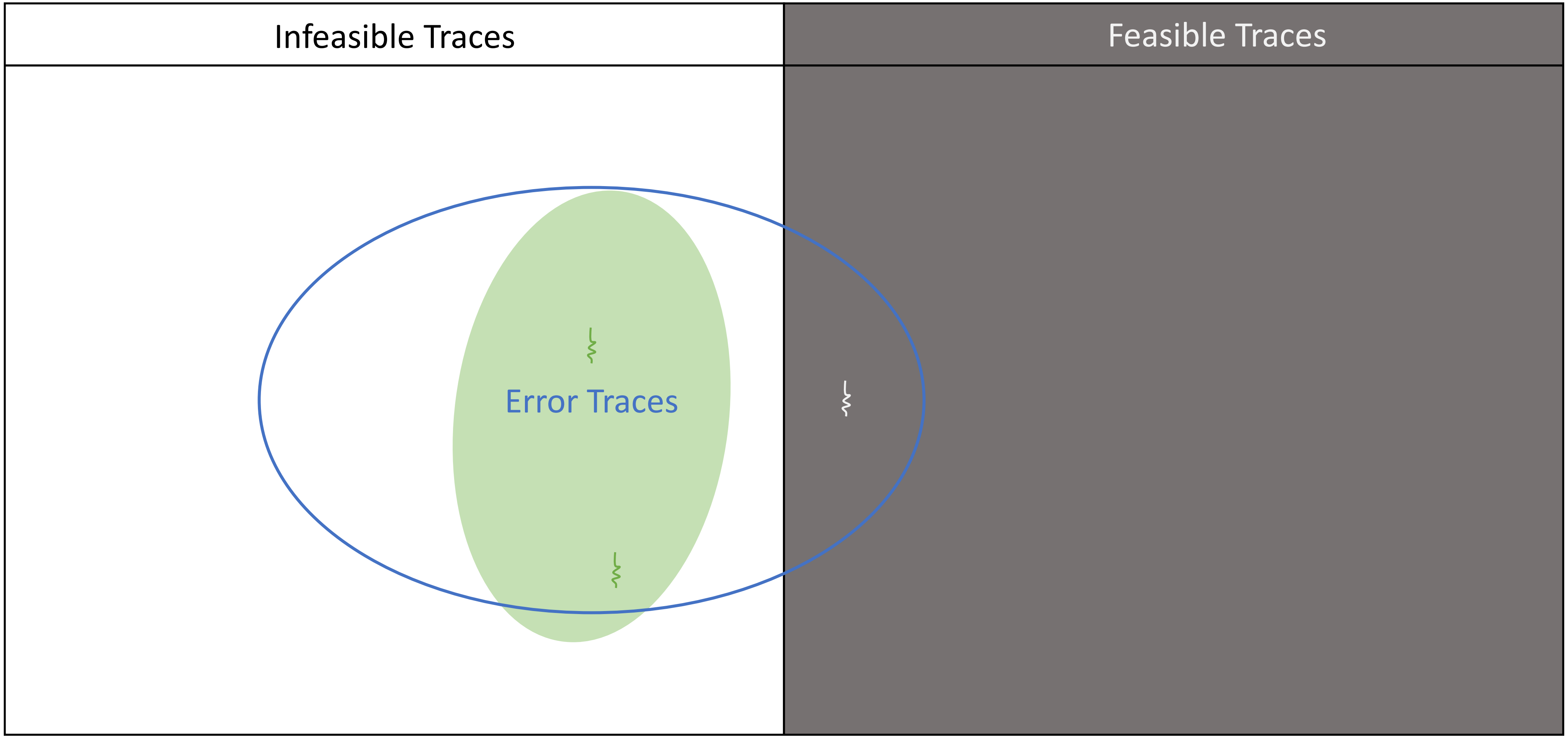

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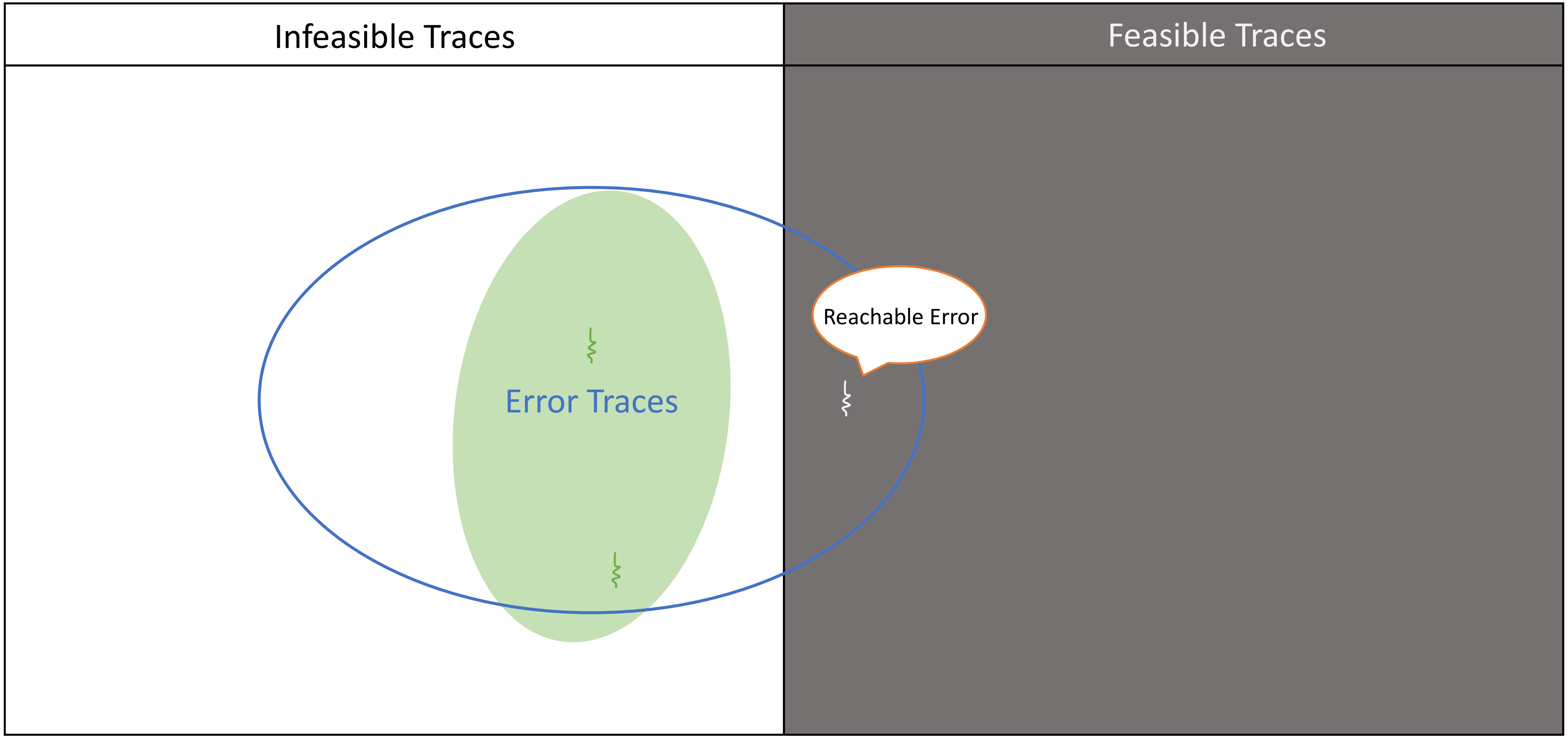

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$$\{\text{pre}\}\ \tau\ \{\text{false}\} \in H$$

then the program is safe.

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Proof space inclusion then reduces to PA emptiness:

$$\forall \tau \in \text{Error Trace}.\,\{pre\}\,\tau\,\{false\} \in H$$
$$\Leftrightarrow$$
$$Err \cap \overline{A(H)} = \emptyset$$

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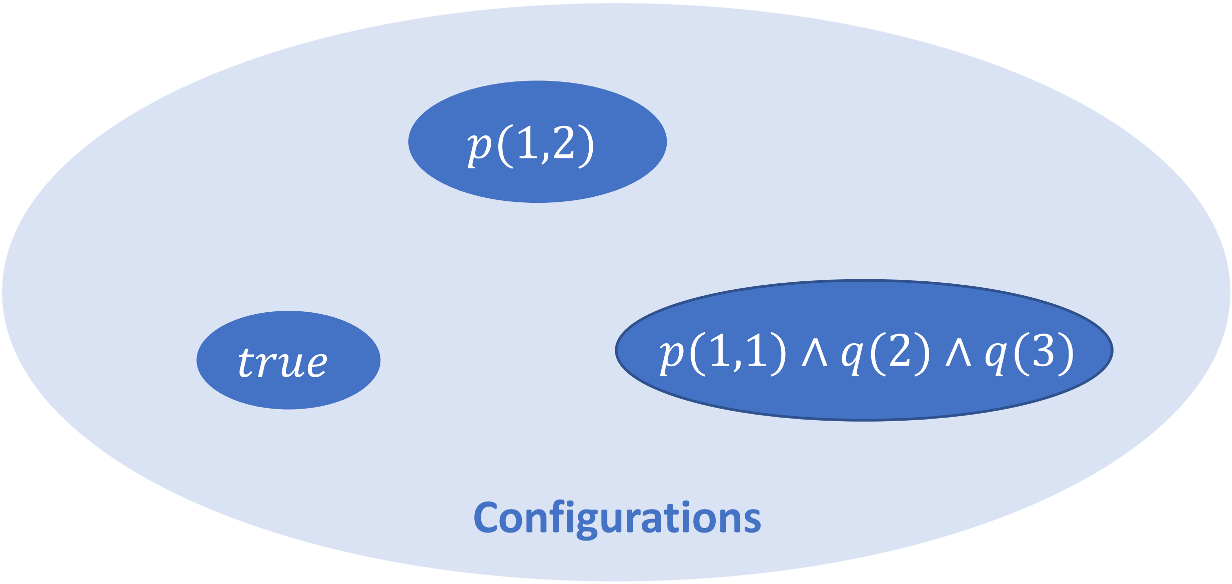

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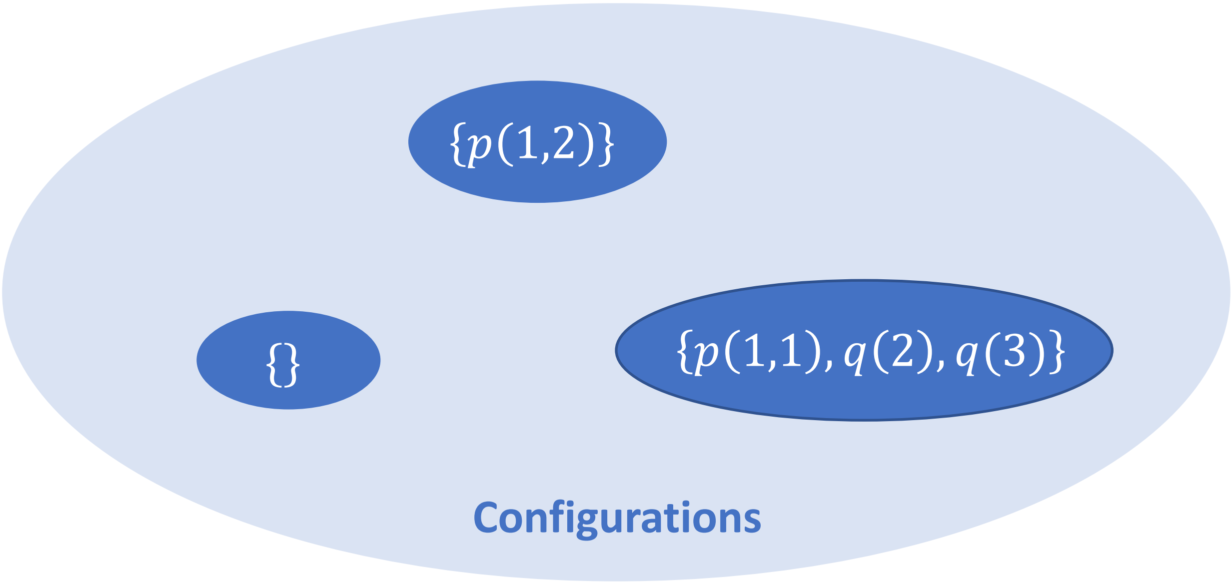

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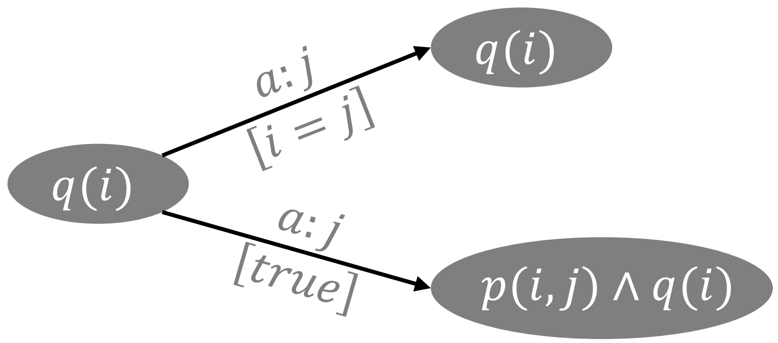

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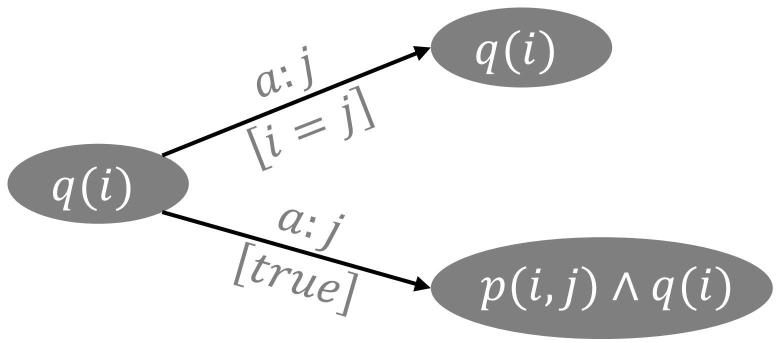

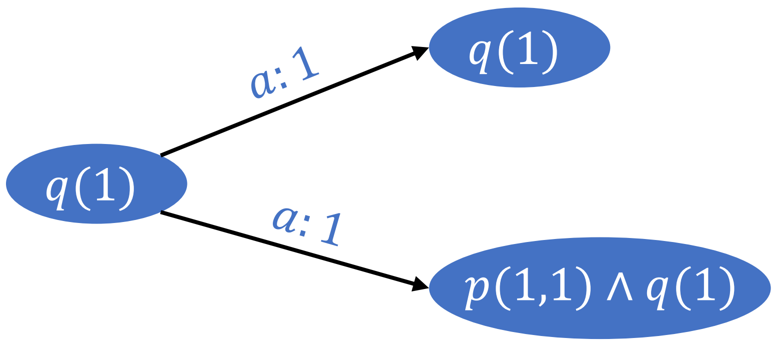

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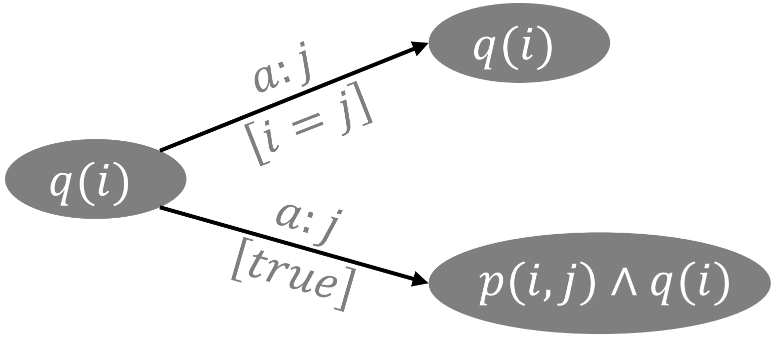

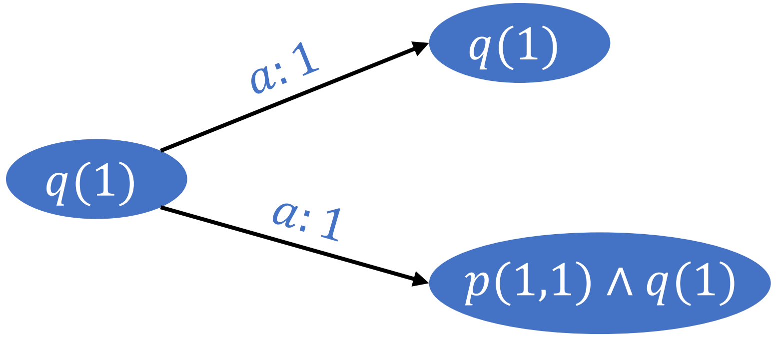

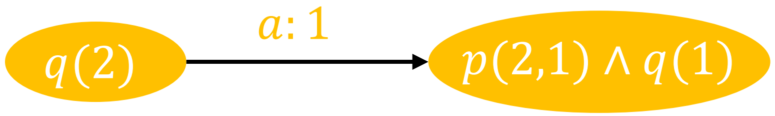

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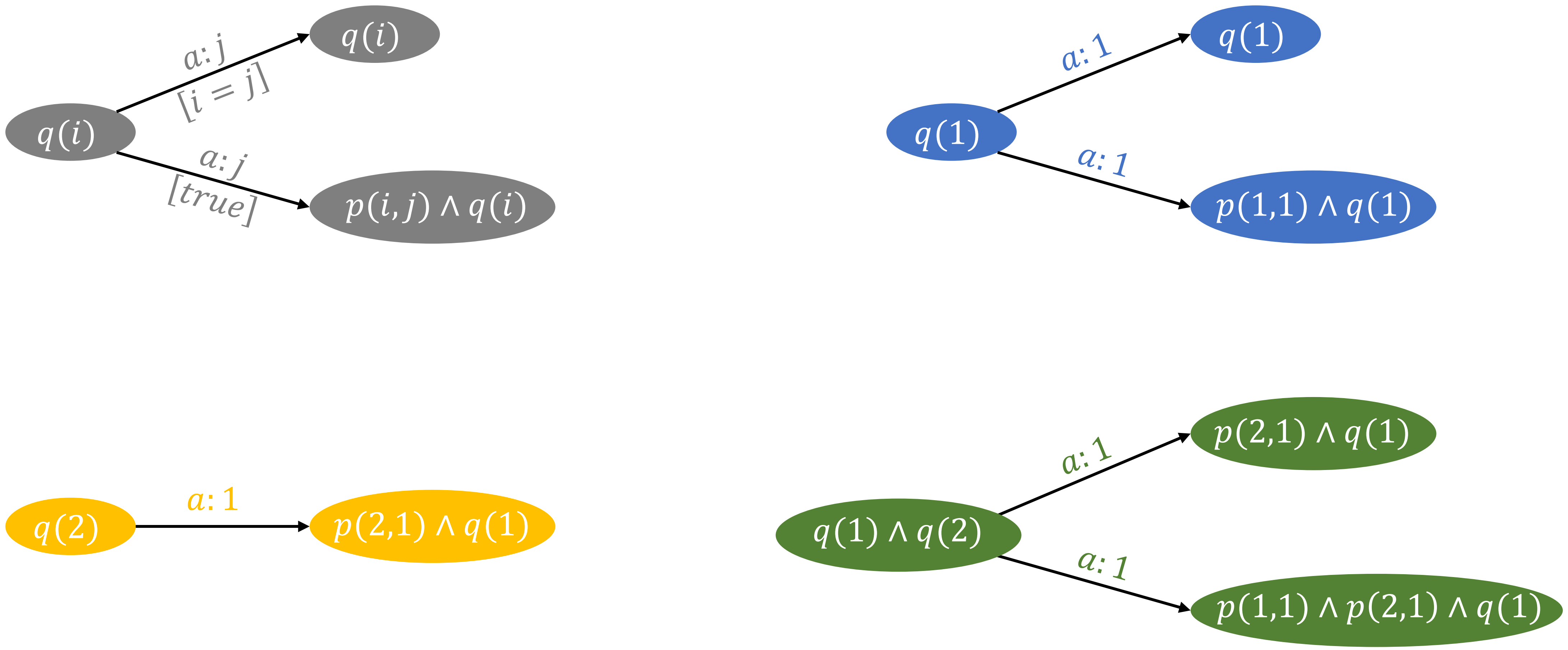

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    - Q : Finite set of predicate symbols
    - $\mathrm{ar} : \mathrm{Q} \rightarrow \mathbb{N}$
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Predicate Automata

- Infinite State Automata over Infinite Alphabet ($\Sigma \times \mathbb{N}$)
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    - $F \subseteq \mathrm{Q}$: Set of accepting predicate symbols.
    - $\delta : \mathrm{Q} \times \Sigma \rightarrow \mathcal{F}(Q, ar)$ the only free variables of $\delta(q, \sigma)$ are the free variables of q and σ

Emptiness Algorithm

$Closed \leftarrow \emptyset$
$N \leftarrow \emptyset$
$E \leftarrow \emptyset$
$wl \leftarrow dnf(\varphi_{start})$
while $wl \neq []$ **do**
  $C \leftarrow head(wl)$
  $wl \leftarrow tail(wl)$
  if $\neg\exists C' \in Closed\ s.t. C' \preccurlyeq C$ **then**
    foreach $i \in supp(C) \cup \{1 + \max supp(c)\}$ **do**
      foreach $\sigma \in \Sigma$ **do**
        foreach $C' s.t. C \xrightarrow{\sigma:i} C'$ and $C' \notin N$ **do**
          $N \leftarrow N \cup \{C'\}$
          $E \leftarrow E \cup \{C \xrightarrow{\sigma:i} C'\}$
          if C is accepting **then**
            return a word w labeling a path in the graph (N, E) from C to a root
          else
            $wl \leftarrow wl \mathbin{++} [C']$
  $Closed \leftarrow Closed \cup \{C\}$
return Empty

Configurations and Coverings

- A Configuration, C, Accepts iff $\{q | q(i_0, \cdots, i_{ar(q)}) \in C\} \subseteq F$
- $C \overset{\sigma:k}{\longrightarrow} C'$ iff C' is a cube of (in DNF)

$$\bigwedge_{q(i_1,\cdots,i_{ar(q)})\in C} \delta(q,\sigma)[\mathrm{i}_o \mapsto k, \mathrm{i}_1 \mapsto i_1, \cdots, \mathrm{i}_{ar(q)} \mapsto i_{ar(q)}]$$

- If $C \preccurlyeq C'$,

  If C' is accepting then C must be accepting

  If $C' \overset{\sigma:j}{\longrightarrow} \overline{C'}$ then $\exists k, C \overset{\delta:k}{\longrightarrow} \overline{C}$ and $\overline{C} \preccurlyeq \overline{C'}$

  Therefore, if C' can reach an accepting state then so must C

Covering Relation (≼)

$$C = \{q(1,2), q(1,3), r(2)\}$$

Covering Relation ($\preccurlyeq$)

$$C = \{q(1,2), q(1,3), r(2)\}$$

$$supp(C) = \{1,2,3\}$$

Covering Relation (≼)

$C = \{q(1,2), q(1,3), r(2)\}$ $\quad$ $C' = \{q(8,7), q(8,6), r(7), r(6)\}$

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$$\pi = \{1 \mapsto 8, 2 \mapsto 6, 3 \mapsto 7, \dots\}$$

Covering Relation (≼)

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Covering Relation (≼)

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$$C \not\preccurlyeq C'$$

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π must be a permutation (injective)

Covering Relation (≼)

- For configurations C and C’, C **covers** C’ ($C \preccurlyeq C'$)

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- Downward Compatibility with PA[1,2]

[Kincaid et. al. 2015][1] [Finkel and Schnoebelen. 2001][2]